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DISPLACEMENT STRUCTURE OF GENERALIZED INVERSE $A_{T,S}^{(1,2)}$

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Abstract. It is well known that matrices with a UV-displacement structure possess generalized inverse with a VU-displacement structure. Estimation for the displacement rank of $A_{T,S}^{(1,2)}U - VA_{T,S}^{(1,2)}$ are presented, where $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of A with prescribed range T and null space S. We extend the results due to G. Heinig and F. Hellinger, Wei and Ng, Cai and Wei for the Moore-Penorse inverse, group inverse and weighted Moore-Penrose inverse, respectively.

1. Introduction and Preliminaries

The aim of the present paper is to study the generalized inverse of a structured matrix. To begin with, we recall some facts concerning the regular inversion of structured matrices, which are motivated by [5, 6]. If the rank of a matrix's displacement is small, fast algorithms for the matrix are available.

A matrix is called *matrix with displacement structure* [2, 3, 8] if and only if the rank of the matrix $AU - VA$ or $A - VAU$ is small compared with the order of the matrix A. The rank of $AU - VA$ is said to be *Sylvester UV-displacement rank* and the rank of $A - VAU$ is called the *Stein UV-displacement rank* of A, since A is the solution of a *Sylvester* or *Stein* equation, respectively.

As is well known, fast inversion algorithms for matrix A can be constructed if A is a matrix with displacement structure. We are interested in the generalized inverse with as small as possible displacement rank. We present the estimate for the rank of $A_{T,S}^{(1,2)}U - VA_{T,S}^{(1,2)}$, where $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of A, in Section 2. Then we give an explicit estimate for *general displacement*, which is defined in the paper, Section 3. Our results cover the previous results [2, 4, 14] for the Moore-Penrose inverse, weighted Moore-Penrose, inverse and group inverse, respectively.

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A matrix X is said to be a $(1,2)$-inverse of A if the following condition is fulfilled:

$$XAX = X, \qquad AXA = A.$$

Let $A \in C_r^{m\times n}$, T and S are subspaces of C^n and C^m, respectively, the dimensions of T and S are r and $m-r$, then A has a unique $(1,2)$-inverse X satisfying $R(X) = T$, $N(X) = S$ if and only if $R(A) \bigoplus S = C^m$, $Ker(A) \bigoplus T = C^n$. We denote X by $A_{T,S}^{(1,2)}$.

If $G \in C^{n\times m}$, and G satisfies $R(G) = T$, $N(G) = S$, then we get [9]

$$A_{T,S}^{(1,2)} = (GA)_g G = G(AG)_g, \quad AA_{T,S}^{(1,2)} = AG(AG)_g, \quad A_{T,S}^{(1,2)} A = (GA)_g GA,$$

where $(AG)_g$ and $(GA)_g$ are group inverses of AG and GA.

From Jordan canonical form theory, we get that for any complex $m\times n$ matrix A with $rank(A) = r$ with prescribed range T and null space S, there exist nonsingular matrices R and N [13] such that

$$A = R^{-1}\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} N, \quad G = N^{-1}\begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} R \tag{1}$$

where R is an $m\times m$ nonsingular matrix and N is $n\times n$. Note that A_{11}, G_{11} are nonsingular matrices. Now we can write $A_{T,S}^{(1,2)}$ of A in the form [13],

$$A_{T,S}^{(1,2)} = N^{-1}\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} R \tag{2}$$

Let

$$Q = A_{T,S}^{(1,2)} A, \quad P = I - Q, \quad Q_* = AA_{T,S}^{(1,2)}, \quad P_* = I - Q_*. \tag{3}$$

Obviously, Q and P are oblique projections, where $Im(A)$ denotes the range of A and $Ker(A)$ is the null space of A. It is easy to check that

$$\begin{aligned} Im(Q) = Im(A_{T,S}^{(1,2)}) = Ker(P) = Im(G), \qquad Im(Q_*) = Ker(P_*) = Im(A), \\ Im(P_*) = Ker(A_{T,S}^{(1,2)}) = Ker(Q_*) = Ker(G), \quad Im(P) = Ker(Q) = Ker(A). \end{aligned} \tag{4}$$

For recent results on the generalized inverse $A_{T,S}^{(1,2)}$, we refer to [10, 11, 12, 15].

