МАТЕМАТИЧКИ ВЕСНИК
65, 1 (2013), 14–28
March 2013

оригинални научни рад
research paper

CERTAIN SUBCLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS DEFINED BY CONVOLUTION WITH NEGATIVE COEFFICIENTS

M.K. Aouf, A.A. Shamandy, A.O. Mostafa and A.K. Wagdy

Abstract. The aim of this paper is to obtain coefficient estimates, distortion theorems, convex linear combinations and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $TS(g,\lambda;\alpha,\beta)$. Furthermore partial sums $f_n(z)$ of functions $f(z)$ in the class $TS(g,\lambda;\alpha,\beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f_n'(z)$ are determined.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $g \in A$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (g * f)(z). \tag{1.3}$$

Following Goodman ([6] and [7]), Ronning ([11 and [12]) introduced and studied the following subclasses:

(i) A function $f(z)$ of the form (1.1) is said to be in the class $S_p(\alpha,\beta)$ of β-uniformly starlike functions if it satisfies the condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{U}), \tag{1.4}$$

where $-1 \le \alpha < 1$ and $\beta \ge 0$.

2010 AMS Subject Classification: 30C45.

Keywords and phrases: Analytic function; Hadamard product; distortion theorems; partial sums.

(ii) A function $f(z)$ of the form (1.1) is said to be in the class $UCV(\alpha,\beta)$ of β-uniformly convex functions if it satisfies the condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in\mathbb{U}), \tag{1.5}$$

where $-1\le\alpha<1$ and $\beta\ge 0$.

It follows from (1.4) and (1.5) that

$$f(z)\in UCV(\alpha,\beta)\Leftrightarrow zf'(z)\in S_p(\alpha,\beta). \tag{1.6}$$

For $-1\le\alpha<1$, $0\le\lambda\le 1$ and $\beta\ge 0$, we let $S(g,\lambda;\alpha,\beta)$ be the subclass of A consisting of functions $f(z)$ of the form (1.1) and functions $g(z)$ of the form (1.2) and satisfying the analytic criterion:

$$\operatorname{Re}\left\{\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z\ (f*g)'(z)}-\alpha\right\}>\beta\left|\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right|. \tag{1.7}$$

Remark 1. (i) Putting $g(z)=\frac{z}{(1-z)}$ in the class $S(g,\lambda;\alpha,\beta)$, we obtain the class $S_p(\lambda,\alpha,\beta)$ defined by Murugusundaramoorthy and Magesh [10].

(ii) Putting $g(z)=\frac{z}{(1-z)^2}$ in the class $S(g,\lambda;\alpha,\beta)$, we obtain the class $UCV(\lambda,\alpha,\beta)$ defined by Murugusundaramoorthy and Magesh [10].

Let T denote the subclass of A consisting of functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k\ (a_k\ge 0). \tag{1.8}$$

Further, we define the class $TS(g,\lambda;\alpha,\beta)$ by

$$TS(g,\lambda;\alpha,\beta)=S(g,\lambda;\alpha,\beta)\cap T \tag{1.9}$$

We note that:

(i) $TS(\frac{z}{(1-z)},0;\alpha,1)=TS_p(\alpha)$ and $TS(\frac{z}{(1-z)^2},0;\alpha,1)=UCT(\alpha)$ (see Bharati et al. [2]);

(ii) $TS(\frac{z}{(1-z)},0;\alpha,\beta)=TS_p(\alpha,\beta)$ and $TS(\frac{z}{(1-z)^2},0;\alpha,\beta)=UCT(\alpha,\beta)$ (see Bharati et al. [2]);

(iii) $TS(\frac{z}{(1-z)},0;\alpha,0)=T^*(\alpha)$ and $TS(\frac{z}{(1-z)^2},0;\alpha,0)=C\ (\alpha)$ (see Silverman [15]);

(iv) $TS(z+\sum_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k,0;\alpha,\beta)=TS(\alpha,\beta)\ \ (c\ne 0,-1,-2,\dots)$ (see Murugusundaramoorthy and Magesh [8, 9]);

(v) $TS(z+\sum_{k=2}^{\infty}k^n z^k,0;\alpha,\beta)=TS(n,\alpha,\beta)\ \ (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, where $\mathbb{N}=\{1,2,\dots\})$ (see Rosy and Murugusundaramoorthy [13]);

(vi) $TS(\frac{z}{(1-z)},\lambda;\alpha,\beta)=TS_p(\lambda,\alpha,\beta)$ and $TS(\frac{z}{(1-z)^2},\lambda;\alpha,\beta)=UCT(\lambda,\alpha,\beta)$ (see Murugusundaramoorthy and Magesh [10]);

(vii) $TS(z+\sum_{k=2}^{\infty}\binom{k+\delta-1}{\delta}z^k,0;\alpha,\beta)=D(\beta,\alpha,\delta)\ \ (\delta>-1)$ (see Shams et al. [14]);

(viii) $TS(z+\sum_{k=2}^{\infty}\left[1+\delta\left(k-1\right)\right]^{n}z^{k},0;\alpha,\beta)=TS_{\delta}(n,\alpha,\beta)$ $(\delta\geq 0,\ n\in\mathbb{N}_{0})$ (see Aouf and Mostafa [1]).

