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FUZZY PARTITIONS OF UNITY

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Abstract. In General Topology, there exist useful theorems on the existence of partitions of unity for paracompact regular spaces (and also for normal spaces). In this paper, we define the notion of fuzzy partition of unity and we obtain some results about this concept.

It is known that, in Topology and in Differential Geometry and Mathematical Analysis, the notion of partition of unity is particularly interesting, because this technique allows us "to glue" local properties in some spaces. The classic definition is due to E. Michael [5] who proved an interesting characterization of paracompactness in terms of partitions of unity.

In this paper, we define the notion of fuzzy partition of unity and we prove some theorems on its existence.

We consider fuzzy topological spaces in the Chang's sense. We will denote by q the quasi-coincident relation [7], by $\mathcal{Q}(\mu)$ the Q-neighborhood system of a fuzzy set μ [7], and by χ_A the characteristic function of a set A.

For regularity and normality of a fuzzy topological space we refer to [1].

Definition 1. [3] Let $\mathcal{U}$, $\mathcal{V}$ be two families of sets in a fuzzy topological space (X,T). $\mathcal{V}$ is called a refinement of $\mathcal{U}$ if for any $\nu \in \mathcal{V}$, there exists a $\mu \in \mathcal{U}$ such that $\nu \leqslant \mu$.

Definition 2. [3] Let $\mathcal{A}$ be a family of fuzzy sets and μ be a fuzzy set in a fuzzy topological space (X,T). We say that $\mathcal{A}$ is locally finite in μ if for each fuzzy point e in μ, there exists a $\nu \in \mathcal{Q}(e)$ such that ν is quasi-coincident with at most a finite number of sets of $\mathcal{A}$; we omit the word "in μ" when μ is the total.

Definition 3. [3] A family of sets $\mathcal{U}$ is called a Q-cover of a fuzzy set μ, if for each $x \in \operatorname{supp}\mu$, there exists a $\nu \in \mathcal{U}$ such that ν and μ are quasi-coincident at x. Let $r \in (0,1]$; then $\mathcal{U}$ is called an r-Q-cover of μ if $\mathcal{U}$ is a Q-cover of $r\chi_{\{x\in X \mid \mu(x)\geqslant r\}}$.

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Definition 4. [3] Let $r \in (0,1]$, μ be a fuzzy set in a fuzzy topological space (X,T). We say that μ is r-paracompact if for each r-open Q-cover of μ there exists an open refinement of it which is both locally finite in μ and an r-Q-cover of μ. If for every $r \in (0,1]$ we have that μ is r-paracompact, then μ is called S-paracompact.

Definition 5. Let $\{\mu_j\}_{j\in J}$ be a family of open fuzzy sets in a fuzzy topological space. We will say that $\{\mu_j\}_{j\in J}$ is a fuzzy partition of unity subordinated to a Q-cover $\mathcal{U}$, if $\{\mu_j\}_{j\in J}$ is a locally finite Q-cover, $\sum_{j\in J}\mu_j = 1$, and for each $j \in J$ we have that $\operatorname{supp}\mu_j \subset \operatorname{supp}\mu$ for some $\mu \in \mathcal{U}$.

Remark. By the above definition, if $\{\mu_j\}_{j\in J}$ is a fuzzy partition of unity in an induced fuzzy topological space, for each $j \in J$ we have that $\mu_j^{-1}((0,1])$ is open in the original topology, because $\mu_j\colon X \to [0,1]$ is lower semicontinuous.

Proposition 1. *If (X,T) is a fuzzy topological space such that for each open 1-Q-cover there exists a fuzzy partition of unity subordinated to it, then (X,T) is 1-paracompact.*

Proof. For an arbitrary open 1-Q-cover $\mathcal{U}$, there exists a fuzzy partition of unity $\{\mu_j\}_{j\in J}$ subordinated to $\mathcal{U}$. Thus $\{\mu_j\}_{j\in J}$ is both a locally finite open refinement of $\mathcal{U}$ and 1-Q-cover of X, because for all $x \in X$, there is $j_0 \in J$ such that $\mu_{j_0}(x) > 0$ (for $\sum_{j\in J}\mu_j(x) = 1$), and so $\mu_{j_0}(x) + 1 > 1$, thus $\mu_{j_0}\, q\, \chi_X$ and (X,T) is 1-paracompact. ■

Theorem 1. *Let (X,T) be an S-paracompact regular weakly induced fuzzy topological space. Then, for each $r \in (0,1)$ and for each open r-Q-cover $\mathcal{U}$, there exists a fuzzy partition of unity subordinated.*

Proof. Let $[T] = \{\, A \subset X \mid \chi_A \in T \,\}$, the original topology of T. The topological space $(X,[T])$ is regular because (X,T) is regular fuzzy. Moreover, for every $x \in X$ there is a $\mu \in \mathcal{U}$ such that $\mu(x) + r > 1$ (because $\mathcal{U}$ is r-Q-cover of X), so $\{\, W_\mu \mid \mu \in \mathcal{U} \,\}$ where $W_\mu = \{\, x \in X \mid \mu(x) > 1 - r \,\}$ is an open cover of $(X,[T])$.

