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RELATIONS BETWEEN SOME TOPOLOGIES

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Abstract. Generalizations of openness, such as semi-open, preopen, semi-pre-open, α-open, etc. are important in topological spaces and in particular in topological spaces on which ideals are defined. α-equivalent topologies and $*$-equivalent topologies with respect to an ideal have some common properties. Relations between these aforementioned notions of openness are investigated within the framework of α-equivalence and $*$-equivalence.

1. Introduction

The subject of ideals in general topological spaces was introduced by Kuratowski [8] and Vaidyanathaswamy [18]. An ideal $\mathcal{I}$ on a set X is a nonempty collection of subsets of X which satisfies

(i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$,

(ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

By $(X, \tau, \mathcal{I})$ we will denote a topological space (X, τ) with an ideal $\mathcal{I}$ on X. No separation properties are assumed on X. For a space $(X, \tau, \mathcal{I})$ and a subset $A \subset X$,

$$A^*(\tau, \mathcal{I}) = \{\, x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \,\}$$

(where $\tau(x) = \{U \in \tau : x \in U\}$) is called the *local function of A with respect to $\mathcal{I}$ and τ* [8]. Note that $\mathrm{cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ [15] on X. If there is no chance of confusion, we simply write A^* or $A^*(\mathcal{I})$ instead of $A^*(\tau, \mathcal{I})$, and τ^* instead of $\tau^*(\mathcal{I})$.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on X, then $\mathcal{I} \vee \mathcal{J} = \{I \cup J : I \in \mathcal{I} \text{ and } J \in \mathcal{J}\}$ is also an ideal on X [6].

In a topological space (X, τ), for any subset A, A^o, $\operatorname{int} A$ or $\tau \operatorname{int} A$ will stand for the interior of A and $\overline{A}$, $\operatorname{cl} A$, or $\tau \operatorname{cl} A$ will stand for the closure of A. A subset A of a space (X, τ) is said to be *semi-open* (*pre-open*, *α-open*, *semi-pre-open*, *regular open*, *nowhere dense*, *codense*) if $A \subset A^{\overline{o}}$ ($A \subset \overline{A}^{o}$, $A \subset \overline{A^{o}}^{o}$, $A \subset \overline{\overline{A}^{o}}$, $A = \overline{A}^{o}$, $\overline{A}^{o} = \emptyset$, $A^o = \emptyset$), respectively.

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A point x of X is called a *θ-interior point of A* if there exists an open set U such that $x \in U \subset \bar{U} \subset A$. Also, θ-int A will stand for the set of θ-interior points of A. A is θ-open iff $A \subset \theta\text{-int}\, A$ [10]. The family of all α-open sets in (X, τ) is a topology on X which is finer than τ and it is denoted by τ^α. Topologies τ and σ on X are called *α-equivalent* if they have the same α-open sets [14].

A supratopology $\mathcal{A}$ on X is a nonempty collection of subsets of X which satisfies

(i) $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$,

(ii) $\mathcal{A}$ is closed under arbitrary unions [10].

The *$\mathcal{A}$-interior* (shortly, $\mathcal{A}$-int) of a subset A of X is defined as

$$\mathcal{A}\text{-}\operatorname{int} A = \bigcup \{U : U \subset A, U \in \mathcal{A}\}$$

[9]. It is well known that the family of semi-open (pre-open, semi-pre-open) sets of a topological space is a supratopology on this space. If $\mathcal{A}$ is a supratopology on X, then

$$\mathcal{T}_{\mathcal{A}} = \{A \subset X : A \cap B \in \mathcal{A} \text{ for each } B \in \mathcal{A}\}$$

is a topology and $\mathcal{T}_{\mathcal{A}} \subset \mathcal{A}$ [19].

We will use the following notational conventions:

$\tau \in Top\,(X) \iff \tau$ is a topology on X,

$\mathcal{I} \in Id(X) \iff \mathcal{I}$ is an ideal on X,

$A \in D(X) \iff A$ is dense in X,

$A \in CD(X) \iff A^o = \emptyset$ (i.e. A is codense),

$A \in NO(X) \iff A^{\overset{o}{-}} = \emptyset$ (i.e. A is nowhere dense),

$A \in SO(X) \iff A \subset A^{\overline{o}}$,

$A \in PO(X) \iff A \subset A^{\overset{o}{-}}$,

$A \in \alpha O(X) \iff A \subset A^{\overset{o}{\underset{o}{-}}}$,

$A \in SPO(X) \iff A \subset A^{\overline{\overset{o}{-}}}$,

$A \in RO(X) \iff A = A^{\overset{o}{-}}$,

$A \in \theta O(X) \iff A$ is θ-open,

$A \in SC(X) \iff X - A$ is semi-open (i.e. A is semi-closed),

$A \in SR(X) \iff A$ is semi-open and semi-closed (i.e. A is semi-regular),

$\sigma \in [\tau]^\alpha \iff \sigma \in Top\,(X)$, and $\sigma^\alpha = \tau^\alpha$ (i.e. τ and σ are α-equivalent).

I_n (or $I_n(\tau)$) and $I_n(\sigma)$ will stand for the family of nowhere dense sets in X with respect to τ and σ, respectively. From now on, A^o and $\bar{A}$ will be reserved for the interior and closure of A with respect to topology τ, respectively.

In a topological space scl, sint, pcl, spcl, etc. will stand for the operations semi-closure, semi-interior, pre-closure, semi-pre-closure, respectively. Where it is necessary to indicate the topology, we will write, for example, τ-scl or σ-scl.

In the following theorem we recall some known results in the literature which will be used in this paper. They appear in [1, 2, 5, 6, 13–18].