2. Sylvester displacement rank

Throughout the paper, $U \in C^{n\times n}$ and $V \in C^{m\times m}$ are some fixed matrices. The operator

$$d(U,V) = AU - VA$$

is called the *UV-displacement* of A. To distinguish this displacement concept from the more general case in Section 3 we call it *the Sylvester UV-displacement*, since A is the solution of a certain Sylvester equation. We can easily find that for a nonsingular matrix A with *UV-displacement* structure the inverse matrix A^{-1} possesses a *VU-displacement* structure, and the relation

$$rank(A^{-1}V - UA^{-1}) = rank(AU - VA)$$

holds.

We want to get an estimate for the *VU-displacement* rank of the generalized inverse of a matrix with displacement structure. First we show that the displacement structure of $A_{T,S}^{(1,2)}$ is the following representation.

PROPOSITION 2.1. *Let $A \in C^{m\times n}$, $U \in C^{n\times n}$ and $V \in C^{m\times m}$, and $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of A. Then*

$$A_{T,S}^{(1,2)}V - UA_{T,S}^{(1,2)} = A_{T,S}^{(1,2)}VP_* - PUA_{T,S}^{(1,2)} - A_{T,S}^{(1,2)}(AU - VA)A_{T,S}^{(1,2)}, \tag{5}$$

where Q, P, Q_* and P_* are defined in (4).

Proof. The relation is immediately proved by the following equation

$$A_{T,S}^{(1,2)}(AU - VA)A_{T,S}^{(1,2)} = (I-P)UA_{T,S}^{(1,2)} - A_{T,S}^{(1,2)}V(I-P_*). \quad \blacksquare \tag{6}$$

From (4) we obtain the following Corollary.

COROLLARY 2.1. *The VU-displacement rank of $A_{T,S}^{(1,2)}$ satisfies the following estimate:*

$$rank(A_{T,S}^{(1,2)}V - UA_{T,S}^{(1,2)}) \le rank(AU - VA) + rank(Q_*VP_*) + rank(PUQ). \tag{7}$$

Proof. We will prove $dim(PUA_{T,S}^{(1,2)}) = rank(PUQ)$, and $rank(A_{T,S}^{(1,2)}VP) = rank(QVP)$.

$$rank(PUA_{T,S}^{(1,2)} = dim[PUIm(A_{T,S}^{(1,2)})] = dim[Im(PUQ)] = rank(PUQ), \tag{8}$$

the second equality can be proved similarly.

Taking these into account, we obtain the above estimate. ■

Now we aim to obtain the estimate for the second and third terms on the right-hand side.

PROPOSITION 2.2. *With the notation above, we get the estimate*

$$rank(Q_*VP_*) + rank(PUQ) \le rank(UG - GV). \tag{9}$$

Proof. We set $F = UG - GV$ and make partitions

$$NUN^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad RVR^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}. \tag{10}$$

From (1) we write the matrix F in the following form

$$\begin{aligned} NFR^{-1} &= NUN^{-1}NGR^{-1} - NGR^{-1}RVR^{-1} \\ &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \\ &= \begin{bmatrix} U_{11}G_{11} - G_{11}V_{11} & -G_{11}V_{12} \\ U_{21}G_{11} & 0 \end{bmatrix}. \end{aligned}$$

From [7] we know that

$$rank\begin{bmatrix} U_{11}G_{11}-G_{11}V_{11} & -G_{11}V_{12} \\ U_{21}G_{11} & 0 \end{bmatrix} \geq rank(-G_{11}V_{12}) + rank(U_{21}G_{11})$$
$$= rank(V_{12}) + rank(U_{21}) = rank\begin{bmatrix} 0 & V_{12} \\ 0 & 0 \end{bmatrix} + rank\begin{bmatrix} 0 & 0 \\ U_{21} & 0 \end{bmatrix}$$
$$= rank(Q_*VP_*) + rank(PUQ).$$

where G_{11} is nonsingular. Then the proof is over. ■

From Propositions 2.1 and 2.2 we derive the first main result.

