МАТЕМАТИЧКИ ВЕСНИК
61 (2009), 149–152

UDK 515.122
оригинални научни рад
research paper

# SUBSPACES OF CS-STARCOMPACT SPACES

**Yan-Kui Song**

**Abstract.** A space $X$ is cs-starcompact if for every open cover $\mathcal{U}$ of $X$, there exists a convergent sequence $S$ of $X$ such that $St(S,\mathcal{U}) = X$, where $St(S,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap S \neq \emptyset\}$. In this note, we investigate the closed subspaces of cs-starcompact spaces.

## 1. Introduction

By a space, we mean a topological space. Let us recall that a space $X$ is countably compact if every countable open cover of $X$ has a finite subcover. Fleischman [4] defined a space $X$ to be starcompact if for every open cover $\mathcal{U}$ of $X$, there exists a finite subset $F$ of $X$ such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$, and he proved that every countably compact space is starcompact. Conversely, van Douwen, Reed, Roscoe and Tree [2] proved that every Hausdorff starcompact space is countably compact, but this does not hold for $T_1$-spaces (see [8, Example 2.5]). As generalizations of starcompactness, the following classes of spaces are given.

Definition 1.1. A space $X$ is cs-starcompact if for every open cover $\mathcal{U}$ of $X$, there exists a convergent sequence $S$ of $X$ such that $St(S,\mathcal{U}) = X$.

Definition 1.2. [5] A space $X$ is $1\frac{1}{2}$-starcompact if for every open cover $\mathcal{U}$ of $X$, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\cup\mathcal{V},\mathcal{U}) = X$.

In [6], a cs-starcompact space is called star determined by convergent sequence and in [2], a $1\frac{1}{2}$-starcompact space is called 1-starcompact. It is known [5] that these properties lie between countable compactness and pseudocompactness. From the above definitions, it is clear that every star-compact space is cs-starcompact and every cs-starcompact space is $1\frac{1}{2}$-starcompact.

*AMS Subject Classification*: 54D20, 54G20.

*Keywords and phrases*: Countable compact; cs-starcompact.

The author acknowledges supports from the NSF of China (Grant No 10571081) and the National Science Foundation of Jiangsu Higher Education Institutions of China (Grant No 07KJB-110055)

In [6], van Mill, Tkachuk and Wilson discussed the relationship between cs-star-compact spaces and related spaces. The purpose of this note is to investigate the closed subspaces of cs-starcompact spaces.

Thorough this paper, the cardinality of a set $A$ is denoted by $|A|$. For a space $X$, the extent $e(X)$ of $X$ is defined as the smallest cardinal number $\kappa$ such that the cardinality of every discrete closed subset of $X$ is not greater than $\kappa$. Let $\omega$ denote the first infinite cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers, For a cardinal $\kappa$, let $\kappa^+$ be the smallest cardinal greater than $\kappa$. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each pair of ordinals $\alpha$, $\beta$ with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$. Other terms and symbols that we do not define will be used as in [3].

## 2. Closed subspaces of cs-starcompact spaces

First, we give a machine that produces a Hausdorff cs-starcompact space. Let $X$ be a Hausdorff separable space with $|X| = \kappa$. For every countable dense subset $D$ of $X$, let $T$ be $D$ with the discrete topology and let

$$Y = T \cup \{\infty\}, \text{ where } \infty \notin T$$

be the one-point compactification of $T$. We define $S(X, D) = X \cup (Y \times \kappa^+)$ and we topologize $S(X, D)$ as follows: $Y \times \kappa^+$ has the usual product topology and is an open subspace of $S(X, D)$, and a basic neighborhood of a point $x$ of $X$ takes the form

$$G(U, \alpha) = U \cup ((U \cap T) \times (\alpha, \kappa^+)),$$

where $U$ is a neighborhood of $x$ in $X$ and $\alpha < \kappa^+$. When it is not necessary to specify $D$, we simply write $S(X)$ instead of $S(X, D)$.

THEOREM 2.1. *Let $\kappa$ be a cardinal. Let $X$ be a separable space with $|X| = \kappa$ and $D$ a countable dense subset of $X$. Then, $S(X, D)$ is cs-starcompact. Moreover, if $X$ is a Hausdorff space, then so is $S(X, D)$.*

*Proof.* If $S = S(X, D)$ then it is easy to see that $X$ is a closed subset of $S$ and $S$ is Hausdorff if $X$ is Hausdorff.