Theorem 1.1. *In any topological space* (X,τ) *we have the following results:*
(1) *For any subset* A *of* X,

$$\operatorname{scl} A = A \cup A^{\overset{o}{-}}, \qquad \operatorname{sint} A = A \cap A^{\overline{o}},$$
$$\operatorname{pcl} A = A \cup A^{\overline{o}}, \qquad \operatorname{pint} A = A \cap A^{\overset{o}{-}},$$
$$\operatorname{spcl} A = A \cup A^{\overset{o}{\underset{o}{-}}}, \qquad \operatorname{spint} A = A \cap A^{\overset{\overline{o}}{-}},$$
$$\alpha\text{-}\operatorname{cl} A = \tau^{\alpha} \operatorname{cl} A = A \cup A^{\overset{\overline{o}}{-}}, \qquad \alpha\text{-}\operatorname{int} A = \tau^{\alpha} \operatorname{int} A = A \cap A^{\overset{o}{\underset{o}{-}}} \ [1,2].$$

(2) $\tau \subset \mathcal{T}_{SO(X)} \cap \mathcal{T}_{PO(X)} \cap \mathcal{T}_{SPO(X)}$ [1,2].
In the remaining results given below, $\sigma \in Top\,(X)$ *and* $\mathcal{I}, \mathcal{J} \in Id(X)$.
(3) *If* $\tau \subset \sigma \subset \tau^{\alpha}$, *then* $\sigma \in [\tau]^{\alpha}$ [14].
(4) $\tau \subset \tau^*(\mathcal{I})$ *and* $(\tau^*(\mathcal{I}))^*(\mathcal{I}) = \tau^*(\mathcal{I})$.
(5) $A^*(\tau,\mathcal{I})$ *is* τ*-closed and* $A^*(\tau,\mathcal{I}) = A^*(\tau^*(\mathcal{I}),\mathcal{I})$ *for each* $A \subset X$,
(6) $I \in \mathcal{I} \implies I$ *is* $\tau^*(\mathcal{I})$*-closed and* $I^* = \emptyset$.
(7) $\mathcal{I} \subset \mathcal{J} \implies A^*(\tau,\mathcal{J}) \subset A^*(\tau,\mathcal{I})$ *for each* $A \subset X$ *and* $\tau^*(\mathcal{I}) \subset \tau^*(\mathcal{J})$.
(8) $\tau \subset \sigma \implies A^*(\sigma,\mathcal{I}) \subset A^*(\tau,\mathcal{I})$ *for each* $A \subset X$ *and* $\tau^*(\mathcal{I}) \subset \sigma^*(\mathcal{I})$.
(8) $\tau \cap \mathcal{I} = \{\emptyset\} \iff \tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\} \iff X = X^* \iff U \subset U^*$ *for each* $U \in \tau$.
(9) $\mathcal{B}(\tau^*(\mathcal{I})) = \{U - I : U \in \tau, I \in \mathcal{I}\}$ *is a base for the topology* $\tau^*(\mathcal{I})$.
(10) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then for each* $U \in \tau$ *and for each* $I \in \mathcal{I}$, *we have* $\bar{U} = U^* = (U - I)^* = (U - I)^- = \tau^*\text{-}\operatorname{cl}(U - I) = \tau^*\text{-}\operatorname{cl} U$.
(11) $\tau^*(\mathcal{I} \vee \mathcal{J}) = (\tau^*(\mathcal{I}))^*(\mathcal{J})$ [6].
(12) $\tau^*(\mathcal{I}_n) = \tau^{\alpha}$, $\tau \cap \mathcal{I}_n = \{\emptyset\}$, $PO(X,\tau) \cap \mathcal{I}_n = \{\emptyset\}$.
(13) $A \subset B \implies A^*(\mathcal{I}) \subset B^*(\mathcal{I})$.
(14) $(A - I)^* = A^*$ *for each* $A \subset X$ *and each* $I \in \mathcal{I}$.
(15) $A^*(\mathcal{I}_n) = A^{\overset{\overline{o}}{-}}$ *for each* $A \subset X$.
(16) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then for each* $U \in \tau^*(\mathcal{I})$ *we have* $\tau\text{-}\operatorname{cl} U = \tau^*(\mathcal{I})\text{-}\operatorname{cl} U$.

2. Relations between topologies and some special sets

Firstly, some relations between families such as $SO(X), PO(X), SPO(X)$, etc. on a set X with two topologies are investigated. Then, these relations will be carried over to topological spaces on which ideals are defined. Some known results will be obtained by a different method.

Theorem 2.1. *Let* $\tau, \sigma, \omega \in Top\,(X)$. *Then we have the following results.*
(1) *If* $\tau \subset \sigma \subset SPO(X,\tau)$, *then:* (a) $SPO(X,\sigma) \subset SPO(X,\tau)$, (b) $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma)$.
(2) *If* $\tau \subset \sigma \subset PO(X,\tau)$, *then* $PO(X,\sigma) \subset PO(X,\tau)$, *and the relations* (*a*) *and* (*b*) *in* (1) *are valid.*
(3) I. *If* $\tau \subset \sigma$, *and* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U$ *for each* $U \in \tau$, *then*

(a) *For each* $A \subset X$ *we have*

$$\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A) = \sigma\,\mathrm{cl}(\tau\,\mathrm{int}\,A) \subset \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A),$$
$$\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A) \subset \sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A),$$
$$\tau\,\mathrm{int}(\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A)) \subset \tau\,\mathrm{int}(\sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A)) \subset \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A)),$$
$$\sigma\,\mathrm{cl}(\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)) \subset \sigma\,\mathrm{cl}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)) = \tau\,\mathrm{cl}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)).$$

(b) *For each* $U \in SO(X,\tau)$, *we have* $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U = \tau\,\mathrm{cl}(\tau\,\mathrm{int}\,U) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,U)$,

(c) $SO(X,\tau) \subset SO(X,\sigma)$,

(d) $PO(X,\sigma) \subset PO(X,\tau)$,

(e) $SPO(X,\sigma) \subset SPO(X,\tau)$,

(f) $\alpha O(X,\tau) \subset \alpha O(X,\sigma)$,

(g) $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma)$,

(h) $RO(X,\tau) \subset RO(X,\sigma)$.

