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CONVERGENCE OF THE ISHIKAWA ITERATES FOR MULTI-VALUED MAPPINGS IN METRIC SPACES OF HYPERBOLIC TYPE

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Abstract. Let (X,d) be a metric spaces of hyperbolic type, C be a closed and of hyperbolic type subset of X and let $B(C)$ be the family of all nonempty bounded subsets of C. In this paper some results on the convergence of the Ishikawa iterates associated with a pair of multi-valued mappings $S,T : C \to B(C)$ which satisfy the condition (2.1) below, are obtained.

1. Introduction

Let X be a Banach space, C a closed convex subset of X and T a self-mapping on X. In [2] Ćirić considered a wide class of self-mappings on C which satisfy the following condition:

$$d(Tx,Ty) \le q \max\{ad(x,y), d(x,Tx)+d(y,Ty),\ d(x,Ty)+d(y,Tx)\}, \tag{1.1}$$

where $0 < q < 1$ and a is an arbitrary positive number. Ćirić proved that if the sequence of Mann iterates [10] with constant coefficients converges, then it converges strongly to a fixed point of T. This result for single-valued mappings satisfying (1.1) have been extended and generalized by Ghosh [5], Naimpally and Singh [11], Rashwan [12], [13], Ray [14] and Rhoades [15].

Recently Kubiaczyk and Ali [9], Hu et al. [16] have studied the convergence of the sequence of Ishikawa iterates associated with a pair of multi-valued mappings which map a closed convex subset C of a Banach space into a family $CB(C)$ of all nonempty closed and bounded subsets of C and satisfy the following condition

$$H(Sx,Ty) \le q \max\{ad(x,y),\ D(x,Sx)+D(y,Ty),\ D(x,Ty)+D(y,Sx)\}, \tag{1.2}$$

where H denotes the Hausdorff metric on $CB(C)$ induced by d.

In [3] Ćirić and Ume have studied more general class of multi-valued mappings and generalized the result of Kubiaczyk and Ali [9], Hu et al. [16]. Throughout our consideration we suppose that (X,d) is a metric space which contains a family L of metric segments (isometric images of real line segments) such that

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(a) each two points x, y in X are endpoints of exactly one member $seg[x, y]$ of L, and

(b) if $u, x, y \in X$ and if $z \in seg[x, y]$ satisfies $d(x, z) = \lambda d(x, y)$ for $\lambda \in [0, 1]$, then

$$d(u, z) \leq (1 - \lambda) d(u, x) + \lambda d(u, y).$$

A space of this type is said to be a *metric space of hyperbolic type* (Takahashi [17] has used the term *convex metric space)*. This class includes all normed linear spaces, as well as all spaces with a hyperbolic metric (see [8] for a discussion).

Clearly a Banach space, or any convex subset of it, is a metric space of hyperbolic type with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. More generally, if X is a linear space with a translation invariant metric satisfying

$$d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0),$$

then X is a metric space of hyperbolic type. There are many other examples but we consider these as paradigmatic.

Definition 1. Let X be a nonempty set. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T\colon X \to 2^X$ if $x \in T(x)$. If $Tx = \{x\}$, then x is called a stationary point (or a strict fixed point) of F.

Many authors have considered fixed points in metric spaces of hyperbolic type (see [1–5], [7], [9], [11–16] and references there in).

The purpose of this paper is to continue investigation of the convergence of the Ishikawa iterates associated with a pair of multi-valued mappings which map a closed and of hyperbolic type subset C of a metric space of hyperbolic type X into a family $B(C)$ of all nonempty bounded subsets of C and satisfy the condition (2.1) below, slightly different than the condition (1.2). Our main result is a modified generalization of the corresponding results of Ćirić and Ume [3], Ghosh [5], Kubiaczyk and Ali [9], Naimpally and Singh [11], Ray [14] and Rhoades [15].

2. Main results

Let (X, d) be a metric space and and $B(X)$ be the family of all nonempty bounded subsets of X. Denote

$$\delta(A, B) = \sup\{d(a, b) : \ a \in A, \ b \in B\}, \quad A, B \in B(X).$$

Now we prove our main result.

