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ON A CLASS OF MULTIVALENT FUNCTIONS DEFINED BY A MULTIPLIER TRANSFORMATION

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Abstract. In the present paper, the authors investigate starlikeness and convexity of a class of multivalent functions defined by multiplier transfomation. As a consequence, a number of sufficient conditions for starlikeness and convexity of analytic functions are also obtained.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $p \in \mathbf{N} = \{1, 2, \dots\}$, which are analytic in the open unit disc $E = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order α $(0 \le \alpha < p)$ in E if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in E.$$

We denote by $S_p^*(\alpha)$, the class of all such functions. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order α $(0 \le \alpha < p)$ in E if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in E.$$

Let $K_p(\alpha)$ denote the class of all those functions $f \in \mathcal{A}_p$ which are multivalently convex of order α in E. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order α, $0 \le \alpha < 1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \quad (\lambda \ge 0, \ \ n \in \mathbf{Z}). \tag{1}$$

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The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et. al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Srivastava [2] and Cho and Kim [3], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [9]. $I_1(n, 0)$ is the well-known Sălăgean [7] derivative operator D^n, defined as: $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_n z^n$, $n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $f \in \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $S_n(p, \lambda, \alpha)$ for all z in E if it satisfies

$$\operatorname{Re}\left[\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)}\right] > \frac{\alpha}{p}, \tag{2}$$

for some α $(0 \leq \alpha < p, p \in \mathbf{N})$. We note that $S_0(1, 0, \alpha)$ and $S_1(1, 0, \alpha)$ are the usual classes $S^*(\alpha)$ and $K(\alpha)$ of starlike functions of order α and convex functions of order α, respectively.

In 1989, Owa, Shen and Obradović [6] obtained a sufficient condition for a function $f \in \mathcal{A}$ to belong to the class $S_n(1, 0, \alpha) = S_n(\alpha)$, say. In fact, they proved the following result:

Theorem A. *For $n \in \mathbf{N}_0$, if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right|^{1-\beta} \left|\frac{D^{n+2} f(z)}{D^{n+1} f(z)} - 1\right|^{\beta} < (1-\alpha)^{1-2\beta}(1 - \frac{3}{2}\alpha + \alpha^2)^{\beta}, z \in E,$$

for some real numbers $\alpha (0 \leq \alpha \leq \frac{1}{2})$ and $\beta (0 \leq \beta \leq 1)$, then $f \in S_n(\alpha)$, i.e. $\operatorname{Re}\left[\frac{D^{n+1} f(z)}{D^n f(z)}\right] > \alpha$ *in E.*

This result was, later on, extended by Li and Owa [5] for all $\alpha, 0 \leq \alpha < 1$ and $\beta \geq 0$. They proved

Theorem B. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right|^{\gamma} \left|\frac{D^{n+2} f(z)}{D^{n+1} f(z)} - 1\right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma}\left(\frac{3}{2} - \alpha\right)^{\beta}, & 0 \leq \alpha \leq 1/2, \\ 2^{\beta}(1-\alpha)^{\beta+\gamma}, & 1/2 \leq \alpha < 1. \end{cases}$$

for some reals α $(0 \leq \alpha < 1)$, $\beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$, then $f \in S_n(\alpha)$, $n \in \mathbf{N}_0$.

In the present paper, our aim is to determine sufficient conditions for a function $f \in \mathcal{A}_p$ to be a member of the class $S_n(p, \lambda, \alpha)$. As a consequence of our main result, we get a number of sufficient conditions for starlikeness and convexity of analytic functions.

2. Main result

To prove our result, we shall make use of the famous Jack's lemma which we state below.

Lemma 2.1. (Jack [4]) *Suppose $w(z)$ be a nonconstant analytic function in E with $w(0)=0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in E$ on the circle $|z|=r<1$, then $z_0 w'(z_0)=mw(z_0)$, where m, $m \geq 1$, is some real number.*

We, now, state and prove our main result.

Theorem 2.1. *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}-1\right|^{\gamma}\left|\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}-1\right|^{\beta}<M(p,\lambda,\alpha,\beta,\gamma),\ z \in E, \tag{3}$$

for some real numbers α, β and γ such that $0 \leq \alpha < p$, $\beta \geq 0$, $\gamma \geq 0$, $\beta+\gamma>0$, then $f \in S_n(p,\lambda,\alpha)$, where $n \in \mathbf{N}_0$ and

$$M(p,\lambda,\alpha,\beta,\gamma)=\begin{cases}\left(1-\frac{\alpha}{p}\right)^{\gamma}\left(1-\frac{\alpha}{p}+\frac{1}{2(p+\lambda)}\right)^{\beta}, & 0 \leq \alpha \leq p/2,\\ \left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}\left(1+\frac{1}{p+\lambda}\right)^{\beta}, & p/2 \leq \alpha < p.\end{cases}$$