We show that $S$ is cs-starcompact. To this end, let $\mathcal{U}$ be an open cover of $S$. Without loss of generality, we assume that $\mathcal{U}$ consists of basic open sets of $S$. For each $x \in X$, there exists a $U_x \in \mathcal{U}$ such that $x \in U_x$. Hence, there exist $\alpha_x < \kappa^+$ and an open neighborhood $V_x$ of $x$ in $X$ such that $G(V_x, \alpha_x) \subseteq U_x$. If we put $\alpha_0 = \sup\{\alpha_x : x \in X\}$, then $\alpha_0 < \kappa^+$, since $|X| < \kappa^+$. Let $F_1 = Y \times \{\alpha_0\}$. Then, $F_1$ is a convergent sequence of $S(X, \kappa)$. Since $U_x \cap F_1 \neq \emptyset$ for every $x \in X$, we have

$$X \subseteq St(F_1, \mathcal{U}).$$

On the other hand, since $Y$ is compact and $\kappa^+$ is countably compact, the space $Y \times \kappa^+$ is countably compact, and hence there exists a finite subset $F_2$ of $Y \times \kappa^+$ such that

$$Y \times \kappa^+ \subseteq St(F_2, \mathcal{U}).$$

If we put $S' = F_1 \cup F_2$, then $S'$ is a convergent sequence of $S$ and $S = St(S', \mathcal{U})$, which shows that $S$ is cs-starcompact. This completes the proof. ∎

Corollary 2.2. *Every Hausdorff separable space can be represented in a Hausdorff cs-starcompact space as a closed subspace.*

It is well-known that the extent of every countably compact space is countable. However, a similar result does not hold for cs-starcompactness. In fact, the following example shows that the extent of a cs-starcompact space may be equal $2^{\mathfrak{c}}$.

For a Tychonoff space $X$, let $\beta X$ denote the Čech-Stone compactification of $X$. We construct an example of a Hausdorff cs-starcompact space $X$ such that $e(X) = 2^{\mathfrak{c}}$ by using an example from [1]. We include the construction here for the sake of completeness.

Example 2.3. There exists a Hausdorff cs-starcompact space $X$ such that $e(X) = 2^{\mathfrak{c}}$.

*Proof.* Let $Y = \beta\omega$. We define another topology on $Y$ as follows: every point of $\omega$ is isolated and a basic neighborhood of a point $x \in \beta\omega \setminus \omega$ takes the from

$$\{x\} \cup (U_x \cap \omega),$$

where $U_x$ is a neighborhood of $x$ in $\beta\omega$. Clearly, $Y$ is Hausdorff and separable, since $\omega$ is a countable dense subspace of $Y$. Since $\beta\omega \setminus \omega$ is discrete and closed in $X$ with $|\beta\omega \setminus \omega| = 2^{\mathfrak{c}}$, we have $e(Y) = 2^{\mathfrak{c}}$, which completes the construction.

Let $X = S(Y)$. Then $\beta\omega \setminus \omega$ is discrete and closed in $X$ by the construction of $X$, hence $e(X) = 2^{\mathfrak{c}}$. The space $X$ is cs-starcompact by Theorem 2.1, which completes the proof. ∎

In [6], van Mill, Tkachuk and Wilson showed that the Tychonoff plank is cs-starcompact, which shows that a closed subset of a cs-starcompact space need not be cs-starcompact. In fact, an example below shows that cs-starcompactness is not preserved by passing to a regular closed subset. We will need the following lemma from [5].

Lemma 2.4. *If a regular space $X$ contains a closed discrete subspace $Y$ such that $|Y| = |X| \geq \omega$, then $X$ is not $1\frac{1}{2}$-starcompact* (*hence, not cs-starcompact.*)

Example 2.5. There exists a Hausdorff cs-starcompact space $X$ with a regular-closed subset which is not cs-starcompact.

*Proof.* Let $S_1 = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [7], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of $\omega$ with $|\mathcal{R}| = \mathfrak{c}$. Then, $S_1$ is separable, since $\omega$ is countable dense in $S_1$ while it is not cs-starcompact by Lemma 2.4. If $S_2 = S(S_1)$ then $S_2$ is cs-starcompact by Theorem 2.1.