Theorem 1. *Let (X, d) be a metric space of hyperbolic type and let C be a closed and of hyperbolic type subset C of X. Let $S, T\colon C \to B(C)$ be mappings satisfying the following condition:*

$$\delta(Sx_n, Ty_n) \leq \varphi(\max\{ad(x, y), \ d(x, y) + \delta(x, Ty) + \delta(y, Sx), \ \delta(x, Ty) + \delta(y, Sx) + \delta(y, Ty)\}) \quad (2.1)$$

for all x, y *in* K, *where* $a > 1$ *is an arbitrary positive real number and* $\varphi\colon [0, +\infty) \to [0, +\infty)$ *is a nondecreasing function such that* $t - \varphi(t) > 0$ *for each* $t > 0$ *and*

$$t - \varphi(t) \to 0 \quad \textit{implies} \quad t \to 0. \tag{2.2}$$

Let $x_0 \in X$ *be arbitrary and let* $\{x_n\}$ *be the sequence of Ishikawa iterates associated with a pair* $\{S, T\}$, *defined as follows:*

$$y_n \in W(x_n^*, x_n, \beta_n), \qquad n \geq 0, \tag{2.3}$$

$$x_{n+1} = W(y_n^*, x_n, \alpha_n), \qquad n \geq 0, \tag{2.4}$$

where $0 \leq \alpha_n < 1$, $0 \leq \beta_n < 1$,

$$\liminf_{n\to\infty} \alpha_n = \alpha > 0, \tag{2.5}$$

$$\limsup_{n\to\infty} \beta_n < \frac{1}{a} \tag{2.6}$$

and for each $n \geq 0$, $x_n^* \in Sx_n$ *and* $y_n^* \in Ty_n$ *are such that*

$$\delta(y_n, Sx_n) \leq d(y_n, x_n^*) + \epsilon_n; \qquad \delta(y_n, Ty_n) \leq d(y_n, y_n^*) + \epsilon_n, \tag{2.7}$$

where $\epsilon_n > 0$ *and* $\lim_{n\to\infty} \epsilon_n = 0$. *If* $\{x_n\}$ *converges, then its limit point, say* $z \in C$, *is a stationary point of* S, *and if* T *is a closed mapping, then* z *is a common fixed point of* S *and* T, *that is,* $Sz = \{z\} \in Tz$.

Proof. It is clear that

$$d(x, y) \leq d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y) \leq (1 - \lambda)d(x, y) + \lambda d(x, y) = d(x, y)$$

implies the following:

$$d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y), \qquad d(y, W(x, y, \lambda)) = \lambda d(x, y).$$

From (2.4) we have, as $y_n^* \in Ty_n$,

$$d(x_n, x_{n+1}) = d(x_n, W(y_n^*, x_n, \alpha_n)) = \alpha_n d(x_n, y_n^*) \leq \alpha_n \delta(x_n, Ty_n). \tag{2.8}$$

Since $\lim_{n\to\infty} x_n = z$, then $d(x_n, x_{n+1}) \to 0$. Thus from (2.8) and (2.5),

$$\lim_{n\to\infty} \delta(x_n, Ty_n) = 0. \tag{2.9}$$

From (2.7) and (2.1) with $x = x_n$ and $y = y_n$, and using that φ is a nondecreasing function, we get

$$\begin{aligned}
\delta(Sx_n, Ty_n) &\leq \varphi(\max\{ad(x_n, y_n), d(x_n, y_n) + \delta(x_n, Ty_n) + \delta(y_n, Sx_n), \\
&\qquad \delta(x_n, Ty_n) + \delta(y_n, Sx_n) + \delta(y_n, Ty_n)\}) \\
&\leq \varphi(\max\{ad(x_n, y_n),\ d(x_n, y_n) + \delta(x_n, Ty_n) + d(x_n^*, y_n) + \epsilon_n, \\
&\qquad \delta(x_n, Ty_n) + d(x_n^*, y_n) + d(y_n, y_n^*) + 2\epsilon_n\}).
\end{aligned} \tag{2.10}$$