II. *If* $\tau \subset \sigma$, *and* $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U$ *for each* $U \in \sigma$, *then in addition to the results given in (3.I) above, we have* $RO(X,\tau) = RO(X,\sigma)$, $SR(X,\tau) = SR(X,\sigma)$, *and* $\theta O(X,\tau) = \theta O(X,\sigma)$,

(4) *If* $\tau \subset \sigma \subset SO(X,\tau)$ *and* $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U$ *for each* $U \in \tau$, *then* $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U$ *for each* $U \in \sigma$. *So, the results in* (3) *are valid.*

(5) $\tau \subset \omega \subset \sigma$, *and* $\tau clU = \sigma\,\mathrm{cl}\,U$ *for each* $U \in \sigma$, *then* $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U = \omega\,\mathrm{cl}\,U$ *for each* $U \in \sigma$ *(hence for each* $U \in \omega$*).*
So, results similar to those given in (3) *are valid for* τ, ω *and* σ.

Proof. (1a) Let $\tau \subset \sigma \subset SPO(X,\tau)$ and $U \in SPO(X,\sigma)$. We have:

$$U \subset \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,U)) \subset \tau\,\mathrm{cl}(\sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,U)) \subset \tau\,\mathrm{cl}(\tau\,\mathrm{cl}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}(\sigma\,\mathrm{int}(\tau clU)))) \subset \tau\,\mathrm{cl}(\tau\,\mathrm{int}(\tau clU)).$$

(1b) The proof is clear from Corollary 2.13 below.

2. Let $\tau \subset \sigma \subset PO(X,\tau)$ and $U \in PO(X,\sigma)$. We have:

$$U \subset \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,U) \subset \sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,U) \subset \tau\,\mathrm{int}(\tau\,\mathrm{cl}(\sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,U))) \subset \tau\,\mathrm{int}(\tau\,\mathrm{cl}(\tau\,\mathrm{cl}\,U)) = \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,U).$$

So, $U \in PO(X,\tau)$. Since $PO(X,\tau) \subset SPO(X,\tau)$, the results in (1) are valid.

(3Ib) Let $U \in SO(X,\tau)$. Since $U \subset \tau\,\mathrm{cl}(\tau\,\mathrm{int}\,U)$ and $\sigma clU \subset \tau\,\mathrm{cl}\,U$, we have

$$\sigma\,\mathrm{cl}\,U \subset \tau\,\mathrm{cl}\,U = \tau\,\mathrm{cl}(\tau\,\mathrm{int}\,U) = \sigma\,\mathrm{cl}(\tau\,\mathrm{int}\,U) \subset \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,U) \subset \sigma\,\mathrm{cl}\,U.$$

Hence we have $\sigma\,\mathrm{cl}\,U = \tau\,\mathrm{cl}\,U = \tau\,\mathrm{cl}(\tau\,\mathrm{int}\,U) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,U)$.

(3Ic–f). These are clear from (3Ia).

(3Ig) This is clear from (1a) since $\tau \subset \sigma \subset SPO(X,\tau)$.

(3Ih) Let $U \in RO(X,\tau)$. We have $U = \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,U)$ and $U \in SO(X,\tau)$. Now if we use (3Ib), we obtain that $\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,U) = \sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,U) = \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,U) = U$.

(3II) Under the hypothesis, since $RO(X,\tau) = RO(X,\sigma)$[11], it is clear that $SR(X,\tau) = SR(X,\sigma)$ [7], and $\theta O(X,\tau) = \theta O(X,\sigma)$[12].

(4). The proof is clear from (3Ib). ■

Corollary 2.2. *If $\tau \cap \mathcal{I} = \{\emptyset\}$ for $\tau \in Top\,(X)$ and $\mathcal{I} \in Id(X)$, then*

(a) *The results* (3) *in Theorem 2.1. are valid by taking $\tau^*(\mathcal{I})$ instead of σ.*

(b) *For $\mathcal{J} \in Id(X)$ and $\omega \in Top\,(X)$, if $\omega \cap \mathcal{J} = \{\emptyset\}$ and $\omega^*(\mathcal{J}) = \tau^*(\mathcal{I})$, then $(X,\tau), (X,\omega), (X,\omega^*(\mathcal{J}))$ and $(X,\tau^*(\mathcal{I}))$ have the same $RO(X)$, $SR(X)$ and $\theta O(X)$ sets.*

(c) *If $\tau^*(\mathcal{I}) = \sigma^*(\mathcal{I})$ for $\sigma \in Top\,(X)$, then the results* (3) *in Theorem 2.1. are valid by taking $\sigma^*(\mathcal{I})$ instead of σ, and then $(X,\tau), (X,\tau^*(\mathcal{I}), (X,\sigma)$ and $(X,\sigma^*(\mathcal{I}))$ have the same $RO(X)$, $SR(X)$ and $\theta O(X)$ sets.*

The following theorem and corollaries can be obtained by using Lemma 2.7. below and the results of Andrijević given in [1,2,3]. We note that Corollary 2.6.(1) was given by Rose and Hamlett using a different method [16]. We will obtain these results by using the results given here.

Theorem 2.3. *Let $\tau, \sigma \in Top\,(X)$. If $\tau \subset \sigma \subset SO(X,\tau)$, and $\tau\,\mathrm{cl}\,U = \sigma\,\mathrm{cl}\,U$ for each $U \in \sigma$, then we have the following results.*

(1) *For each $A \subset X$, we have*

(a) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$,

(b) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A)$,

(c) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)) = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A))$,

(d) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A)) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}(\sigma int A))$.