We can consider that $S_1 \cap S_2 = \emptyset$. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let $f : \mathcal{R} \to \mathcal{R}$ be a map defined by $f(r_\alpha) = r_\alpha$ for each $\alpha < \mathfrak{c}$. Let $X$ be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying $r_\alpha$ of $S_1$ with $r_\alpha$ of $S_2$ for each $\alpha < \mathfrak{c}$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map.

Let $Y = \pi(S_1)$. It is easy to check that $Y$ is a regular closed in $X$, however, it is not cs-starcompact, since it is homeomorphic to $S_1$.

Next, we show that $X$ is cs-starcompact. For this end, let $\mathcal{U}$ be an open cover of $X$. Since $\pi(S_2)$ is cs-starcompact, there exists a convergent sequence $F_1$ of $\pi(S_2)$ such that $\pi(S_2) \subseteq St(F_1,\mathcal{U})$. On the other hand, since $\pi(S_1)$ is homeomorphic to $S_1$, every infinite subset of $\pi(\omega)$ has an accumulation point in $\pi(S_1)$. We claim that there exists a finite subset $F_2$ of $\pi(\omega)$ such that $\pi(\omega) \subseteq St(F_2,\mathcal{U})$. For, if $\pi(\omega) \nsubseteq St(B,\mathcal{U})$ for any finite subset $B$ of $\pi(\omega)$ then by induction, we can define a sequence $\{x_n : n \in \omega\}$ in $\pi(\omega)$ such that

$$x_n \notin St(\{x_i : i < n\},\mathcal{U}) \text{ for each } n \in \omega.$$

By the property of $\pi(\omega)$ mentioned above, the sequence $\{x_n : n \in \omega\}$ has a limit point $x'$ in $\pi(S_1)$. Pick $U \in \mathcal{U}$ such that $x' \in U$. Choose $n < m < \omega$ such that $x_n \in U$ and $x_m \in U$. Then $x_m \in St(\{x_i : i < m\},\mathcal{U})$, which contradicts the definition of the sequence $\{x_n : n \in \omega\}$. Let $S = F_1 \cup F_2$. Then $S$ is a convergent sequence of $X$ and $X = St(S,\mathcal{U})$. Hence $X$ is cs-starcompact, which completes the proof. ∎

Remark. We give below the list of questions that we could not answer while working on this paper.

(1) Can every Tychonoff separable space be represented in a Tychonoff cs-starcompact space as a closed subspace?

(2) Can the extent of a Hausdorff cs-starcompact space be greater than $2^{\mathfrak{c}}$ and the extent of a Tychonoff cs-starcompact space $X$ equal $\mathfrak{c}$ or be greater than $\mathfrak{c}$?

(3) which closed subspaces of Tychonoff cs-starcompact spaces are cs-starcompact?

Acknowledgement. The author is most grateful to the referee for his kind help and valuable comments.

REFERENCES

[**1**] A.V. Arhangel'skii and I.Y. Gordienko, *Relatively locally finite Hausdorff spaces*, Questions Answers Gen. Topology., **12** (1994), 15–25.

[**2**] E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree, *Star covering properties*, Topology Appl. **39** (1991), 71–103.

[**3**] R. Engelking, *General Topology*, Revised and completed edition, Heldermann Verlag, Berlin, 1989.

[**4**] W.M. Fleischman, *A new extension of countable compactness*, Fund. Math. **67** (1970), 1–7.

[**5**] M.V. Matveev, *A survey on star-covering properties*, Topological Atlas **330**, 1998.

[**6**] J. van Mill, V.V. Tkachuk and R.G. Wilson, *Classes defined by stars and neighbourhood assignments*, Topology Appl. **154**.

[**7**] S. Mrówka, *On complete regular spaces*, Fund. Math. **41** (1954), 105–106.

[**8**] Y-K. Song, *On countable star-covering properties*, Applied General Topology **8**, 2 (2007), 249–258.

(received 26.02.2008, in revised form 08.07.2008)

Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing, 210097, P.R of China

*E-mail*: songyankui@njnu.edu.cn