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SPECTRUM OF THE COZERO-DIVISOR GRAPH OF THE RING $\mathbb{Z}_n$

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Abstract. Consider a commutative ring R with an identity element $1 \neq 0$. The set $Z(R)'$ is defined as the collection of all elements in the ring R that are non-zero and non units. The undirected graph $\Gamma'(R)$ with vertex set $Z(R)'$, known as the cozero-divisor graph of R. In this graph, two distinct vertices, denoted by x and y, are connected by an edge if and only if x is not an element of the ideal yR and y is not an element of the ideal xR. In this article, we find the Laplacian eigenvalues of the graphs $\Gamma'(\mathbb{Z}_n)$ for $n = q_1^Q q_2 q_3$ and $q_1^{Q_1} q_2^{Q_2} q_3$, where q_1, q_2, q_3 are distinct primes and Q, Q_1, Q_2 are positive integers.

1. Introduction

In this article, unless specifically stated otherwise, we will use the symbol R to represent a commutative ring that possesses an identity element 1, where 1 is distinct from 0. The notation wR signifies the ideal generated by an element w within the ring R, defined as the set $wR = \{wa : a \in R\}$. The collection $Z(R)'$ represents the set of elements in the ring R that are neither zero nor units.

Introducing the terminology used in graph theory, we denote the vertex set of a graph G as V and its edge set as E, which allows us to represent the graph as $G = (V, E)$. The number of vertices within a graph G is referred to as its *order*, while the number of edges in the same graph signifies its *size*. A complete graph with t vertices is symbolized as K_t, and its complement is denoted by $\overline{K}_t$. The *neighbourhood* of a vertex v, denoted by $N_G(v)$, is defined as the set of vertices in graph G that are adjacent to vertex v. The number of edges that are incident to a particular vertex $v \in V$ is represented by $deg(v)$, which signifies the *degree* of that vertex. If, for every vertex v, the degree $deg(v)$ equals s, then we characterize the graph G as *s-regular*.

Now, consider any square matrix denoted by $\mathscr{Q}$ and let the distinct eigenvalues of this matrix be $\zeta_1, \zeta_2, \ldots, \zeta_t$ each with multiplicities $\mu_1, \mu_2, \ldots, \mu_t$ respectively. We

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represent the *spectrum* of matrix $\mathscr{Q}$ as $\sigma(\mathscr{Q})$ and it is defined as the set of these distinct eigenvalues $\sigma(\mathscr{Q}) = \left\{\zeta_1^{\mu_1}, \zeta_2^{\mu_2}, \ldots, \zeta_t^{\mu_t}\right\}$. For a graph G, $L(G)$ denotes the *Laplacian* matrix of G and is defined as

$$L(G) = (l_{st}) = \begin{cases} \deg(w_s), & s = t, \\ -1, & s \neq t \text{ and } w_s \sim w_t, \\ \ \ 0, & \text{otherwise.} \end{cases}$$

A graph G is termed Laplacian *integral* if all of its Laplacian eigenvalues are integers. The spectrum associated with the Laplacian matrix is referred to as the Laplacian spectrum of the graph G and is denoted by $\sigma_L(G)$.

The concept of the cozero-divisor graph was introduced by Afkhami et al. [1], in the context of commutative rings. Denoted by $\Gamma'(R)$, the cozero-divisor graph of the ring R is an undirected graph with a vertex set comprised of elements from the set $Z(R)'$. Two distinct vertices x and y are connected by an edge in this graph if and only if $x \notin yR$ and $y \notin xR$. For more in-depth information about the cozero-divisor graph, reader can refer to the references provided, such as [2–5, 7, 8], where one can also find additional sources for further study.

In their work, Parveen et al. [6] computed the Laplacian eigenvalues of the graph $\Gamma'(\mathbb{Z}_n)$ for a specific case where n is expressed as $n = p^{n_1}q^{n_2}$, with p and q being distinct prime numbers and n_1 and n_2 being natural numbers. This article focuses on extending their findings by exploring the Laplacian eigenvalues of the graphs $\Gamma'(\mathbb{Z}_n)$ for various values of n. Section 2 provides a review of essential concepts that are employed to establish our primary results. In Section 3, we delve into the Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$, specifically considering cases where n can be expressed as $n = q_1^{Q}q_2q_3$ or $n = q_1^{Q_1}q_2^{Q_2}q_3$.

2. Preliminaries

We initiate our discourse with the following definition and a few established results.

Definition 2.1. Consider a graph $G(V, E)$ with vertex set $\{w_1, w_2, \ldots, w_r\}$ of order r. Let $\Psi_k(V_m, E_m)$ represent a collection of disjoint graphs, each with an order of r_m, where m ranges from 1 to r. The generalized join graph $G[\Psi_1, \Psi_2, \ldots, \Psi_m]$ is formed by taking the graphs $\Psi_1, \Psi_2, \ldots, \Psi_m$ and each vertex of Ψ_{w_i} is connected to every vertex of Ψ_{w_j} whenever there exists an adjacency between vertices w_i and w_j within the original graph G.

To indicate that an integer q divides m, we use the notation $q|m$ and (s, m) represents the greatest common divisor of s and m. An integer q is a proper divisor of m if and only if q divides m for $1 < q < m$. The *Euler's phi function*, symbolized as $\phi(m)$, represents the number of positive integers less than or equal to m that are relatively prime to m. We say that m can be expressed in a prime decomposition if it can be represented as $m = {q_1}^{k_1}{q_2}^{k_2}\cdots{q_l}^{k_l}$, where $q_1, q_2, \ldots, q_l$ are distinct prime numbers and $k_1, k_2, \ldots, k_l$ are positive integers.