Proof. *Case* (i). Let $0 \leq \alpha \leq \frac{p}{2}$. Writing $\frac{\alpha}{p}=\mu$, we see that $0 \leq \mu \leq \frac{1}{2}$. Define a function $w(z)$ as

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}=\frac{1+(1-2\mu)w(z)}{1-w(z)}, \quad z \in E. \tag{4}$$

Then w is analytic in E, $w(0)=0$ and $w(z) \neq 1$ in E. By a simple computation, we obtain from (4),

$$\frac{z(I_p(n+1,\lambda)f(z))'}{I_p(n+1,\lambda)f(z)}-\frac{z(I_p(n,\lambda)f(z))'}{I_p(n,\lambda)f(z)}=\frac{2(1-\mu)zw'(z)}{(1-w(z))(1+(1-2\mu)w(z))} \tag{5}$$

By making use of the identity

$$(p+\lambda)I_p(n+1,\lambda)f(z)=z(I_p(n,\lambda)f(z))'+\lambda I_p(n,\lambda)f(z) \tag{6}$$

we obtain from (5)

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}=\frac{1+(1-2\mu)w(z)}{1-w(z)}+\frac{2(1-\mu)zw'(z)}{(p+\lambda)(1-w(z))(1+(1-2\mu)w(z))}$$

Thus, we have

$$\begin{aligned}&\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}-1\right|^{\gamma}\left|\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}-1\right|^{\beta}\\ &=\left|\frac{2(1-\mu)w(z)}{1-w(z)}\right|^{\gamma}\left|\frac{2(1-\mu)w(z)}{1-w(z)}+\frac{2(1-\mu)zw'(z)}{(p+\lambda)(1-w(z))(1+(1-2\mu)w(z))}\right|^{\beta}\\ &=\left|\frac{2(1-\mu)w(z)}{1-w(z)}\right|^{\gamma+\beta}\left|1+\frac{zw'(z)}{(p+\lambda)w(z)(1+(1-2\mu)w(z))}\right|^{\beta}.\end{aligned}$$

Suppose that there exists a point $z_0 \in E$ such that $\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $w(z_0) = e^{i\theta}$, $0 < \theta \leq 2\pi$ and $z_0 w'(z_0) = mw(z_0)$, $m \geq 1$. Therefore

$$\left|\frac{I_p(n+1,\lambda)f(z_0)}{I_p(n,\lambda)f(z_0)} - 1\right|^{\gamma} \left|\frac{I_p(n+2,\lambda)f(z_0)}{I_p(n+1,\lambda)f(z_0)} - 1\right|^{\beta}$$

$$= \left|\frac{2(1-\mu)w(z_0)}{1-w(z_0)}\right|^{\gamma+\beta} \left|1 + \frac{m}{(p+\lambda)(1+(1-2\mu)w(z_0))}\right|^{\beta}$$

$$= \frac{2^{\gamma+\beta}(1-\mu)^{\gamma+\beta}}{|1-e^{i\theta}|^{\beta+\gamma}} \left|1 + \frac{m}{(p+\lambda)(1+(1-2\mu)e^{i\theta}}\right|^{\beta}$$

$$\geq (1-\mu)^{\beta+\gamma}\left(1 + \frac{m}{2(p+\lambda)(1-\mu)}\right)^{\beta} \geq (1-\mu)^{\beta+\gamma}\left(1 + \frac{1}{2(p+\lambda)(1-\mu)}\right)^{\beta}$$

$$= (1-\mu)^{\gamma}\left(1 - \mu + \frac{1}{2(p+\lambda)}\right)^{\beta}$$

which contradicts (3) for $0 \leq \alpha \leq \frac{p}{2}$. Therefore, we must have $|w(z)| < 1$ for all $z \in E$, and hence $f \in S_n(p,\lambda,\alpha)$.

Case (ii). When $\frac{p}{2} \leq \alpha < p$. In this case, we must have $\frac{1}{2} \leq \mu < 1$, where $\mu = \frac{\alpha}{p}$. Let w be defined by

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = \frac{\mu}{\mu-(1-\mu)w(z)}, \quad z \in E,$$

where $w(z) \neq \frac{\mu}{1-\mu}$ in E. Then w is analytic in E and $w(0) = 0$. Proceeding as in Case (i) and using identity (6), we obtain

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right|^{\gamma} \left|\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - 1\right|^{\beta}$$

$$= \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)}\right|^{\gamma} \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)} + \frac{(1-\mu)zw'(z)}{(p+\lambda)(\mu-(1-\mu)w(z))}\right|^{\beta}$$

$$= \left|\frac{1-\mu}{\mu-(1-\mu)w(z)}\right|^{\gamma+\beta} |w(z)|^{\gamma} \left|w(z) + \frac{zw'(z)}{p+\lambda}\right|^{\beta}.$$