(2) *(X,τ) and (X,σ) have the same $SO(X)$, $PO(X)$, $SPO(X)$, $RO(X)$, $SR(X)$, $NO(X)$, $D(X)$, $\alpha O(X)$, $CD(X)$ and $\theta O(X)$ sets.*

Proof. (1a) Let $A \subset X$. Then $\sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A) \subset \sigma\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)$. Since $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) \in \sigma$ and $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) \subset A \cup \tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau\,\mathrm{scl}\,A \subset \sigma\,\mathrm{cl}\,A$, we have $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A)) \subset \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$. Hence $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$.

The remaining proofs are clear. ■

Corollary 2.4. *Let $\tau \in Top\,(X), \mathcal{I} \in Id(X)$. If $\tau \cap \mathcal{I} = \{\emptyset\}$ and $\tau^*(\mathcal{I}) \subset SO(X,\tau)$, then the results of Theorem 2.3 are satisfied by taking $\tau^*(\mathcal{I})$ instead of σ.*

Corollary 2.5. *Since $\tau \cap \mathcal{I}_n = \{\emptyset\}$ and $\tau^*(\mathcal{I}_n) = \tau^\alpha \subset SO(X,\tau)$, the results of the above theorem are satisfied by taking τ^α instead of σ.*

Corollary 2.6. *If $\sigma^\alpha = \tau^\alpha$ (i.e. if $\sigma \in [\tau]^\alpha$ in the sense of Njåstad* [14]), *then we have the following results.*

(1) *For each $A \subset X$, we have*

(a) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}\,A) = \tau^\alpha\,\mathrm{int}(\tau^\alpha\,\mathrm{cl}\,A) = \sigma^\alpha\,\mathrm{int}(\sigma^\alpha\,\mathrm{cl}\,A) = \sigma\,\mathrm{int}(\sigma\,\mathrm{cl}\,A)$,

(b) $\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A) = \tau^\alpha\,\mathrm{cl}(\tau^\alpha\,\mathrm{int}\,A) = \sigma^\alpha\,\mathrm{cl}(\sigma^\alpha\,\mathrm{int}\,A = \sigma\,\mathrm{cl}(\sigma\,\mathrm{int}\,A)$,

(c) $\tau\,\mathrm{int}(\tau\,\mathrm{cl}(\tau\,\mathrm{int}\,A)) = \tau^\alpha\,\mathrm{int}(\tau^\alpha\,\mathrm{cl}(\tau^\alpha\,\mathrm{int}\,A)) = \sigma^\alpha\,\mathrm{int}(\sigma^\alpha\,\mathrm{cl}(\sigma^\alpha\,\mathrm{int}\,A))$.

(d) $\tau \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} A)) = \tau^{\alpha} \operatorname{cl}(\tau^{\alpha} \operatorname{int} A) = (\sigma^{\alpha} \operatorname{cl} A(\sigma^{\alpha} \operatorname{int}(\sigma^{\alpha} \operatorname{cl} A) = \sigma \operatorname{cl}(\sigma \operatorname{int}(\sigma \operatorname{cl} A))$.

(2) $(X,\tau), (X,\tau^{\alpha}), (X,\sigma^{\alpha})$ *and* (X,σ) *have the same* $SO(X)$, $PO(X)$, $SPO(X)$, $NO(X)$, $D(X)$, $CD(X)$, $\alpha O(X)$, $RO(X)$, $SR(X)$ *and* $\theta O(X)$ *sets* [3].

Lemma 2.7. *Let* $\tau, \sigma \in Top(X)$ *and* $\tau \subset \sigma$. *Then,* $\sigma \subset SO(X,\tau)$ *and* $\tau \operatorname{cl} U = \sigma \operatorname{cl} U$ *for each* $U \in \sigma$ *iff* $\sigma \in [\tau]^{\alpha}$.

Njåstad defined α-equivalent topologies and $*$-equivalent topologies in [14], [15], respectively. Njåstad showed that if $\tau \subset \sigma \subset \tau^{\alpha}$ for $\tau, \sigma \in Top(X)$, then τ and σ are α-equivalent. For $\tau, \sigma \in Top(X)$, $\mathcal{I} \in Id(X)$, if $\tau^*(\mathcal{I}) = \sigma^*(\mathcal{I})$, then we say that σ and τ are $^*\mathcal{I}$-equivalent.

The α-equivalence or $^*\mathcal{I}$-equivalence of topologies on a set on which ideals are defined is important.

For any ideal $\mathcal{I}$ on (X,τ), $\tau^*(\mathcal{I})$ and $\tau^*(\mathcal{I})^{\alpha}$ are α-equivalent. We know that $\tau^*(\mathcal{I})^{\alpha} = (\tau^*(\mathcal{I}))^*(\mathcal{I}_n(\tau^*(\mathcal{I}))) = \tau^*(\mathcal{I} \text{ V } \mathcal{I}_n(\tau^*(\mathcal{I})))$. Hence, for each ideal $\mathcal{J}$ such that $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$, $\tau^*(\mathcal{I})$, $\tau^*(\mathcal{I} \vee \mathcal{J})$ and $\tau^*(\mathcal{I} \vee \mathcal{I}_n(\tau^*(\mathcal{I})))$ are α-equivalent.

At the same time, for any ideal $\mathcal{I}$, since $\tau^*(\mathcal{I}) \cap \mathcal{I}_n(\tau^*(\mathcal{I}))) = \{\emptyset\}$ and $\tau \subset \tau^*(\mathcal{I})$, we have that that $\tau \cap \mathcal{I}_n(\tau^*(\mathcal{I})) = \{\emptyset\}$. Hence for any ideal $\mathcal{J}$ such that $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$ we have $\tau \cap \mathcal{J} = \{\emptyset\}$. And, if $\tau \cap \mathcal{I} = \{\emptyset\}$, then we know that $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$. Now, we can give the following result.