Let $y_1, y_2, \ldots, y_r$ be the distinct proper divisors of n, consider the sets A_{y_k}, defined as $A_{y_k} = \{s \in \mathbb{Z}_n : (s, n) = y_k\}$. We observe that these sets, A_{y_k} and A_{y_l} have no common elements ($A_{y_k} \cap A_{y_l} = \phi$) when k is not equal to l. This observation leads to an important implication, the sets $A_{y_1}, A_{y_2}, \ldots, A_{y_r}$ are pairwise disjoint, and collectively, they partition the vertex set of the graph $\Gamma'(\mathbb{Z}_n)$, which can be denoted by $V(\Gamma'(\mathbb{Z}_n)) = A_{y_1} \cup A_{y_2} \cup \cdots \cup A_{y_r}$. Parveen et. al. in [6, Lemma 3.3] used the proper divisors set as $\tau(n) - 2$, where $\tau(n)$ is the divisors set of n. Here, we write proper divisors as $y_1, y_2, \ldots, y_r$ and rewrite the [6, Lemma 3.3] as follows.

Lemma 2.2 ([6, Lemma 3.3]). *Let $x \in A_{y_k}, z \in A_{y_l}$, where $k, l \in \{1, 2, \ldots, r\}$. Then $x \sim z$ in $\Gamma'(\mathbb{Z}_n)$ if and only if $y_k \nmid y_l$ and $y_l \nmid y_k$.*

Let $y_1, y_2, \ldots, y_r$ be the distinct proper divisors of m and Ω_m be a simple graph with vertex set $y_1, y_2, \ldots, y_r$. In this graph, two distinct vertices are considered adjacent if and only if $y_k \nmid y_l$ and $y_l \nmid y_k$. Now, if m can be expressed as its prime decomposition, such as $m = {p_1}^{n_1}{p_2}^{n_2}\cdots{p_r}^{n_r}$, then the order of the graph Ω_m is determined as follows: $|V(\Omega_m)| = \prod_{i=1}^{r}(n_i + 1) - 2$.

Theorem 2.3 ([6, Theorem 4.1]). *Let $y_1, y_2, \ldots, y_r$ be the distinct proper divisors of n. Then the Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ can be calculated as*

$$\sigma_L(\Gamma'(\mathbb{Z}_n)) = \left(\bigcup_{k=1}^{r}\left(N_{y_k} + \left(\sigma_L(\Gamma'(A_{y_k})) \setminus \{0\}\right)\right)\right)\bigcup \sigma(\mathscr{L}(\Omega_n)),$$

where $\Gamma'(A_{y_k})$ are r_k regular graphs and $N_{y_k} + \left(\sigma_L(\Gamma'(A_{y_k})) \setminus \{0\}\right)$ represents that N_{y_k} is added to each element of the multiset $\left(\sigma_L(\Gamma'(A_{y_k})) \setminus \{0\}\right)$ and the matrix $\mathscr{L}(\Omega_n)$ is given in (1).

3. Main results

In the results section, we will present the main findings of this paper. We consider the proper divisors of the positive integer n, denoted by $y_1, y_2, \ldots, y_r$. For each $1 \leq t \leq r$, we assign a weight of $\phi\left(\frac{n}{y_t}\right) = |A_{y_t}|$ to the vertex y_t in the graph Ω_n. Define the integer $N_{y_s} = \sum\limits_{y_t \in \mathcal{N}_{\Omega_n}(y_s)} \phi\left(\frac{n}{y_t}\right)$. Then the vertex weighted Laplacian matrix $\mathscr{L}(\Omega_n)$ of Ω_n is given by

$$\mathscr{L}(\Omega_n) = z_{s,t} = \begin{cases} \sum\limits_{y_t \in \mathcal{N}_{\Omega_n}(y_s)} \phi\left(\frac{n}{y_t}\right), & s = t, \\ -\phi\left(\frac{n}{y_t}\right), & s \neq t \text{ and } y_s \sim y_t \text{ in } \Omega_n, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

By [6, Corollary 3.4], $\Gamma'(A_{y_k})$ is isomorphic to $\overline{K}_{\phi(\frac{n}{y_k})}$ for $k \in \{1, 2, \ldots, r\}$. Hence, based on Theorem 2.3, we can assert the existence of $n - \phi(n) - 1$ Laplacian eigenvalues for the graph $\Gamma'(\mathbb{Z}_n)$. Among these, the r Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ are determined by the roots of the characteristic polynomial of the matrix $\mathscr{X}(\Omega_n)$, as defined in (1) and $n - \phi(n) - 1 - r$ are already known.

We present and prove the primary result of this paper, which provides the Laplacian spectrum of the cozero-divisor graph associated with $\Gamma'(\mathbb{Z}_{q_1^{Q} q_2^{Q_2} q_3})$.

Theorem 3.1. *Let Q_1, Q_2 are positive integers and q_1, q_2, q_3 are distinct primes. Then the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{q_1^{Q_1} q_2^{Q_2} q_3})$ consists of the eigenvalues*