Suppose that there exists a point $z_0 \in E$ such that $\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$, then by Lemma 2.1, we obtain $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = mw(z_0), m \geq 1$. Therefore

$$\left|\frac{I_p(n+1,\lambda)f(z_0)}{I_p(n,\lambda)f(z_0)} - 1\right|^{\gamma} \left|\frac{I_p(n+2,\lambda)f(z_0)}{I_p(n+1,\lambda)f(z_0)} - 1\right|^{\beta}$$

$$= \frac{(1-\mu)^{\gamma+\beta}(1+\frac{m}{p+\lambda})^{\gamma+\beta}}{|\mu-(1-\mu)e^{i\theta}|^{\gamma+\beta}} \geq \left(1 - \frac{\alpha}{p}\right)^{\gamma+\beta}\left(1 + \frac{1}{p+\lambda}\right)^{\beta}$$

which contradicts (3) for $\frac{p}{2} \leq \alpha \leq p$. Therefore we must have $|w(z)| < 1$ for all $z \in E$, and hence $f \in S_n(p,\lambda,\alpha)$. This completes the proof of our theorem. ■

3. Deductions

For $p = 1$, Theorem 2.1 reduces to the following result:

Corollary 3.1. *If, for all $z \in E$, a function $f \in \mathcal{A}$ satisfies*

$$\left|\frac{I_1(n+1,\lambda)f(z)}{I_1(n,\lambda)f(z)} - 1\right|^{\gamma} \left|\frac{I_1(n+2,\lambda)f(z)}{I_1(n+1,\lambda)f(z)} - 1\right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma}\left(1-\alpha+\frac{1}{2(1+\lambda)}\right)^{\beta}, & 0 \le \alpha \le 1/2, \\ (1-\alpha)^{\gamma+\beta}\left(1+\frac{1}{1+\lambda}\right)^{\beta}, & 1/2 \le \alpha < 1, \end{cases}$$

for some reals α $(0 \le \alpha < 1)$, $\beta \ge 0$ and $\gamma \ge 0$ with $\beta + \gamma > 0$, then $f \in S_n(1,\lambda,\alpha)$, where $n \in \mathbf{N}_0$.

Remark 3.1. Setting $\lambda = 0$ in Corollary 3.1, we obtain Theorem B.

Recently, Sivaprasad Kumar et. al. [8] proved the following result:

Theorem C. *Let ψ be univalent in E, $\psi(0) = 1$, $\operatorname{Re}\psi(z) > 0$ and $\frac{z\psi'(z)}{\psi(z)}$ be starlike in E. Suppose $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} \prec \psi(z) + \frac{z\psi'(z)}{(p+\lambda)\psi(z)}.$$

Then, $\dfrac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec \psi(z)$.

Set $\psi(z) = \frac{1+(1-\frac{2\alpha}{p})z}{1-z}$, $0 \le \alpha < p$, $p \in \mathbf{N}$, in Theorem C. Clearly ψ satisfies all the conditions of above theorem. Thus, we obtain the following result:

If $f \in \mathcal{A}_p$ satisfies

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} \prec \frac{1+(1-\frac{2\alpha}{p})z}{1-z} + \frac{(1-\frac{2\alpha}{p})z}{(p+\lambda)(1+(1-\frac{2\alpha}{p})z)},$$

then

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec \frac{1+(1-\frac{2\alpha}{p})z}{1-z},$$

i.e.

$$\operatorname{Re}\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \frac{\alpha}{p}.$$

Compare this result with the result below, which we get by writing $\gamma = 0$ and $\beta = 1$ in Theorem 2.1:

Corollary 3.2. *If, for all $z \in E$, a function $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - 1\right| < \begin{cases} 1-\frac{\alpha}{p}+\frac{1}{2(p+\lambda)}, & 0 \le \alpha \le p/2, \\ (1-\frac{\alpha}{p})(1+\frac{1}{p+\lambda}), & p/2 \le \alpha < p, \end{cases}$$

then $\operatorname{Re}\left(\dfrac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \dfrac{\alpha}{p}$, *$z \in E$.*

Setting $p = 1$, $\lambda = 1$ and $n = 0$ in Theorem 2.1, we obtain the following result:

COROLLARY 3.3. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{\gamma} \left|\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} - 1\right|^{\beta} < \left(\frac{3}{2}\right)^{\beta} (1-\alpha)^{\gamma+\beta}, \quad 0 \le \alpha < 1,\, z \in E,$$

for some $\beta \ge 0$ and $\gamma \ge 0$ with $\beta + \gamma > 0$, then $f \in S^*(\alpha)$.