Theorem 2.8. *Let* $\mathcal{I}$ *be an ideal on* (X,τ) *and* $\mathcal{J}$ *any ideal such that* $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$. *Then the following are equivalent.*

(a) $\tau \cap \mathcal{I} = \{\emptyset\}$
(b) $\mathcal{I} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(c) $\mathcal{I} \vee \mathcal{I}_n \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(d) $\mathcal{I} \vee \mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(e) $\mathcal{I} \vee \mathcal{I}_n \vee \mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$
(f) $\tau \cap (\mathcal{I} \vee \mathcal{J}) = \{\emptyset\}$
(g) $\tau \cap (\mathcal{I} \vee \mathcal{J} \vee \mathcal{I}_n) = \{\emptyset\}$
(h) $\tau \cap (\mathcal{I} \vee \mathcal{I}_n) = \{\emptyset\}$.

Proof. (a) $\Longrightarrow$ (b) Let $I \in \mathcal{I}$. Then I is $\tau^*(\mathcal{I})$-closed and since $\tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I})\text{-int}\, I = \emptyset$. So, $\tau^*(\mathcal{I})\text{-int}(\tau^*(\mathcal{I})\text{-cl}\, I) = \emptyset$ and $I \in \mathcal{I}_n(\tau^*(\mathcal{I}))$.

The remaining proofs are clear. ∎

We deduce that if $\tau \cap \mathcal{I} = \{\emptyset\}$ for an ideal $\mathcal{I}$, then $\mathcal{I} \vee \mathcal{I}_n(\tau^*(\mathcal{I})) = \mathcal{I}_n(\tau^*(\mathcal{I}))$, and $(\tau^*(\mathcal{I}))^{\alpha} = \tau^*(\mathcal{I}_n(\tau^*(\mathcal{I})))$.

Corollary 2.9. *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then for any ideal* $\mathcal{J}$ *satisfying* $\mathcal{J} \subset \mathcal{I}_n(\tau^*(\mathcal{I}))$ *we have that* $\tau^*(\mathcal{I})$, $\tau^*(\mathcal{I} \vee \mathcal{J})$, $\tau^*(\mathcal{I} \vee \mathcal{J} \vee \mathcal{I}_n(\tau^*(\mathcal{I})))$ *and* $\tau^*(\mathcal{I}_n(\tau^*(\mathcal{I})))$ *are all* α*-equivalent.*

Several statements equivalent to $\mathcal{I} \subset \mathcal{I}_n$ have been given in the literature. Since τ and $\tau^*(\mathcal{I})$ are α-equivalent when $\mathcal{I} \subset \mathcal{I}_n$, we give some further conditions for $\mathcal{I} \subset \mathcal{I}_n$, in the following theorem.

Theorem 2.10. *Let $\mathcal{I}$ be an ideal on (X,τ) and $\mathcal{I}_n$ the ideal of nowhere dense sets in (X,τ). Then the following are equivalent.*

(1) $\mathcal{I}\subset\mathcal{I}_n$,
(2) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* τ *and* $\tau^*(\mathcal{I})$ *are* α-*equivalent,*
(3) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $\tau^*(\mathcal{I})\subset\tau^\alpha$,
(4) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $\tau^*(\mathcal{I})\subset SO(X,\tau)$,
(5) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $SO(X,\tau^*(\mathcal{I}))\subset SO(X,\tau)$,
(6) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $PO(X,\tau)\subset PO(X,\tau^*(\mathcal{I}))$,
(7) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $SPO(X,\tau)\subset SPO(X,\tau^*(\mathcal{I}))$,
(8) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $D(X,\tau)\subset D(X,\tau^*(\mathcal{I}))$,
(9) $A^*(\mathcal{I}_n)\subset A^*(\mathcal{I})$ *for each* $A\subset X$,
(10) $A^{\underline{o}}\subset A^*(\mathcal{I})$ *for each* $A\in D(X,\tau)$,
(11) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and*$A^{\underline{o}}\subset\tau^*(\mathcal{I})\text{-cl}\,A$ *for each* $A\subset X$,
(12) $\tau\cap\mathcal{I}=\{\emptyset\}$ *and* $\mathcal{I}\subset SC(X,\tau)$.

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{I}\subset\mathcal{I}_n$. We have, $\tau\subset\tau^*(\mathcal{I})\subset\tau^*(\mathcal{I}_n)=\tau^\alpha$, so τ and $\tau^*(\mathcal{I})$ are α-equivalent from Theorem 1.1.(3).

(2) $\Longrightarrow$ (1) If $\tau^*(\mathcal{I})\in[\tau]^\alpha$, then we have $\mathcal{I}_n(\tau)=\mathcal{I}_n(\tau^*(\mathcal{I}))$. If $\tau\cap\mathcal{I}=\{\emptyset\}$, then $\mathcal{I}\subset\mathcal{I}_n(\tau^*(\mathcal{I}))$. Hence, $\mathcal{I}\subset\mathcal{I}_n$, if $\tau\cap\mathcal{I}=\{\emptyset\}$ and $\tau^*(\mathcal{I})\in[\tau]^\alpha$.

(2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) Clear.

(2) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8) Clear from [3, Theorem 1] and Corollary 2.2

(1) $\Longleftrightarrow$ (9) Known from the literature.

(9) $\Longrightarrow$ (10) Clear.

(10) $\Longrightarrow$ (8) Since $X\in D(X,\tau)$, we have $X^{\underline{o}}=X\subset X^*(\mathcal{I})$, $X=X^*(\mathcal{I})$ and hence $\tau\cap\mathcal{I}=\{\emptyset\}$. For $A\in D(X,\tau)$ we deduce $\bar{A}=X$, $A^{\underline{o}}=X$, $A^*(\mathcal{I})=X$ and $\tau^*(\mathcal{I})\text{-cl}\,A=A\cup A^*(\mathcal{I})=X$. Hence we have $D(X,\tau)\subset D(X,\tau^*(\mathcal{I}))$.

(4) $\Longrightarrow$ (11) We have $\mathrm{scl}\,A\subset\tau^*(\mathcal{I})\text{-cl}\,A$ for each $A\subset X$. Since $\mathrm{scl}\,A=A\cup A^{\underline{o}}$, the result is clear.