$$[N_{q_1}]^{\phi(q_1^{Q_1-1} q_2^{Q_2} q_3)-1}, [N_{q_1^2}]^{\phi(q_1^{Q_1-2} q_2^{Q_2} q_3)-1}, \ldots, [N_{q_1^{Q_1}}]^{\phi(q_2^{Q_2} q_3)-1}, [N_{q_2}]^{\phi(q_1^{Q_1} q_2^{Q_2-1} q_3)-1}, \ldots,$$
$$[N_{q_2^{Q_2}}]^{\phi(q_1^{Q_1} q_3)-1}, [N_{q_1 q_2}]^{\phi(q_1^{Q_1-1} q_2^{Q_2-1} q_3)-1}, \ldots, [N_{q_1^{Q_1} q_2}]^{\phi(q_2^{Q_2-1} q_3)-1}, [N_{q_1 q_2^2}]^{\phi(q_1^{Q_1-1} q_2^{Q_2-2} q_3)-1},$$
$$\ldots, [N_{q_1 q_2^{Q_2}}]^{\phi(q_1^{Q_1-1} q_3)-1}, \ldots, [N_{q_1^{Q_1} q_2^{Q_2}}]^{\phi(q_3)-1}, [N_{q_1 q_2 q_3}]^{\phi(q_1^{Q_1-1} q_2^{Q_2-1})-1}, \ldots, [N_{q_1^{Q_1} q_2 q_3}]^{\phi(q_2^{Q_2-1})-1},$$
$$[N_{q_1 q_2^2 q_3}]^{\phi(q_1^{Q_1-1} q_2^{Q_2-2})-1}, \ldots, [N_{q_1 q_2^{Q_2} q_3}]^{\phi(q_1^{Q_1-1})-1}, \ldots, [N_{q_1^{Q_1-1} q_2^{Q_2} q_3}]^{\phi(q_1)-1}, [N_{q_1 q_3}]^{\phi(q_1^{Q_1-1} q_2^{Q_2})-1},$$
$$\ldots, [N_{q_1^{Q_1-1} q_3}]^{\phi(q_2^{Q_2})-1}, [N_{q_2 q_3}]^{\phi(q_1^{Q_1} q_2^{Q_2-1})-1}, \ldots, [N_{q_2^{Q_2} q_3}]^{\phi(q_1^{Q_1})-1}, [N_{q_3}]^{\phi(q_1^{Q_1} q_2^{Q_2})-1}.$$

The $2(Q_1+1)(Q_2+1) - 2$ Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{q_1^{Q_1} q_2^{Q_2} q_3})$ are precisely the roots of the characteristic polynomial associated with the matrix presented in equation (1).

Proof. Let $n = q_1^{Q_1} q_2^{Q_2} q_3$, where Q_1, Q_2 are positive integers and q_1, q_2, q_3 are distinct primes. The proper divisors of n are

$$q_1, q_1^2, \ldots, q_1^{Q_1}, q_2, q_2^2, \ldots, q_2^{Q_2}, q_3, q_1 q_3, q_1^2 q_3, \ldots, q_1^{Q_1} q_3, q_2 q_3, q_2^2 q_3, \ldots, q_2^{Q_2} q_3, q_1 q_2, q_1^2 q_2, \ldots,$$
$$q_1^{Q_1} q_2, q_1 q_2^2, q_1^2 q_2^2, \ldots, q_1^{Q_1} q_2^2, \ldots, q_1 q_2^{Q_2}, q_1^2 q_2^{Q_2}, \ldots, q_1^{Q_1} q_2^{Q_2}, q_1 q_2 q_3, q_1^2 q_2 q_3, \ldots, q_1^{Q_1} q_2 q_3,$$
$$q_1 q_2^2 q_3, q_1^2 q_2^2 q_3, \ldots, q_1^{Q_1} q_2^2 q_3, \ldots, q_1 q_2^{Q_2} q_3, q_1^2 q_2^{Q_2} q_3, \ldots, q_1^{Q_1-1} q_2^{Q_2} q_3.$$

We have the following adjacency relations by the definition of Ω_n:

$$q_1^i \sim q_2^j,\ q_3,\ q_2^j q_3 \text{ for all } i, j, \qquad q_1^{i_1} \sim q_1^{i_2} q_2^j \text{ for all } i_1 > i_2, j > 0,$$
$$q_1^{i_1} \sim q_1^{i_2} q_3,\ q_1^{i_2} q_2^j q_3, \text{ for all } i_1 > i_2, \qquad q_2^j \sim q_1^i q_3,\ q_3 \text{ for all } i, j,$$
$$q_2^{j_1} \sim q_1^i q_2^{j_2} \text{ for all } j_1 > j_2, i > 0, \qquad q_2^{j_1} \sim q_2^{j_2} q_3,\ q_1^i q_2^{j_2} q_3, \text{ for all } j_1 > j_2,$$
$$q_1^{i_1} q_2^{j_1} \sim q_1^{i_2} q_2^{j_2} \text{ if either } i_1 < i_2, j_1 > j_2 \text{ or } j_1 < j_2, i_1 > i_2,$$
$$q_1^i q_2^j \sim q_1^i q_3,\ q_2^j q_3,\ q_3 \text{ for all } i, j, \qquad q_1^{i_1} q_2^{j_1} \sim q_1^{i_2} q_2^{j_2} q_3 \text{ either } i_2 < i_1 \text{ or } j_2 < j_1,$$
$$q_1^i q_3 \sim q_2^j q_3 \text{ for all } i, j, \qquad q_1^{i_1} q_3 \sim q_1^{i_2} q_2^j q_3 \text{ for all } i_1 > i_2,$$
$$q_2^{j_1} q_3 \sim q_1^i q_2^{j_2} q_3 \text{ for all } j_1 > j_2,$$
$$q_1^{i_1} q_2^{j_1} q_3 \sim q_1^{i_2} q_2^{j_2} q_3 \text{ if either } i_1 < i_2, j_1 > j_2 \text{ or } j_1 < j_2, i_1 > i_2.$$

By result of [6, Lemma 3.6], we have

$$\Gamma'(\mathbb{Z}_{q_1^{Q_1} q_2^{Q_2} q_3}) = \Omega_{q_1^{Q_1} q_2^{Q_2} q_3}[\Gamma'(A_{q_1}), \Gamma'(A_{q_1^2}), \ldots, \Gamma'(A_{q_1^{Q_1}}), \Gamma'(A_{q_2}), \Gamma'(A_{q_2^2}), \ldots,$$