From (2.7),

$$\begin{aligned} d(x_n, y_n) &= d(x_n, W(x_n^*, x_n, \beta_n)) = \beta_n d(x_n^*, x_n), \\ d(x_n^*, y_n) &= d(x_n^*, W(x_n^*, x_n, \beta_n)) = (1-\beta_n) d(x_n^*, x_n) \\ d(y_n, y_n^*) &\leq d(y_n, x_n) + d(x_n, y_n^*) \leq \beta_n d(x_n^*, x_n) + d(x_n, y_n^*) \\ &\leq \beta_n d(x_n^*, x_n) + \delta(x_n, Ty_n). \end{aligned}$$

and hence

$$\begin{aligned} d(x_n, y_n) + d(x_n^*, y_n) &= d(x_n, x_n^*); \\ d(x_n^*, y_n) + d(y_n, y_n^*) &\leq d(x_n, x_n^*) + \delta(x_n, Ty_n). \end{aligned} \tag{2.11}$$

Thus from (2.10) and (2.6) we get

$$\delta(Sx_n, Ty_n) \leq \varphi(d(x_n, x_n^*) + 2(\delta(x_n, Ty_n) + \epsilon_n)). \tag{2.12}$$

Since $y_n^* \in Ty_n$, by the triangle inequality we have

$$d(x_n, x_n^*) \leq d(x_n, y_n^*) + d(x_n^*, y_n^*) \leq \delta(x_n, Ty_n) + d(x_n^*, y_n^*). \tag{2.13}$$

Because $d(x_n^*, y_n^*) \leq \delta(Sx_n, Ty_n)$, by (2.13),

$$d(x_n^*, y_n^*) \leq \varphi(d(x_n, x_n^*) + 2(\delta(x_n, Ty_n) + \epsilon_n)).$$

Inserting this in (2.13) we obtain

$$d(x_n, x_n^*) \leq \varphi(d(x_n, x_n^*) + 2(\delta(x_n, Ty_n) + \epsilon_n)) + \delta(x_n, Ty_n)). \tag{2.14}$$

Set

$$d(x_n, x_n^*) + 2(\delta(x_n, Ty_n) + \epsilon_n) = w_n. \tag{2.15}$$

Then from (2.14), $w_n \leq \varphi(w_n) + 3\delta(x_n, Ty_n) + 2\epsilon_n$. Hence

$$w_n - \varphi(w_n) \leq 3\delta(x_n, Ty_n) + 2\epsilon_n.$$

Taking the limit as $n \to \infty$, and using (2.9), we get $\lim_{n\to\infty}(w_n - \varphi(w_n)) = 0$. Hence, by (2.2), $\lim_{n\to\infty} w_n = 0$. Thus from (2.15) and (2.9),

$$\lim_{n\to\infty} d(x_n, x_n^*) = 0. \tag{2.16}$$

Since $x_n \to z$, from (2.16) $x_n^* \to z$.

From (2.16), (2.9) and (2.11) we get $d(x_n^*, y_n) \to 0$, $d(y_n, y_n^*) \to 0$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n^* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_n^* = z. \tag{2.17}$$

Now we show that z is a stationary point of S. Since $y_n^* \in Ty_n$, from the triangle inequality and (2.1) we get

$$\begin{aligned} \delta(z, Sz) &\leq d(z, x_n) + \delta(x_n, Ty_n) + \delta(Sz, Ty_n) \\ &\leq d(z, x_n) + \delta(x_n, Ty_n) + \varphi(\max\{ad(z, y_n), \\ &\qquad d(z, y_n) + \delta(z, Ty_n) + \delta(y_n, Sz), \delta(z, Ty_n) + \delta(y_n, Sz) + \delta(y_n, Ty_n)\}) \\ &\leq d(z, x_n) + \delta(x_n, Ty_n) \\ &\qquad + \varphi((2+a)d(z, y_n) + d(z, x_n) + d(z, x_n) + \delta(x_n, Ty_n) + \delta(z, Sz)). \end{aligned}$$

Hence

$$v_n - \varphi(v_n) \leq (2+a)d(z,y_n) + d(z,x_n) + 2d(z,x_n) + 2\delta(x_n,Ty_n), \tag{2.18}$$

where

$$v_n = \delta(z,Sz) + (2+a)d(z,y_n) + d(z,x_n) + d(z,x_n) + \delta(x_n,Ty_n). \tag{2.19}$$

Taking the limit as n tends to infinity we get, by (2.18) and (2.2),

$$\lim_{n\to\infty} v_n = 0.$$

Then from (2.19), $\delta(z,Sz) = 0$. Hence $Sz = \{z\}$.