(11) $\Longrightarrow$ (3) Under the hypothesis of (11) we obtain $A^{\overline{\underline{o}}}=\tau^*(\mathcal{I})\text{-cl}\,A^{\underline{o}}\subset\tau^*(\mathcal{I})\text{-cl}\,A$ and $\tau^\alpha\text{-cl}\,A=A\cup A^{\overline{\underline{o}}}\subset\tau^*(\mathcal{I})\text{-cl}\,A$ for each subset A. Hence we have $\tau^*(\mathcal{I})\subset\tau^\alpha$.

(4) $\Longrightarrow$ (12) We know that each $I\in\mathcal{I}$ is $\tau^*(\mathcal{I})$-closed. Since $\tau^*(\mathcal{I})\subset SO(X,\tau)$, it follows that each $I\in\mathcal{I}$ is τ-semiclosed.

(12) $\Longrightarrow$ (4) If $\mathcal{I}\subset SC(X,\tau)$, then $U-I\in SO(X,\tau)$ for any $U\in\tau$ and any $I\in\mathcal{I}$. So, $SO(X,\tau)$ contains a base of $\tau^*(\mathcal{I})$ (from Theorem 1.1.(10)). Hence $\tau^*(\mathcal{I})\subset SO(X,\tau)$. ■

Corollary 2.11. *If $\mathcal{I}\subset\mathcal{I}_n$, then we have the following results.*

(a) $\mathcal{I}_n(\tau^*(\mathcal{I}))=\mathcal{I}_n(\tau)$,
(b) $\sigma\in[\tau]^\alpha$ *iff* $\sigma\in[\tau^*(\mathcal{I})]^\alpha$,
(c) *If* $\sigma^*(\mathcal{I})=\tau^*(\mathcal{I})$, *then* τ, $\tau^*(\mathcal{I})$ *and* $\sigma^*(\mathcal{I})$ *are all* α-*equivalent.*

Some other statements equivalent to $\mathcal{I} \subset \mathcal{I}_n$ can be seen from Corollary 2.13. and Corollary2.15.

If $\mathcal{A}$ is a supratopology and $\mathcal{I}$ an ideal on X, then it is clear that $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ iff $\mathcal{A}$-int $I = \emptyset$ for each $I \in \mathcal{I}$. In the following theorem, the results are clear and almost all of them are known.

Theorem 2.12. *Let (X, τ) be a topological space. Then we have the following results for any $A \subset X$.*

(1) $A^{\overset{o}{-}} = \emptyset \iff \text{pre-}\operatorname{int} A = \emptyset \iff A^{\overset{\overline{o}}{-}} = \emptyset \iff \text{semi-pre-}\operatorname{int} A = \emptyset$,

(2) $A^{\overset{o}{o}} = \emptyset \iff \alpha\text{-}\operatorname{int} A = \emptyset \iff A^{o} = \emptyset \iff A^{\overline{o}} = \emptyset \iff \text{semi-}\operatorname{int} A = \emptyset$,

(3) $A^{\overline{o}} = X \iff \text{pre-}\operatorname{cl} A = X \iff A^{\overset{o}{o}} = X \iff \text{semi-pre-}\operatorname{cl} A = X$,

(4) $A^{\overset{\overline{o}}{-}} = X \iff \alpha\text{-}\operatorname{cl} A = X \iff \bar{A} = X \iff A^{\overset{o}{-}} = X \iff \operatorname{scl} A = X$.

Clearly, in a topological space (X, τ), for any $x \in X$, $\{x\} \notin \mathcal{I}_n$ iff pre-int $\{x\} \neq \emptyset$ iff semi-pre-int $\{x\} \neq \emptyset$ iff $\{x\}$ is pre-open iff $\{x\}$ is semi-pre-open.

In the following corollary we assume that the necessary ideals are defined on the topological space (X, τ).

Corollary 2.13. *We have the following results.*

(1) $\mathcal{I}_n = \{A : A^{\overset{o}{-}} = \emptyset\} = \{A : \text{pre-}\operatorname{int} A = \emptyset\} = \{A : \text{semi-pre-}\operatorname{int} A = \emptyset\} = \{A : A^{\overset{\overline{o}}{-}} = \emptyset\} = CD(X, \tau) \cap SC(X, \tau)$.

(2) $\mathcal{I}_n \cap PO(X, \tau) = \{\emptyset\}$ *and* $\mathcal{I}_n \cap SPO(X, \tau) = \{\emptyset\}$.

(3) $\mathcal{I} \subset \mathcal{I}_n$ *iff* $\mathcal{I} \cap PO(X, \tau) = \{\emptyset\}$ [4] *iff* $\mathcal{I} \cap SPO(X, \tau) = \{\emptyset\}$.

(4) *For $\sigma \in Top\,(X)$, if $PO(X, \sigma) \subset PO(X, \tau)$ or $SPO(X, \sigma) \subset SPO(X, \tau)$, then $\mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma)$.*

(5) $CD(X) = \{A : A^{o} = \emptyset\} = \{A : A^{\overset{o}{o}} = \emptyset\} = \{A : \alpha\text{-}\operatorname{int} A = \emptyset\} = \{A : \text{semi-}\operatorname{int} A = \emptyset\} = \{A : A^{\overline{o}} = \emptyset\}$. (6) *For $\sigma \in Top\,(X)$, if $SO(X, \tau) \subset SO(X, \sigma)$ or $\tau^{\alpha} \subset \sigma^{\alpha}$, then $D(X, \sigma) \subset D(X, \tau)$ and $CD(X, \sigma) \subset CD(X, \tau)$.*

(7)(a) $\tau \cap \mathcal{I} = \{\emptyset\}$ *iff* $SO(X, \tau) \cap \mathcal{I} = \{\emptyset\}$ [4] *iff* $\tau^{\alpha} \cap \mathcal{I} = \{\emptyset\}$ *iff* $\tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\}$ *iff* $SO(X, \tau^*(\mathcal{I})) \cap \mathcal{I} = \{\emptyset\}$ *iff* $\tau^*(\mathcal{I})^{\alpha} \cap \mathcal{I} = \{\emptyset\}$.