$\Gamma'(A_{q_2^{Q_2}}), \Gamma'(A_{q_3}), \Gamma'(A_{q_1 q_2}), \Gamma'(A_{q_1^2 q_2}), \ldots, \Gamma'(A_{q_1^{Q_1} q_2}), \ldots, \Gamma'(A_{q_1 q_2^{Q_2}}),$
$\Gamma'(A_{q_1^2 q_2^{Q_2}}), \ldots, \Gamma'(A_{q_1^{Q_1} q_2^{Q_2}}), \Gamma'(A_{q_1 q_2 q_3}), \Gamma'(A_{q_1^2 q_2 q_3}), \ldots,$
$\Gamma'(A_{q_1^{Q_1} q_2 q_3}), \ldots, \Gamma'(A_{q_1 q_2^{Q_2} q_3}), \Gamma'(A_{q_1^2 q_2^{Q_2} q_3}), \ldots, \Gamma'(A_{q_1^{Q_1-1} q_2^{Q_2} q_3}),$
$\Gamma'(A_{q_1 q_3}), \Gamma'(A_{q_1^2 q_3}), \ldots, \Gamma'(A_{q_1^{Q_1} q_3}), \Gamma'(A_{q_2 q_3}), \Gamma'(A_{q_2^2 q_3}), \ldots, \Gamma'(A_{q_2^{Q_2} q_3})].$

In view of [9, Proposition 2.1] and [6, Corollary 3.4] we obtain

$$\begin{aligned}
&\Gamma'(A_{q_1^y}) = \overline{K}_{\phi(q_1^{Q_1-y} q_2^{Q_2} q_3)}, \quad \Gamma'(A_{q_1^y q_3}) = \overline{K}_{\phi(q_1^{Q_1-y} q_2^{Q_2})}, \quad \text{for } 1 \le y \le Q_1,\\
&\Gamma'(A_{q_2^y}) = \overline{K}_{\phi(q_1^{Q_1} q_2^{Q_2-y} q_3)}, \quad \Gamma'(A_{q_2^y q_3}) = \overline{K}_{\phi(q_1^{Q_1} q_2^{Q_2-y})}, \quad \text{for } 1 \le y \le Q_2,\\
&\Gamma'(A_{q_1^y q_2^z}) = \overline{K}_{\phi(q_1^{Q_1-y} q_2^{Q_2-z} q_3)}, \quad \text{for } 1 \le y \le Q_1,\ 1 \le z \le Q_2,\\
&\Gamma'(A_{q_1^y q_2^z q_3}) = \overline{K}_{\phi(q_1^{Q_1-y} q_2^{Q_2-z})}, \quad \text{for } 1 \le y \le Q_1-1,\ 1 \le z \le Q_2,\\
&\Gamma'(A_{q_3}) = \overline{K}_{\phi(q_1^{Q_1} q_2^{Q_2})}.
\end{aligned}$$

Also, the values of N_{y_s} are as follows

$$\begin{aligned}
N_{q_1} &= \sum_{p=1}^{Q_2} \phi(q_1^{Q_1} q_2^{Q_2-p} q_3) + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1} q_2^{Q_2-p}) + \phi(q_1^{Q_1} q_2^{Q_2}),\\
N_{q_1^2} &= \sum_{p=1}^{Q_2} \phi(q_1^{Q_1} q_2^{Q_2-p} q_3) + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1} q_2^{Q_2-p}) + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1-1} q_2^{Q_2-p} q_3)\\
&\quad + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1-1} q_2^{Q_2-p}) + \phi(q_1^{Q_1-1} q_2^{Q_2}) + \phi(q_1^{Q_1} q_2^{Q_2}),\\
&\vdots\\
N_{q_1^{Q_1}} &= \sum_{p=1}^{Q_2} \phi(q_1^{Q_1} q_2^{Q_2-p} q_3) + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1} q_2^{Q_2-p}) + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1-1} q_2^{Q_2-p} q_3)\\
&\quad + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1-2} q_2^{Q_2-p} q_3) + \cdots + \sum_{p=1}^{Q_2} \phi(q_1 q_2^{Q_2-p} q_3) + \sum_{p=1}^{Q_2} \phi(q_1^{Q_1-1} q_2^{Q_2-p})\\
&\quad + \cdots + \sum_{p=1}^{Q_2} \phi(q_1 q_2^{Q_2-p}) + \sum_{p=1}^{Q_1-1} \phi(q_1^{Q_1-p} q_2^{Q_2}) + \phi(q_1^{Q_1} q_2^{Q_2}),\\
N_{q_2} &= \sum_{p=1}^{Q_1} \phi(q_1^{Q_1-p} q_2^{Q_2} q_3) + \sum_{p=1}^{Q_1} \phi(q_1^{Q_1-p} q_2^{Q_2}) + \phi(q_1^{Q_1} q_2^{Q_2}),\\
&\vdots\\
N_{q_2^{Q_2}} &= \sum_{p=1}^{Q_1} \phi(q_1^{Q_1-p} q_2^{Q_2} q_3) + \sum_{p=1}^{Q_1} \phi(q_1^{Q_1-p} q_2^{Q_2}) + \sum_{p=1}^{Q_1} \phi(q_1^{Q_1-p} q_2^{Q_2-1} q_3)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-2}q_3)+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2q_3)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1})\\
&+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2)+\sum_{p=1}^{Q_2-1}\phi(q_1^{Q_1}q_2^{Q_2-p})+\phi(q_1^{Q_1}q_2^{Q_2}),\\
N_{q_1q_2}=&\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})\\
&+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p})+\phi(q_1^{Q_1}q_2^{Q_2}),\\
&\vdots\\
N_{q_1^{Q_1}q_2}=&\sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p})\\
&+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p}q_3)+\cdots+\sum_{p=2}^{Q_2}\phi(q_1q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p})\\
&+\cdots+\sum_{p=1}^{Q_2}\phi(q_1q_2^{Q_2-p})+\phi(q_1^{Q_1}q_2^{Q_2}),\\
N_{q_1q_2^2}=&\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=3}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1}q_3)\\
&+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1})+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p})+\phi(q_1^{Q_1}q_2^{Q_2}),\\
&\vdots\\
N_{q_1q_2^{Q_2}}=&\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1}q_3)+\cdots+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2q_3)\\
&+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1})+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})\\
&+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p})+\phi(q_1^{Q_1}q_2^{Q_2}),\\
&\vdots\\
N_{q_1^{Q_1}q_2^{Q_2}}=&\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p})+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-2}q_2^{Q_2-p})+\cdots+\sum_{p=1}^{Q_2-1}\phi(q_2^{Q_2-p})\\
&+\phi(q_1)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p})+\phi(q_1^{Q_1}q_2^{Q_2}),
\end{aligned}
$$