THEOREM 2.1. *Let $A \in C^{m\times n}$ and $A^{(1,2)}_{T,S}$ be its $(1,2)$-inverse. Then*

$$rank(A^{(1,2)}_{T,S}V - UA^{(1,2)}_{T,S}) \leq rank(AU - VA) + rank(UG - GV).$$

PROPOSITION 2.3. *If $AU = UG, VA = GV$, then we get the estimate:*

$$rank(A^{(1,2)}_{T,S}V - UA^{(1,2)}_{T,S}) \leq 2\ rank(AU - VA). \tag{11}$$

3. Displacement structure for generalized displacement

In order to generalize Theorem 2.1 we introduce a generalized displacement concept [2]. Let $a = [a_{ij}]_0^1$ denote a nonsingular 2×2 matrix. We associate a with the polynomial in two variables

$$a(\lambda,\mu) = \sum_{i,j=0}^{1} a_{ij}\lambda^i\mu^j$$

and the linear fractional function

$$f_a(\lambda) = \frac{a_{10}+a_{11}\lambda}{a_{00}+a_{01}\lambda}. \tag{12}$$

For any fixed $U \in C^{n\times n}$ and $V \in C^{m\times m}$, the generalized (a, U, V) displacement of $A \in C^{m\times n}$ generated by $a(\lambda,\mu)$ is defined by

$$a(V,U)A = \sum_{i,j=0}^{1} a_{ij}V^iAU^j.$$

If $a = d \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we just get *Sylvester displacement* that we have discussed. If $a = d \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we get *Stein displacement.*

LEMMA 3.1. [2] *Let $a = [a_{ij}]_0^1$, $b = [b_{ij}]_0^1$, $c = [c_{ij}]_0^1$, $d = [d_{ij}]_0^1$ be nonsingular 2×2 matrices such that*

$$a = b^Tdc. \tag{13}$$

Then

$$(b_{00}+b_{01}\lambda)^{-1}a(\lambda,\mu)(c_{00}+c_{01}\mu)^{-1} = d(f_b(\lambda), f_c(\mu)) \tag{14}$$

for all λ,μ with $b_{00}+b_{01}\lambda \neq 0$ and $c_{00}+c_{01}\mu \neq 0$.

LEMMA 3.2. [2] *If* $d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, *then there exist* 2×2 *matrices* b, c *such that (13) holds and* $b_{00} + b_{01}V$ *and* $c_{00} + c_{01}U$ *are invertible.*

Taking Lemma 3.1 and Lemma 3.2 together, we obtain the following

PROPOSITION 3.1. [2] *If* b *and* c *be matrices satisfying the conditions in Lemma 3.2, then for* $A \in C^{m \times n}$,

$$a(V,U)A = (b_{00} + b_{01}V)[Af_c(U) - f_b(V)A](c_{00} + c_{01}U).$$

The following is very important to generalize Theorem 2.1 for general (a, U, V) displacement.

PROPOSITION 3.2. *(a) If* $\psi = [\psi_{ij}]_0^1$ *is nonsingular and* $\psi_{00} + \psi_{01}V$ *is invertible, then* $rank(Q_*VP_*) = rank(Q_*\tilde{V}P_*)$, *where* $\tilde{V} \equiv f_\psi(V)$ *is defined in (12).*

(b) If $\phi = [\phi_{ij}]_0^1$ *is nonsingular and* $\phi_{00} + \phi_{01}U$ *is invertible, then*

$$rank(PUQ) = rank(P\tilde{U}Q),$$

where $\tilde{U} \equiv f_\phi(U)$ *is defined in (12).*

Proof. We define

$$\mathcal{S} = Ker(G) \cap Ker(GV), \quad \mathcal{S}_1 = Ker(G) \ominus \ CalS.$$

We show that Q_*VP_* is one-to-one on $\mathcal{S}_1$. If $Q_*VP_*x = 0$ and $x \in \mathcal{S}_1$, then $VP_*x \in Ker(Q_*) = Ker(G)$. That means $GVP_*x = GVx = 0$. Noting that $x \in Ker(G)$, we conclude $x \in \mathcal{S}$. Thus $x = 0$.