(b) *Let $\sigma \in Top\,(X)$ and $\sigma^*(\mathcal{I}) = \tau^*(\mathcal{I})$, then we have that $\tau \cap \mathcal{I} = \{\emptyset\}$ iff $\sigma \cap \mathcal{I} = \{\emptyset\}$.*

(c) *Let $\sigma \in Top\,(X)$ and $\sigma \in [\tau^*(\mathcal{I})]^{\alpha}$, then we have that $\tau \cap \mathcal{I} = \{\emptyset\}$ iff $\sigma \cap \mathcal{I} = \{\emptyset\}$.*

Now, by combining the results given above with the following facts,

(i) If $\tau \subset \sigma \subset \tau^*(\mathcal{I})$ for $\sigma \in Top\,(X)$, then $\sigma^*(\mathcal{I}) = \tau^*(\mathcal{I})$,

(ii) If $\tau^*(\mathcal{I}) \subset \sigma \subset (\tau^*(\mathcal{I}))^{\alpha} = (\tau^*(\mathcal{I}))^*(\mathcal{I}_n(\tau^*(\mathcal{I})))$ then $\sigma \in [\tau^*(\mathcal{I}))]^{\alpha}$,

we can obtain several conditions equivalent to $\tau \cap \mathcal{I} = \{\emptyset\}$.

For a supratopology $\mathcal{A}$ on X, $\mathcal{T}_{\mathcal{A}}$ will stand for the topology

$$\{U : A \in \mathcal{A} \implies U \cap A \in \mathcal{A}\} \text{ [19]}.$$

We know that $\tau \subset \mathcal{T}_{PO(X,\tau)}$, $\tau \subset \mathcal{T}_{SO(X,\tau)}$, $\tau \subset \mathcal{T}_{SPO(X,\tau)}$ and $\tau \subset \tau^{\alpha} = \mathcal{T}_{\tau^{\alpha}}$ [1–3].

THEOREM 2.14. *Let (X,τ) be a topological space, $\mathcal{I} \in Id(X)$ and $\mathcal{A}$ a supratopology on X. Then we have the following results.*

(1) *If $\tau \cap \mathcal{I} = \{\emptyset\}$ and $A^o \neq \emptyset$ for each $A \in \mathcal{A} - \{\emptyset\}$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$.*

(2) *If $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ and $\tau \subset \mathcal{T}_{\mathcal{A}}$, then*

(a) *$A \subset A^*$ for each $A \in \mathcal{A}$,*

(b) *$\bar{A} = A^* = \tau^*\text{-cl}\, A$ for each $A \in \mathcal{A}$.*

(3)(a) *If $A^*(\mathcal{I}) \neq \emptyset$ for each $A \in \mathcal{A} - \{\emptyset\}$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$,*

(b) *If $\tau \subset \mathcal{T}_{\mathcal{A}}$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.*

(4) *$\mathcal{A} \cap \mathcal{I}_n = \{\emptyset\}$ iff $A^{\underline{o}} \neq \emptyset$ for each $A \in \mathcal{A} - \{\emptyset\}$.*

(5) *If $\mathcal{A} \cap \mathcal{I}_n = \{\emptyset\}$ and $\tau \subset \mathcal{T}_{\mathcal{A}}$, then $\mathcal{A} \subset SPO(X,\tau)$.*

(6) *If $PO(X,\tau) \subset \mathcal{A} \subset SPO(X,\tau)$, then $\mathcal{A} \cap \mathcal{I}_n = \{\emptyset\}$ iff $\mathcal{I} \subset \mathcal{I}_n$ iff $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.*

(7) *If $\tau \subset \mathcal{A} \subset SO(X,\tau^*(\mathcal{I}))$, then $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ iff $\tau \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.*

Proof. (2a) Let $A \in \mathcal{A}$ and $x \in A$. If $x \in U \in \tau$, then since $\tau \subset \mathcal{T}_{\mathcal{A}}$, we have $\emptyset \neq U \cap A \in \mathcal{A}$. Hence, $U \cap A \notin \mathcal{I}$. So, we have $x \in A^*$.

(2b) We know that A^* is τ-closed, $A^* \subset \bar{A}$ and $\tau^* clA = A \cup A^*$. Result is clear from (a).

(3a) It is known that $I^* = \emptyset$ for any ideal $\mathcal{I}$ and for each $I \in \mathcal{I}$. Now, result is clear.

(3b) Clear from (3a) and (2a)

(4) Clear from Corollary 2.13.

(5) Let $A \in \mathcal{A}$. From (3)(b) we have $A \subset A^*(\mathcal{I}_n) = A^{\bar{\underline{o}}}$. Hence, $A \in SPO(X,\tau)$.

(6) Let $PO(X,\tau) \subset \mathcal{A} \subset SPO(X,\tau)$. If $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$ then $PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ and hence $\mathcal{I} \subset \mathcal{I}_n$. If $\mathcal{I} \subset \mathcal{I}_n$, then $SPO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ and hence $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$.

If $\mathcal{I} \subset \mathcal{I}_n$, then $SPO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$. Now, from Corollary 2.15.(1) below, and since $\mathcal{A} \subset SPO(X,\tau)$, we have $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.

If $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$, then we have $A \subset A^*(\mathcal{I})$ for each $A \in PO(X,\tau)$. So, from Corollary 2.15(1), we have $\mathcal{I} \subset \mathcal{I}_n$.

(7) Let $\tau \subset \mathcal{A} \subset SO(X,\tau^*(\mathcal{I}))$. If $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$, then $\tau \cap \mathcal{I} = \{\emptyset\}$. If $\tau \cap \mathcal{I} = \{\emptyset\}$, then from Corollary 2.13(7) we have $SO(X,\tau^*(\mathcal{I})) \cap \mathcal{I} = \{\emptyset\}$. So, $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$.