$$\begin{aligned}
N_{q_1q_2q_3} =& \sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)\\
&+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1}q_3)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-2}q_3)\\
&+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_3)+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}),\\
&\vdots\\
N_{q_1^{Q_1}q_2q_3} =& \sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p}q_3)+\cdots+\sum_{p=2}^{Q_2}\phi(q_2^{Q_2-p}q_3)\\
&+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p})+\cdots+\sum_{p=2}^{Q_2}\phi(q_1q_2^{Q_2-p})+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}),\\
N_{q_1q_2^2q_3} =& \sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=3}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1}q_3)\\
&+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-2}q_3)+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_3)+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1})\\
&+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2})+\sum_{p=3}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}),\\
&\vdots\\
N_{q_1q_2^{Q_2}q_3} =& \sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-2}q_2^{Q_2-p}q_3)+\cdots+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p}q_3)\\
&+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1})+\cdots+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2)+\sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}),\\
&\vdots\\
N_{q_1^{Q_1-1}q_2^{Q_2}q_3} =& \phi(q_2^{Q_2}q_3)+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p})+\phi(q_2^{Q_2}),\\
N_{q_1q_3} =& \sum_{p=2}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p}q_3)\\
&+\cdots+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}),\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
N_{q_1^{Q_1-1}q_3} =& \sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p}q_3)+\cdots+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p}q_3)\\
&+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p})+\cdots+\sum_{p=1}^{Q_2}\phi(q_1q_2^{Q_2-p})+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}),\\
N_{q_2q_3} =& \sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=2}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)\\
&+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p}q_3)+\cdots+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}),\\
&\vdots\\
N_{q_2^{Q_2}q_3} =& \sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1-1}q_2^{Q_2-p}q_3)+\cdots+\sum_{p=1}^{Q_2}\phi(q_2^{Q_2-p}q_3)\\
&+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1})+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}),\\
N_{q_3} =& \sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2}q_3)+\sum_{p=1}^{Q_2}\phi(q_1^{Q_1}q_2^{Q_2-p}q_3)+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-1}q_3)\\
&+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_2^{Q_2-2}q_3)+\cdots+\sum_{p=1}^{Q_1}\phi(q_1^{Q_1-p}q_3).
\end{aligned}$$

By Theorem 2.3, the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{q_1^{Q_1}q_2^{Q_2}q_3})$ is given by

$$\begin{aligned}
\sigma_L(\Gamma'(\mathbb{Z}_{q_1^{Q_1}q_2^{Q_2}q_3})) =& \Big(N_{q_1}+\Big(\sigma_L(\Gamma'(A_{q_1}))\backslash\{0\}\Big)\Big)\bigcup\Big(N_{q_1^2}+\Big(\sigma_L(\Gamma'(A_{q_1^2}))\backslash\{0\}\Big)\Big)\\
&\bigcup\cdots\bigcup\Big(N_{q_1^{Q_1}}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q_1}}))\backslash\{0\}\Big)\Big)\bigcup\Big(N_{q_2}+\Big(\sigma_L(\Gamma'(A_{q_2}))\backslash\{0\}\Big)\Big)\bigcup\cdots\\
&\bigcup\Big(N_{q_2^{Q_2}}+\Big(\sigma_L(\Gamma'(A_{q_2^{Q_2}}))\backslash\{0\}\Big)\Big)\bigcup\Big(N_{q_1q_2}+\Big(\sigma_L(\Gamma'(A_{q_1q_2}))\backslash\{0\}\Big)\Big)\\
&\bigcup\cdots\bigcup\Big(N_{q_1^{Q_1}q_2}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q_1}q_2}))\backslash\{0\}\Big)\Big)\bigcup\Big(N_{q_1q_2^2}+\Big(\sigma_L(\Gamma'(A_{q_1q_2^2}))\backslash\{0\}\Big)\Big)\\
&\bigcup\cdots\bigcup\Big(N_{q_1q_2^{Q_2}}+\Big(\sigma_L(\Gamma'(A_{q_1q_2^{Q_2}}))\backslash\{0\}\Big)\Big)\bigcup\Big(N_{q_1q_2q_3}+\Big(\sigma_L(\Gamma'(A_{q_1q_2q_3}))\backslash\{0\}\Big)\Big)\\
&\bigcup\cdots\bigcup\Big(N_{q_1^{Q_1}q_2q_3}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q_1}q_2q_3}))\backslash\{0\}\Big)\Big)\bigcup\Big(N_{q_1q_2^2q_3}+\Big(\sigma_L(\Gamma'(A_{q_1q_2^2q_3}))\backslash\{0\}\Big)\Big)\\
&\bigcup\cdots\bigcup\Big(N_{q_1q_2^{Q_2}q_3}+\Big(\sigma_L(\Gamma'(A_{q_1q_2^{Q_2}q_3}))\backslash\{0\}\Big)\Big)\bigcup\cdots\\
&\bigcup\Big(N_{q_1^{Q_1-1}q_2^{Q_2}q_3}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q_1-1}q_2^{Q_2}q_3}))\backslash\{0\}\Big)\Big)\\
&\bigcup\Big(N_{q_1^{Q_1}q_2^{Q_2}}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q_1}q_2^{Q_2}}))\backslash\{0\}\Big)\Big)\\
&\bigcup\Big(N_{q_1q_3}+\Big(\sigma_L(\Gamma'(A_{q_1q_3}))\backslash\{0\}\Big)\Big)\bigcup\cdots\Big(N_{q_1^{Q_1}q_3}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q_1}q_3}))\backslash\{0\}\Big)\Big)
\end{aligned}$$