Furthermore, Q_*VP_* vanishes on $\mathcal{S}$. Since $Q_*VP_*x = Q_*Vx = (AG)_gAGVx = 0$ for all $x \in \mathcal{S}$ we have

$$rank(Q_*VP_*) = dim(\mathcal{S}_1). \tag{15}$$

Analogously we define $\tilde{\mathcal{S}} = Ker(G) \cap Ker(G\tilde{V})$, $\tilde{\mathcal{S}}_1 = Ker(G) \ominus \tilde{\mathcal{S}}$, and we get

$$rank(Q_*\tilde{V}P_*) = dim(\tilde{\mathcal{S}}_1). \tag{16}$$

Now we show that the invertible matrix $\overline{\psi}_{00} + \overline{\psi}_{01}V$ bijectively maps $\mathcal{S}$ onto $\tilde{\mathcal{S}}$. Suppose that $x \in \mathcal{S}$. Then $x, Vx \in Ker(G)$. Hence $y \equiv (\overline{\psi}_{10} + \overline{\psi}_{11}V)x$ and $z \equiv (\overline{\psi}_{00} + \overline{\psi}_{01}V)x$ are all contained in $Ker(G)$. Thus $y = \tilde{V}z$ and we conclude that $z, \tilde{V}z \in Ker(G)$, which implies $z \in \tilde{\mathcal{S}}$. Convesely, with the same arguments we get $(\overline{\psi}_{00} + \overline{\psi}_{01}V)^{-1}z \in \mathcal{S}$ for $z \in \tilde{\mathcal{S}}$.

This implies

$$dim(\mathcal{S}_1) = dim[Ker(G)] - dim(\mathcal{S}) = dim[Ker(G)] - dim(\tilde{\mathcal{S}}) = dim(\tilde{\mathcal{S}}_1).$$

According to (15) and (16), we get assertion (a).

Assertion (b) is proved analogously. ■

Now we can generalize Theorem 2.1 for general (a, U, V) displacement.

THEOREM 3.1. *If* a, b *are* 2×2 *nonsingular matrices, then*

$$rank[a(U,V)A_{T,S}^{(1,2)})] \leq rank[a^T(V,U)A] + rank[b(U,V)G]. \tag{17}$$

Proof. According to Lemma 3.2 there exist 2×2 matrices w, x, y, z such that $w_{00} + w_{01}U$, $x_{00} + x_{01}V$, $y_{00} + y_{01}U$, $z_{00} + z_{01}V$ are invertible and $a = w^T dz$, $b = x^T dy$. Hence,

$$\begin{aligned} &rank[a(U,V)A_{T,S}^{(1,2)}] - rank[a^T(V,U)A] \\ &\quad = rank[f_w(U)A_{T,S}^{(1,2)} - A_{T,S}^{(1,2)} f_z(V)] - rank[f_z(V)A - Af_w(U)] \\ &\le rank[Pf_w(U)Q] + rank[Q_* f_z(V)P_*] = rank[Pf_y(U)Q] + rank[Q_* f_x(V)P_*] \\ &\quad \le rank[f_y(U)G - Gf_x(V)] = rank[b(U,V)G]. \quad \blacksquare \end{aligned}$$

4. Concluding remarks

In this paper we study the displacement structure of $(1,2)$-inverse of a singular matrix. It is natural to ask if we can extend our results to linear operators in Hilbert space. This will be the future research.

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