МАТЕМАТИЧКИ ВЕСНИК
54 (2002), 9–12

UDK 517.546
оригинални научни рад
research paper

THE RADIUS OF CONVEXITY FOR THE CLASS OF JANOWSKI CONVEX FUNCTIONS OF COMPLEX ORDER

Yaşar Polatoğlu and Metin Bolcal

Abstract. In the present paper we shall give the sharp bound of radius of convexity for the class of Janowski convex functions of complex order.

1. Introduction

Let $\mathcal{A}$ be the family of functions $\omega(z)$ regular in the unit disc $D = \{z : |z| < 1\}$ and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

Next, for arbitrary fixed numbers A, B, $-1 < B \leqslant A \leqslant 1$ denote by $p(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.1}$$

regular in D and such that $p(z) = \dfrac{1 + A\omega(z)}{1 + B\omega(z)}$ for some function $\omega(z) \in \mathcal{A}$ and every $z \in D$. The class $p(A, B)$ was introduced by W. Janowski [3].

Moreover let $\mathcal{C}(A, B, b)$ ($b \neq 0$ complex) denote the family of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{1.2}$$

regular in D such that $1 + \dfrac{1}{b} z \dfrac{f''(z)}{f'(z)} = p(z)$ for some $p(z) \in p(A, B)$ and all $z \in D$. The class $\mathcal{C}(A, B, b)$ is the class of Janowski convex functions of complex order b.

It can be noticed that by giving specific values to A, B and b, we obtain the following important subclasses studied by various authors in earlier works [2]:

(i) $\mathcal{C}(1, -1, 1)$ is the well known class of convex functions [2].

(ii) $\mathcal{C}(1, -1, b)$ is the class of convex functions of complex order b. This class was introduced by P. Wiatrowski [11].

AMS Subject Classification: 30 C 45

Keywords and phrases: Janowski convex functions of complex order, radius of convexity, convex functions, λ-spirallike functions, Convex functions of order α, λ-spirallike functions of order α.

(iii) $\mathcal{C}(1,-1,1-\alpha)$, $0 \leqslant \alpha < 1$ is the class of convex functions of order α, introduced by M.S. Robertson [7].

(iv) $\mathcal{C}(1,-1,e^{-i\lambda}\cos\lambda)$, $|\lambda| < \pi/2$ is the class of functions for which $zf'(z)$ is λ-spirallike, introduced by M.S. Robertson [8,9].

(v) $\mathcal{C}(1,-1,(1-\alpha)e^{-i\lambda}\cos\lambda)$ is the class of functions for which $zf'(z)$ is λ-spirallike of order α $(0 \leqslant \alpha < 1,\ |\lambda| < \pi/2)$, studied by B. Pinchuk [5] and Sizuk [10].

(vi) $\mathcal{C}(1-2\beta,-1,b)$, $0 \leqslant \beta < 1$ is the set defined by $\Re\left(1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right) > \beta$.

(vii) $\mathcal{C}(1,0,b)$ is the set defined by $\left|\left(1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right)-1\right| < 1$.

(viii) $\mathcal{C}(\beta,0,b)$ is the set defined by $\left|\left(1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right)-1\right| < \beta$.

(ix) $\mathcal{C}(1,-1+\frac{1}{M},b)$, $M > \frac{1}{2}$ is the set defined by $\left|\left(1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right)-M\right| < M$.

(x) $\mathcal{C}(\beta,-\beta,b)$, $0 \leqslant \beta < 1$ is the set defined by $\left|\frac{(1+\frac{1}{b}z\frac{f''(z)}{f'(z)})-1}{(1+\frac{1}{b}z\frac{f''(z)}{f'(z)})+1}\right| < \beta$.

2. Radius of convexity for the class $\mathcal{C}(A,B,b)$

In this section we shall give sharp bound of the radius of convexity for the class of Janowski convex functions of complex order.

THEOREM 2.1 *The radius of convexity for the class $\mathcal{C}(A,B,b)$ is*

$$r_c = \frac{2}{|b|(A-B)+\sqrt{|b|^2(A-B)^2-4[(B^2-AB)\Re b-B^2]}}. \tag{$*$}$$

This radius is sharp because the extremal function is

$$f_*(z) = \begin{cases} \dfrac{B}{b(A-B)+B}(1+Bz)^{\frac{b(A-B)+B}{B}}, & \text{if } B \neq 0, \\ \dfrac{1}{bA}e^{bAz}, & \text{if } B = 0. \end{cases}$$

Really, if we take $w = \dfrac{r(r-\sqrt{\bar{b}/b})}{1-r\sqrt{\bar{b}/b}}$ we obtain $1+w\dfrac{f''(w)}{f'(w)} = r_c$.