$$\bigcup\Big(N_{q_2q_3}+\Big(\sigma_L(\Gamma'(A_{q_2q_3}))\backslash\{0\}\Big)\Big)\bigcup\cdots\bigcup\Big(N_{q_2^{Q_2}q_3}+\Big(\sigma_L(\Gamma'(A_{q_2^{Q_2}q_3}))\backslash\{0\}\Big)\Big)$$
$$\bigcup\Big(N_{q_3}+\Big(\sigma_L(\Gamma'(A_{q_3}))\backslash\{0\}\Big)\Big)\bigcup\sigma(\mathscr{Z}(\Omega_n)).$$

The $2(Q_1+1)(Q_2+1)-2$ Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{q_1^{Q_1}q_2^{Q_2}q_3})$ are precisely the roots of the characteristic polynomial associated with the matrix $\mathscr{Z}(\Omega_n)$ presented in equation (1). □

As a consequence of Theorem 3.1, the following result determines the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{q_1^{Q}q_2q_3})$.

Corollary 3.2. *Let Q be a positive integer and q_1, q_2, q_3 are distinct primes. Then the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{q_1^{Q}q_2q_3})$ is given by*

$$\sum_{w=0}^{y\text{-}1}\Big[\phi(q_1^{Q\text{-}w})+\phi(q_1^{Q\text{-}w}q_2)+\phi(q_1^{Q\text{-}w}q_3)\Big]^{\phi(q_1^{Q\text{-}y}q_2q_3)\text{-}1},$$

$$\Big[\sum_{w=0}^{Q\text{-}(y+1)}\phi(q_1^{w}q_2q_3)+\sum_{x=0}^{y\text{-}1}\phi(q_1^{Q\text{-}x})+\sum_{z=0}^{Q}\phi(q_1^{z}q_2)\Big]^{\phi(q_1^{Q\text{-}y}q_3)\text{-}1},$$

$$\Big[\sum_{w=0}^{Q\text{-}(y+1)}\phi(q_1^{w}q_2q_3)+\sum_{x=0}^{y\text{-}1}\phi(q_1^{Q\text{-}x})+\sum_{z=0}^{Q}\phi(q_1^{z}q_3)\Big]^{\phi(q_1^{Q\text{-}y}q_2)\text{-}1},$$

$$\sum_{w=0}^{Q\text{-}(t+1)}\Big[\phi(q_1^{w}q_2q_3)+\phi(q_1^{w}q_2)+\phi(q_1^{w}q_3)\Big]^{\phi(q_1^{Q\text{-}t})\text{-}1},\quad \sum_{w=0}^{Q\text{-}1}\Big[\phi(q_1^{w}q_2q_3)+\phi(q_1^{w}q_2)+\phi(q_1^{w}q_3)\Big]^{\phi(q_1^{Q})\text{-}1},$$

$$\Big[\sum_{w=0}^{Q\text{-}1}\phi(q_1^{w}q_2q_3)+\sum_{x=0}^{Q}\phi(q_1^{x}q_2)\Big]^{\phi(q_1^{Q}q_3)\text{-}1},\quad \Big[\sum_{w=0}^{Q\text{-}1}\phi(q_1^{w}q_2q_3)+\sum_{x=0}^{Q}\phi(q_1^{x}q_3)\Big]^{\phi(q_1^{Q}q_2)\text{-}1},$$

where $1\le t\le Q-1$ and $1\le y\le Q$. The remaining $4Q+2$ Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{q_1^{Q}q_2q_3})$ are the roots of the characteristic polynomial of the matrix (2).

Proof. Let $n{=}q_1^{Q}q_2q_3$, where Q is a positive integer and q_1, q_2, q_3 are distinct primes. The proper divisors of n are $q_1, q_1^2, q_1^3, \dots, q_1^{Q\text{-}1}, q_1^{Q}, q_2, q_1q_2, q_1^2q_2, \dots, q_1^{Q}q_2, q_3, q_1q_3, q_1^2q_3, \dots, q_1^{Q}q_3, q_2q_3, q_1q_2q_3, q_1^2q_2q_3, \dots, q_1^{Q\text{-}1}q_2q_3$. The adjacency relations for $i{=}1,2,...,Q$ and $j{=}0,1,2,...,Q$ - 1 can be deduced from the definition of Ω_n.

$$q_1^i\sim q_1^jq_2,\ \ q_1^jq_3,\ \ q_1^jq_2q_3\quad\text{for all } i>j.\qquad q_1^iq_2\sim q_1^jq_2q_3\quad\text{for all } i>j.$$
$$q_1^iq_2\sim q_1^jq_3,\ \ q_3,\ \ q_2q_3\quad\text{for all } i,j.\qquad q_1^iq_3\sim q_2,\ \ q_2q_3\quad\text{for all } i.$$
$$q_1^iq_3\sim q_1^jq_2q_3\quad\text{for all } i>j.$$