By [3, Theorem 3.6], $(X,[T])$ is paracompact, so there exists a continuous partition of unity $\{\mu_j\}_{j\in J}$ subordinated to $\{\, W_\mu \mid \mu \in \mathcal{U} \,\}$ by the classical theorem due to Michael [5]. Then all $\mu_j\colon X \to [0,1]$ are continuous maps, $\sum_{j\in J}\mu_j = 1$, $\{\operatorname{supp}\mu_j\}_{j\in J}$ is locally finite in $(X,[T])$ and for each $j \in J$, there is a $\mu \in \mathcal{U}$ such that $\operatorname{supp}\mu_j \subset W_\mu$. Thus, we have that μ_j is open fuzzy, for all $j \in J$.

Also, $\{\mu_j\}_{j\in J}$ is locally finite because for every fuzzy point x_λ, the point $x \in X$ and there is an open neighborhood U such that it meets only a finite number of members of $\{\operatorname{supp}\mu_j\}_{j\in J}$; hence, χ_U is a Q-neighborhood of x_λ which is quasi-coincident with a finite number of members of $\{\mu_j\}_{j\in J}$ (because $\chi_U\, q\, \mu_j \iff 1 + \mu_j(z) > 1$ for some $z \in U$, and so $\mu_j(z) > 0$ and $U \cap \operatorname{supp}\mu_j \neq \varnothing$).

Finally, $\{\mu_j\}_{j\in J}$ is a Q-cover because, for all $x \in X$ there is $j_0 \in J$ such that $\mu_{j_0}(x) > 0$ (by $\sum_{j\in J}\mu_j(x) = 1$), and so $\mu_{j_0}(x) + 1 > 1$, and $\operatorname{supp}\mu_j \subset \operatorname{supp}\mu$ for some $\mu \in \mathcal{U}$ (because $z \in \operatorname{supp}\mu_j$ implies that $z \in W_\mu$). ■

THEOREM 2. *Let (X, T) be a normal weakly induced fuzzy topological space. Then for each $r \in (0,1)$ and for each open locally finite r-Q-cover $\mathcal{U}$, there exists a fuzzy partition of unity subordinated.*

Proof. The topological space $(X, [T])$ is normal by the fuzzy normality of (X, T). Let $\mathcal{U}$ be an open locally finite r-Q-cover of X; then $\{ W_\mu \mid \mu \in \mathcal{U} \}$, where $W_\mu = \{ x \in X \mid \mu(x) > 1 - r \}$ is a locally finite open cover of $(X, [T])$. Indeed, for each $z \in X$, we have that z_{1-r} is a fuzzy point in X. By the hypothesis, there exists an open Q-neighborhood ν of z_{1-r} which is quasi-coincident with only a finite number of members $\mu_1, \dots, \mu_n$ of $\mathcal{U}$. Let $U = \{ x \in X \mid \nu(x) > r \}$; then U is open in $(X, [T])$ and $z \in U$ (because $z_{1-r}\, q\, \nu$ if and only if $1 - r > 1 - \nu(z)$).

If $U \cap W_\mu \neq \varnothing$, then there is $y \in X$ such that $\nu(y) > r$ and $\mu(y) > 1 - r$, thus $\nu(y) + \mu(y) > 1$, $\nu\, q \mu$ and $\mu \in \{\mu_1, \dots, \mu_n\}$. Thus, U is a neighborhood of z in $(X, [T])$ and meets only a finite number of members $W_{\mu_1}, \dots, W_{\mu_n}$ of $\{ W_\mu \mid \mu \in \mathcal{U} \}$. So, by a known theorem of General Topology, there exists a continuous partition of unity $\{\mu_j\}_{j \in J}$ subordinated to $\{ W_\mu \mid \mu \in \mathcal{U} \}$. And, analogously to the proof of the last theorem, this family $\{\mu_j\}_{j \in J}$ is also a fuzzy partition of unity subordinated to $\mathcal{U}$. ■

REFERENCES

[**1**] B. Hutton and I. L. Reilly, *Separation axioms in fuzzy topological spaces*, Fuzzy Sets Syst. **3** (1980), 93–104.

[**2**] Liu Y.-M. and Luo M.-K., *Fuzzy Topology*, World Scientific Publ., Singapore, 1997.

[**3**] Luo M.-K., *Paracompactness in fuzzy topological spaces*, J. Math. Anal. Appl. **130** (1988), 55–77.

[**4**] H. W. Martin, *Weakly induced fuzzy topological spaces*, J. Math. Anal. Appl. **78** (1980), 634–639.

[**5**] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4** (1953), 831–838.

[**6**] Peng Y.-W., *Topological structure of a fuzzy function space – The pointwise convergent topology and compact open topology*, Kexue Tongbao **29** (1984), 289–292.

[**7**] Pu P.-M. and Liu Y.-M., *Fuzzy topology I: Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571–599.

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