Proof. In [3] Janowski proved that, if $p(z) \in p(A,B)$, then

$$\left|p(z)-\frac{1-ABr^2}{1-B^2r^2}\right| \leqslant \frac{(A-B)r}{1-B^2r^2}. \tag{2.1}$$

On the other hand from the definition of the class $\mathcal{C}(A,B,b)$ we write

$$1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = p(z). \tag{2.2}$$

From the relations (2.1) and (2.2) we can write

$$\left|\left[1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right]-\frac{1-ABr^2}{1-B^2r^2}\right|\leqslant\frac{(A-B)r}{1-B^2r^2}. \tag{2.3}$$

After simple calculations from the relation (2.3) we obtain

$$\Re\left(1+z\frac{f''(z)}{f'(z)}\right)\geqslant\frac{[(B^2-AB)\Re b-B^2]r^2-|b|(A-B)r+1}{1-B^2r^2}. \tag{2.4}$$

Hence for $r<r_c$ the left-hand side of preceding inequality is positive which implies the assertion $(*)$.

Also note that the inequality (2.4) becomes an equality for the function $f_*(z)$. It follows that the radius of convexity for the class $\mathcal{C}(A,B,b)$ is r_c. If we take $w=\dfrac{r(r-\sqrt{\bar{b}/b})}{1-r\sqrt{\bar{b}/b}}$ we see that $1+w\dfrac{f_*''(w)}{f_*'(w)}=\dfrac{[(B^2-AB)\Re b-B^2]r^2+|b|(A-B)r-1}{1-r^2B^2}$. ■

(i) For $A=1$, $B=-1$, $r_c=\dfrac{1}{|b|+\sqrt{|b|^2-[2\Re b]+1}}$. This is the radius of convexity for the class of convex functions of complex order. This result was obtained by M.A. Nasr and M.K. Aouf [4].

(ii) For $A=1-2\beta$, $B=-1$, $0\leqslant\beta<1$,

$$r_c=\frac{1}{|b|(1-\beta)+\sqrt{|b|^2(1-\beta)^2-2(1-\beta)\Re b+1}}.$$

This is the radius of convexity for the class $\mathcal{C}(1-\beta,-1,b)$.

(iii) For $A=1$, $B=0$, $r_c=1/|b|$. This is the radius of convexity for the class $\mathcal{C}(1,0,b)$.

(iv) For $A=\beta$, $B=-\beta$, $0\leqslant\beta<1$, $r_c=\dfrac{1}{\beta[|b|+\sqrt{|b|^2-[2\Re b]+1}]}$. This is the radius of convexity for the class $\mathcal{C}(\beta,-\beta,b)$.

(v) For $A=\beta$, $B=0$, $0\leqslant\beta<1$, $r_c=1/(\beta|b|)$. This is the radius of convexity for the class $\mathcal{C}(\beta,0,b)$.

(vi) For $A=1$, $B=\frac{1}{M}-1$, $M>1/2$,

$$r_c=\frac{2}{|b|(2-\frac{1}{M})+\sqrt{|b|^2(2-\frac{1}{M})^2-4[(2-\frac{3}{M}+\frac{1}{M^2})\Re b-(\frac{1}{M}-1)^2]}}.$$

This is the radius of convexity for the class $\mathcal{C}(1,\frac{1}{M}-1,b)$.

We note that by giving specific values to b we obtain the radius of convexity for the corresponding classes.

REFERENCES

[1] M.K. Aouf, H.E. Darwish and A.A. Attiya, *Coefficient estimates for certain classes of analytic functions*, Punjab Univ. J. Math. **32** (2000), 51–64.

[**2**] A.W. Goodman, *Univalent Functions*, Vols. I, II, Mariner Publ. Comp. Inc., Tampa, Florida, 1983.

[**3**] W. Janowski, *Some extremal problems for certain families of analytic functions, I*, Ann. Polon. Math. **28** (1973), 297–326.

[**4**] M.A. Nasr and M.K. Aouf, *On convex functions of complex order*, Bull. Fac. Sci., University of Mansoura, 1982, 566–580.

[**5**] B. Pinchuk, *On starlike and convex functions of order* α, Duke Math. J. **35** (1968), 721–734.

[**6**] C. Pommerenke, *On starlike and convex functions*, J. London Math. Soc. (1962), 209–224.

[**7**] M.S. Robertson, *On the theory of univalent functions*, Ann. Math. **37** (1936), 374–408.

[**8**] M.S. Robertson, *Radii of starlikeness and close-to-convexity*, Proc. Amer. Math. Soc. **16** (1956), 847–882.

[**9**] M.S. Robertson, *Univalent functions* $f(z)$ *for which* $zf(x)$ *is spirallike*, Michigan Math. J. **16** (1969), 97–101.

[**10**] P.I. Sizuk, *Regular functions* $f(z)$ *for which* $zf(z)$ *is* θ*-spiral shaped of order* α, Sibirsk. Math. Zh. **16** (1975), 1286–1290.

[**11**] P. Wiatrowski, *The coefficients of a certain family of holomorphic functions*, Zeszyty Nauk. Univ. Lodz. Nauki Mat. Przyrod., Ser. II. Zesty (39) Math (1971), 57–85.

(received 06.06.2001)

Department of Mathematics, Faculty of Sciences and Arts, Istanbul Kültür University, E5 Karayolu Üzeri (Ataköy Metro Istsymu Karşisi) Şirinevler 34510, Istanbul