By the [6, Lemma 3.6], we have

$$\Gamma'(\mathbb{Z}_{q_1^Qq_2q_3})=\Omega_{q_1^Qq_2q_3}[\Gamma'(A_{q_1}),\Gamma'(A_{q_1^2}),...,\Gamma'(A_{q_1^Q}),\Gamma'(A_{q_2}),\Gamma'(A_{q_1q_2}),\Gamma'(A_{q_1^2q_2}),...,\Gamma'(A_{q_1^Qq_2}),$$
$$\Gamma'(A_{q_3}),\Gamma'(A_{q_1q_3}),\Gamma'(A_{q_1^2q_3}),...,\Gamma'(A_{q_1^Qq_3}),\Gamma'(A_{q_2q_3}),\Gamma'(A_{q_1q_2q_3}),\Gamma'(A_{q_1^2q_2q_3}),...,\Gamma'(A_{q_1^{N-1}q_2q_3})].$$

In view of [6, Corollary 3.4] and [9, Proposition 2.1], we obtain

$$\Gamma'(A_{q_1^y})=\overline{K}_{\phi(q_1^{Q-y}q_2q_3)},\quad \Gamma'(A_{q_1^yq_2})=\overline{K}_{\phi(q_1^{Q-y}q_3)},\quad \Gamma'(A_{q_1^yq_3})=\overline{K}_{\phi(q_1^{Q-y}q_2)}\quad\text{for } 1\le y\le Q,$$

$\Gamma'(A_{q_1^y q_2 q_3}) = \overline{K}_{\phi(q_1^{Q-y})}$ for $1 \leq y \leq Q-1$,

$\Gamma'(A_{q_2}) = \overline{K}_{\phi(q_1^Q q_3)}$, $\Gamma'(A_{q_3}) = \overline{K}_{\phi(q_1^Q q_2)}$, $\Gamma'(A_{q_2 q_3}) = \overline{K}_{\phi(q_1^Q)}$.

Also, the values of N_{y_s} are as follows

$$N_{q_1^y} = \sum_{w=0}^{y\text{-}1} \left[\phi(q_1^{Q\text{-}w}) + \phi(q_1^{Q\text{-}w} q_2) + \phi(q_1^{Q\text{-}w} q_3) \right],$$

$$N_{q_1^y q_2} = \sum_{w=0}^{Q\text{-}(y+1)} \phi(q_1^w q_2 q_3) + \sum_{x=0}^{y\text{-}1} \phi(q_1^{Q\text{-}x}) + \sum_{z=0}^{Q} \phi(q_1^z q_2),$$

$$N_{q_1^y q_3} = \sum_{w=0}^{Q\text{-}(y+1)} \phi(q_1^w q_2 q_3) + \sum_{x=0}^{y\text{-}1} \phi(q_1^{Q\text{-}x}) + \sum_{z=0}^{Q} \phi(q_1^z q_3),$$

$$N_{q_1^y q_2 q_3} = \sum_{w=0}^{Q\text{-}(y+1)} \left[\phi(q_1^w q_2 q_3) + \phi(q_1^w q_2) + \phi(q_1^w q_3) \right],$$

$$N_{q_2 q_3} = \sum_{w=0}^{Q\text{-}1} \left[\phi(q_1^w q_2 q_3) + \phi(q_1^w q_2) + \phi(q_1^w q_3) \right],$$

$$N_{q_2} = \sum_{w=0}^{Q\text{-}1} \phi(q_1^w q_2 q_3) + \sum_{x=0}^{Q} \phi(q_1^x q_2), \quad N_{q_3} = \sum_{w=0}^{Q\text{-}1} \phi(q_1^w q_2 q_3) + \sum_{x=0}^{Q} \phi(q_1^x q_3).$$

With the proper divisors labeled as mentioned, the matrix $\mathscr{L}(\Omega_n)$ takes on the following form:

$$\mathscr{L}(\Omega_n) = \begin{bmatrix} N_{q_1} & \cdots & 0 & \text{-}\phi(q_1^Q q_3) & \cdots & \text{-}\phi(q_1^Q q_2) & \cdots & \text{-}\phi(q_1^Q) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & N_{q_1^Q} & \text{-}\phi(q_1^Q q_3) & \cdots & \text{-}\phi(q_1^Q q_2) & \cdots & \text{-}\phi(q_1^Q) & \cdots & \text{-}\phi(q_1^{Q\text{-}1}) \\ \text{-}\phi(q_1^{Q\text{-}1} q_2 q_3) & \cdots & \text{-}\phi(q_2 q_3) & N_{q_2} & \cdots & \text{-}\phi(q_1^Q q_2) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \text{-}\phi(q_1^{Q\text{-}1} q_2 q_3) & \cdots & \text{-}\phi(q_2 q_3) & \text{-}\phi(q_1^Q q_3) & \cdots & N_{q_3} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \text{-}\phi(q_1^{Q\text{-}1} q_2 q_3) & \cdots & \text{-}\phi(q_2 q_3) & 0 & \cdots & 0 & \cdots & N_{q_2 q_3} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \text{-}\phi(q_2 q_3) & 0 & \cdots & 0 & \cdots & 0 & \cdots & N_{q_1^{(Q\text{-}1)} q_2 q_3} \end{bmatrix}. \quad (2)$$