If $\tau \cap \mathcal{I} = \{\emptyset\}$, then from Corollary 2.15(2) below we have $A \subset A^*(\mathcal{I})$ for each $A \in SO(X,\tau^*(\mathcal{I}))$. So, $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$.

If $A \subset A^*(\mathcal{I})$ for each $A \in \mathcal{A}$, then from Theorem 1.1.(9) we have $U \subset U^*$ for each $U \in \tau$, and $\tau \cap \mathcal{I} = \{\emptyset\}$. ■

COROLLARY 2.15. *We have the following results.*

(1) *$PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in PO(X,\tau)$ [4] iff $A \subset A^*(\mathcal{I})$ for each $A \in SPO(X,\tau)$.*

(2) *$\tau \cap \mathcal{I} = \{\emptyset\}$ iff $A \subset A^*(\mathcal{I})$ for each $A \in SO(X,\tau)$. [4] iff $A \subset A^*(\mathcal{I})$ for*

each $A \in \tau^\alpha$ *iff* $A \subset A^*(\mathcal{I})$ *for each* $A \in SO(X, \tau^*(\mathcal{I}))$ *iff* $A \subset A^*(\mathcal{I})$ *for each* $A \in \tau^*(\mathcal{I})^\alpha$.

Proof. The proofs are clear from Theorem 1.1.(2), Theorem 2.14(3b) and Theorem 1.1(5). ■

COROLLARY 2.16. *We have the following results.*

(1) *If* $PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$, *then* $\tau\text{-cl}\, A = A^* = \tau^*\text{-cl}\, A$ *for each* $A \in SPO(X,\tau)$.

(2) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then* $\tau\text{-cl}\, A = A^* = \tau^*\text{-cl}\, A$ *for each* $A \in SO(X, \tau^*(\mathcal{I}))$.

LEMMA 2.17. *Let* (X,τ) *be topological space,* $\mathcal{A}$ *a supratopology on* X *such that* $\tau \subset \mathcal{T}_{\mathcal{A}}$. *If* $\mathcal{A}\text{-}\operatorname{int} B = \emptyset$ *for a subset* B, *then* $(A \cap B)^- \subset (A - B)^-$ *and* $\bar{A} = (A - B)^-$ *for each* $A \in \mathcal{A}$.

Proof. Let $B \subset X$, $\mathcal{A}\text{-}\operatorname{int} B = \emptyset$, $A \in \mathcal{A}$, $x \in (A \cap B)^-$ and $x \in U \in \tau$. Then $U \cap A \cap B \neq \emptyset$ and $U \cap A \neq \emptyset$. Since $\tau \subset \mathcal{T}_{\mathcal{A}}$, we have $\emptyset \neq U \cap A \in \mathcal{A}$. So $U \cap A \not\subset B$, and $(A - B) \cap U = A \cap U - B \neq \emptyset$. Hence $x \in (A - B)^-$.

This now gives $\bar{A} = (A - B)^- \cup (A \cap B)^- = (A - B)^-$. ■

COROLLARY 2.18. *Let* $\mathcal{I} \in Id(X)$ *and* $\mathcal{A}$ *a supratopology on* X. *If* $\mathcal{A} \cap \mathcal{I} = \{\emptyset\}$, *then we have the following results.*

(a) *If* $\tau \subset \mathcal{T}_{\mathcal{A}}$, *then* $\bar{A} = (A - I)^-$ *for each* $A \in \mathcal{A}$ *and for each* $I \in \mathcal{I}$.

(b) *If* $\tau^*(\mathcal{I}) \subset \mathcal{T}_{\mathcal{A}}$, *then* $\bar{A} = (A - I)^-$ *and* $\tau^*\text{-}clA = \tau^*\text{-cl}(A - I)$ *for each* $A \in A$ *and for each* $I \in \mathcal{I}$.

Proof. (a) If $I \in \mathcal{I}$ we obtain $\mathcal{A}\text{-}\operatorname{int} I = \emptyset$. The proof is now clear from Lemma 2.17.

(b) If $\tau^* \subset \mathcal{T}_{\mathcal{A}}$ then we obtain $\tau \subset \tau^* \subset \mathcal{T}_{\mathcal{A}}$. The proof is now clear from Lemma 2.17. ■

COROLLARY 2.19. *Let* (X,τ) *be a topological space and* $\mathcal{I} \in Id(X)$. *Then we have the following results.*

(1) *If* $PO(X,\tau) \cap \mathcal{I} = \{\emptyset\}$, *then* $\bar{A} = (A - I)^- = A^* = \tau^*\text{-}clA = (A - I)^*$ *for each* $I \in \mathcal{I}$ *and for each* $A \in SPO(X,\tau)$.

(2) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then* $\bar{A} = (A - I)^- = A^* = \tau^*\text{-cl}\, A = \tau^*\text{-cl}(A - I) = (A - I)^*$ *for each* $I \in \mathcal{I}$ *and for each* $A \in SO(X, \tau^*(\mathcal{I}))$.

(3) *If* $\tau \cap \mathcal{I} = \{\emptyset\}$, *then we have* $\tau\text{-scl}\, A = \tau^*\text{-scl}\, A$ *for each* $I \in \mathcal{I}$ *and for each* $A \in SO(X, \tau^*(\mathcal{I}))$.

Proof. (1),(2) Clear from Corollory 2.16, Corollary 2.18 and Theorem 1.1.(15).

(3) From (2) we will have $\overset{\circ}{A} = \tau\text{-int}(\tau^*\text{-cl}\, A) = \tau^*\text{-int}(\tau^*\text{-cl}\, A)$ and $\tau\text{-scl}\, A = A \cup \overset{\circ}{A} = A \cup \tau^*\text{-int}(\tau^*\text{-cl}\, A) = \tau^*\text{-scl}\, A$. ■

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