Assume now that T is a closed mapping (recall that a $T : D \subset X \to 2^X$ is called a closed mapping if for $\{x_n\} \subset D$ the conditions $x_n \to z$, $y_n \in Tx_n$ and $y_n \to w$ imply $w \in Tz$). Since $\{y_n\} \subset C$, $y_n \to z$, $y_n^* \in Ty_n$, $y_n^* \to z$ and T is a closed mapping, it follows that $z \in Tz$. Thus we have

$$Sz = \{z\} \in Tz. \quad \blacksquare$$

Remark. Theorem 1 is a modified and generalized version of Theorem 2.2 of Ćirić and Ume [3] in Banach spaces and also a modified and generalized version of the main theorems of Kubiaczyk and Ali [9] and Hu et al. [16], where they consider a pair of mapping in Banach spaces satisfying (1.1) and for $\{\beta_n\}$ they assumed the condition that $\beta_n \to 0$ as $n \to \infty$. This condition implies that the Ishikawa iterates became the Mann iterates.

REFERENCES

[**1**] I. Beg and A. Azam, *On the iteration methods for multi-valued mappings*, Demonstratio Math. **27** (1994), 493–499.

[**2**] Lj. B. Ćirić, *Quasi-contractions in Banach spaces*, Publ. Inst. Math. (Beograd), **21** (35) (1977), 41–48.

[**3**] Lj. B. Ćirić and J. S. Ume, *Common fixed points of nonlinear contractions*, Acta Math. Hungar. **80** (1998), 31–38.

[**4**] L. B. Ćirić and J. S. Ume, *Iterative approximation of a common fixed point of multi-valued mappings*, Italian J. Pure Appl. Math. **18** (2005), 217–222.

[**5**] K. M. Ghosh, *A note on a theorem of Rhoades*, Mat. Sem. Notes, Kobe Univ. **8** (1980), 505–507.

[**6**] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[**7**] M. S. Khan, H. K. Pathak and M. D. Khan, *Some fixed point theorems in metrically convex spaces*, Georgian Math. J. **7** (2000), 523–530.

[**8**] W. A. Kirk, *Krasnoselskii's iteration process in hyperbolic spaces*, Numer. Func. Anal. Optim. **4** (1982), 371–381.

[**9**] I. Kubiaczyk and M. Ali, *On the convergence of the Ishikawa iterates to a common fixed point for a pair of multi-valued mappings*, Acta Math. Hungar. **75** (1997), 253–257.

[**10**] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[**11**] S. A. Naimpally and K. L. Singh, *Extensions of some fixed point theorems of Rhoades*, J. Math. Anal. Appl. **96** (1983), 437–446.

[**12**] R. A. Rashwan, *On the convergence of Mann iterates to a common fixed point for a pair of mappings*, Demonstratio Math., **23** (1990), 709–712.

[**13**] R. A. Rashwan, *On the convergence of the Ishikawa iterates to a common fixed point for a pair of mappings*, Demonstratio Math. **28** (1995), 271–274.

[**14**] B. K. Ray, *On an extension of a theorem of Ljubomir Ćirić*, Indian J. pure appl. Math. **10** (1979), 145–146.

[**15**] B. E. Rhoades, *Extensions of some fixed point theorems of Ćirić, Maiti and Pal*, Math. Seminar. Notes, Kobe Univ. **6** (1978), 41–46.

[**16**] T. Hu, J. C. Huang and B. E. Rhoades, *A general principle for Ishikawa itertions for multi-valued mappings*, Indian J. pure appl.Math. **28** (8) (1997), 1091–1098.

[**17**] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Math. Sem. Rep. **22** (1970), 142–149.

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