The Laplacian spectrum of $\Gamma'(\mathbb{Z}_{q_1^Q q_2 q_3})$, as a consequence of Theorem 2.3, is given by

$$\sigma_L(\Gamma'(\mathbb{Z}_{q_1^Q q_2 q_3})) = \Big(N_{q_1} + \Big(\sigma_L(\Gamma'(A_{q_1})) \setminus \{0\}\Big)\Big) \bigcup \Big(N_{q_1^2} + \Big(\sigma_L(\Gamma'(A_{q_1^2})) \setminus \{0\}\Big)\Big) \bigcup \cdots$$
$$\bigcup \Big(N_{q_1^Q} + \Big(\sigma_L(\Gamma'(A_{q_1^Q})) \setminus \{0\}\Big)\Big) \bigcup \Big(N_{q_1 q_2} + \Big(\sigma_L(\Gamma'(A_{q_1 q_2})) \setminus \{0\}\Big)\Big)$$
$$\bigcup \Big(N_{q_1^2 q_2} + \Big(\sigma_L(\Gamma'(A_{q_1^2 q_2})) \setminus \{0\}\Big)\Big) \bigcup \cdots \bigcup \Big(N_{q_1^Q q_2} + \Big(\sigma_L(\Gamma'(A_{q_1^Q q_2})) \setminus \{0\}\Big)\Big)$$

$$\bigcup\Big(N_{q_1q_3}+\Big(\sigma_L(\Gamma'(A_{q_1q_3}))\setminus\{0\}\Big)\Big)\bigcup\cdots\bigcup\Big(N_{q_1^Qq_3}+\Big(\sigma_L(\Gamma'(A_{q_1^Qq_3}))\setminus\{0\}\Big)\Big)$$
$$\bigcup\Big(N_{q_1q_2q_3}+\Big(\sigma_L(\Gamma'(A_{q_1q_2q_3}))\setminus\{0\}\Big)\Big)\bigcup\cdots\bigcup\Big(N_{q_1^{Q-1}q_2q_3}+\Big(\sigma_L(\Gamma'(A_{q_1^{Q-1}q_2q_3}))\setminus\{0\}\Big)\Big)$$
$$\bigcup\Big(N_{q_2}+\Big(\sigma_L(\Gamma'(A_{q_2}))\setminus\{0\}\Big)\Big)\bigcup\Big(N_{q_3}+\Big(\sigma_L(\Gamma'(A_{q_3}))\setminus\{0\}\Big)\Big)$$
$$\bigcup\Big(N_{q_2q_3}+\Big(\sigma_L(\Gamma'(A_{q_2q_3}))\setminus\{0\}\Big)\Big)\bigcup\sigma(\mathscr{L}(\Omega_n)).$$

The remaining $4(Q+1)-2$ Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{q_1^Qq_2q_3})$ are obtained from the matrix $\mathscr{L}(\Omega_n)$ given in (2). $\square$

EXAMPLE 3.3. The Laplacian spectrum of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{60})$ consists of eigenvalues

$$\sigma(\Gamma'(\mathbb{Z}_{60}))=\Big\{0^1,11^3,14^7,18^3,19^1,21^7,24^7,26^1,28^1,32^3,9.360^1,12.775^1,$$
$$14.656^1,19.120^1,21.706^1,27.005^1,28.437^1,36.230^1,37.711^1\Big\}.$$

Let $n=2^2.3.5$. The proper divisors of 30 are $2,3,4,5,6,10,12,15,20,30$ and Ω_{60}: $6\sim4\sim10\sim3\sim5\sim2\sim3\sim4\sim5\sim6\sim10\sim12\sim15\sim20\sim30\sim4\sim15\sim2$, $3\sim20\sim6\sim15\sim10$, $5\sim12\sim20$, $30\sim12$. By [6, Lemma 3.6], we have

$$\Gamma'(\mathbb{Z}_{60})=\Omega_{60}[\Gamma'(A_2),\Gamma'(A_3),\Gamma'(A_4),\Gamma'(A_5),\Gamma'(A_6),\Gamma'(A_{10}),$$
$$\Gamma'(A_{12}),\Gamma'(A_{15}),\Gamma'(A_{20}),\Gamma'(A_{30})].$$

In view of [6, Corollary 3.4] and [9, Proposition 2.1], we get

$$\Gamma'(\mathbb{Z}_{60})=G_{10}[\overline{K}_8,\overline{K}_8,\overline{K}_8,\overline{K}_4,\overline{K}_4,\overline{K}_2,\overline{K}_4,\overline{K}_2,\overline{K}_2,\overline{K}_1].$$

And, the values of N_{y_s} are $N_2=14,N_3=24,N_4=21,N_5=32,N_6=18,N_{10}=26,N_{12}=11,N_{15}=28,N_{20}=19,N_{30}=14$. As a consequence of Theorem 2.3, The Laplacian spectrum of $\Gamma'(\mathbb{Z}_{60})$ consists of eigenvalues with their multiplicities

$$\begin{Bmatrix}14&24&21&32&18&26&11&28&19\\7&7&7&3&3&1&3&1&1\end{Bmatrix}.$$

Now the matrix $\mathscr{L}(\Omega_n)$ takes the following form:

$$\mathscr{L}(\Omega_{60})=\begin{bmatrix}14&-8&0&-4&0&0&0&-2&0&0\\-8&24&-8&-4&0&-2&0&0&-2&0\\0&-8&21&-4&-4&-2&0&-2&0&-1\\-8&-8&-8&32&-4&0&-4&0&0&0\\0&0&-8&-4&18&-2&0&-2&-2&0\\0&-8&-8&0&-4&26&-4&-2&0&0\\0&0&0&-4&0&-2&11&-2&-2&-1\\-8&0&-8&0&-4&-2&-4&28&-2&0\\0&-8&0&0&-4&0&-4&-2&19&-1\\0&0&-8&0&0&0&-4&0&-2&14\end{bmatrix}. \tag{3}$$

The eigenvalues of the matrix $\mathscr{L}(\Omega_{60})$ given by (3) are 0, 9.360, 12.775, 14.656, 19.120, 21.706, 27.005, 28.437, 36.230, 37.711.

4. Conclusion

Our findings provide the Laplacian spectrum of the cozero-divisor graph for integers modulo n across various values of n, accomplished through the utilization of the generalized join graph of induced subgraphs. These results can be extended to determine the Laplacian spectrum when n is expressed as $n = p^M q^N r^P$, where M, N, and P are positive integers, and p, q, and r are distinct prime numbers.

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