Setting $\alpha = 0$ in Corollary 3.3, we obtain the following criterion for starlikeness:

COROLLARY 3.4. *For some non-negative real numbers β and γ with $\beta+\gamma > 0$, if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{\gamma} \left|\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} - 1\right|^{\beta} < \left(\frac{3}{2}\right)^{\beta}, \quad z \in E,$$

then $f \in S^*$.

In particular, for $\beta = 1$ and $\gamma = 1$, we obtain the following interesting criterion for starlikeness:

COROLLARY 3.5. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \left|\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} - 1\right| < \frac{3}{2}, \quad z \in E,$$

then $f \in S^*$.

Setting $\lambda = 0$ and $n = 0$ in Theorem 2.1, we obtain the following sufficient condition for a function $f \in \mathcal{A}_p$ to be a p-valent starlike function of order α.

COROLLARY 3.6. *For all $z \in E$, if $f \in \mathcal{A}_p$ satisfies the condition*

$$\left|\frac{zf'(z)}{pf(z)} - 1\right|^{\gamma} \left|\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right|^{\beta} < \begin{cases} \left(1 - \frac{\alpha}{p}\right)^{\gamma} \left(1 - \frac{\alpha}{p} + \frac{1}{2p}\right)^{\beta}, & 0 \le \alpha \le p/2, \\ \left(1 - \frac{\alpha}{p}\right)^{\gamma+\beta} \left(1 + \frac{1}{p}\right)^{\beta}, & p/2 \le \alpha < p, \end{cases}$$

for some real numbers α, β and γ with $0 \le \alpha < p$, $\beta \ge 0$, $\gamma \ge 0$, $\beta + \gamma > 0$, then $f \in S_p^*(\alpha)$.

The substitution $p = 1$ in Corollary 3.6, yields the following result:

COROLLARY 3.7. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{\gamma} \left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma} \left(\frac{3}{2} - \alpha\right)^{\beta} & , 0 \le \alpha \le 1/2, \\ (1-\alpha)^{\gamma+\beta}\, 2^{\beta} & , 1/2 \le \alpha < 1, \end{cases}$$

where $z \in E$ and α, β, γ are real numbers with $0 \le \alpha < 1$, $\beta \ge 0$, $\gamma \ge 0$, $\beta + \gamma > 0$, then $f \in S^*(\alpha)$.

In particular, writing $\beta = 1$,$\gamma = 1$ and $\alpha = 0$ in Corollary 3.7, we obtain the following result of Li and Owa [5]:

COROLLARY 3.8. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{3}{2}, \quad z \in E,$$

then $f \in S^*$.

Taking $\lambda = 0$ and $n = 1$ in Theorem 2.1, we get the following interesting criterion for convexity of multivalent functions:

COROLLARY 3.9. *If, for all $z \in E$, a function $f \in \mathcal{A}_p$ satisfies*

$$\left| \frac{1}{p} \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right|^{\gamma} \left| \frac{1}{p} \left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + f''(z)} \right) - 1 \right|^{\beta}$$
$$< \begin{cases} \left(1 - \frac{\alpha}{p}\right)^{\gamma} \left(1 - \frac{\alpha}{p} + \frac{1}{2p}\right)^{\beta}, & 0 \le \alpha \le p/2, \\ \left(1 - \frac{\alpha}{p}\right)^{\gamma+\beta} \left(1 + \frac{1}{p}\right)^{\beta}, & p/2 \le \alpha < p, \end{cases}$$

for some real numbers α, β and γ with $0 \le \alpha < p$, $\beta \ge 0$, $\gamma \ge 0$, $\beta + \gamma > 0$, then $f \in K_p(\alpha)$.

Taking $p = 1$ in Corollary 3.9, we obtain the following sufficient condition for convexity of univalent functions.

COROLLARY 3.10. *For some non-negative real numbers α, β and γ, with $\beta + \gamma > 0$ and $\alpha < 1$, if $f \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \right|^{\gamma} \left| \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + f''(z)} \right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma} \left(1 - \alpha + \frac{1}{2}\right)^{\beta}, & 0 \le \alpha \le 1/2, \\ (1-\alpha)^{\gamma+\beta} 2^{\beta}, & 1/2 \le \alpha < 1, \end{cases}$$

for all $z \in E$, then $f \in K(\alpha)$.

In particular, writing $\beta = 1$, $\gamma = 1$ and $\alpha = 0$ in Corollary 3.10, we obtain the following sufficient conditon for convexity of analytic functions:

COROLLARY 3.11. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + f''(z)} \right) \right| < \frac{3}{2}, \quad z \in E,$$

then $f \in K$.

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