Also we note that:

(i) $TS(z+\sum\limits_{k=2}^{\infty}\Gamma_{k}(\alpha_{1})z^{k},\lambda;\alpha,\beta)=TS_{q,s}(\alpha_{1};\lambda,\alpha,\beta)$

$=\{f\in T:\operatorname{Re}\left\{\frac{z(H_{q,s}(\alpha_{1},\beta_{1})f(z))'}{(1-\lambda)H_{q,s}(\alpha_{1},\beta_{1})f\ (z)+\lambda z(H_{q,s}(\alpha_{1},\beta_{1})f(z))'}-\alpha\right\}$

$>\beta\left|\frac{z\ (H_{q,s}(\alpha_{1},\beta_{1})f(z))'}{(1-\lambda)H_{q,s}(\alpha_{1},\beta_{1})f(z)+\lambda z\ (H_{q,s}(\alpha_{1},\beta_{1})f(z))'}-1\right|\},$

where $-1\leq\alpha<1$, $0\leq\lambda\leq 1$, $\beta\geq 0$, $z\in U$ and $\Gamma_{k}(\alpha_{1})$ is defined by

$$\Gamma_{k}(\alpha_{1})=\frac{(\alpha_{1})_{k-1}\cdots(\alpha_{q})_{k-1}}{(\beta_{1})_{k-1}\cdots(\beta_{s})_{k-1}(1)_{k-1}} \tag{1.10}$$

$(\alpha_{i}>0,\ i=1,\ldots,q;\ \beta_{j}>0,\ j=1,\ldots,s;\ q\leq s+1,\ q,s\in\mathbb{N}_{0})$, where the operator $H_{q,s}(\alpha_{1},\beta_{1})$ was introduced and studied by Dziok and Srivastava (see [4] and [5]), which is a generalization of many other linear operators considered earlier;

(ii) $TS(z+\sum_{k=2}^{\infty}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^{m}z^{k},\lambda;\alpha,\beta)=TS(m,\mu,\ell,\lambda;\alpha,\beta)$

$=\left\{f\in T:\operatorname{Re}\left\{\frac{z(I^{m}(\mu,\ell)f(z))'}{(1-\lambda)I^{m}(\mu,\ell)f(z)+\lambda z(I^{m}(\mu,\ell)f(z))'}-\alpha\right\}\right.$

$\left.>\beta\left|\frac{z(I^{m}(\mu,\ell)f(z))'}{(1-\lambda)I^{m}(\mu,\ell)f(z)+\lambda z(I^{m}(\mu,\ell)f(z))'}-1\right|\right\},$

where $-1\leq\alpha<1$, $0\leq\lambda\leq 1$, $\beta\geq 0$, $m\in\mathbb{N}_{0}$, $\mu,\ell\geq 0$, $z\in\mathbb{U}$ and the operator $I^{m}(\mu,\ell)$ was defined by Cătaş et al. (see [3]), which is a generalization of many other linear operators considered earlier;

(iii) $TS(z+\sum_{k=2}^{\infty}C_{k}(b,\mu)z^{k},\lambda;\alpha,\beta)=TS(b,\mu,\lambda;\alpha,\beta)=$

$\left\{f\in T:\operatorname{Re}\left\{\frac{z\left(J_{b}^{\mu}f(z)\right)'}{(1-\lambda)J_{b}^{\mu}f\ (z)+\lambda z\left(J_{b}^{\mu}f\ (z)\right)'}-\alpha\right\}>\beta\left|\frac{z\left(J_{b}^{\mu}f(z)\right)'}{(1-\lambda)J_{b}^{\mu}f\ (z)+\lambda z\left(J_{b}^{\mu}f\ (z)\right)'}-1\right|\right\},$

where $-1\leq\alpha<1$, $0\leq\lambda\leq 1$, $\beta\geq 0$, $z\in U$ and $C_{k}(b,\mu)$ is defined by

$$C_{k}(b,\mu)=\left(\frac{1+b}{k+b}\right)^{\mu}(b\in\mathbb{C}\setminus\mathbb{Z}_{0}^{-}\},\ \mathbb{Z}_{0}^{-}=\mathbb{Z}\setminus\mathbb{N}), \tag{1.11}$$

where the operator J_{b}^{μ} was introduced by Srivastava and Attiya (see [18]), which is a generalization of many other linear operators considered earlier.

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $-1\leq\alpha<1$, $0\leq\lambda\leq 1$, $\beta\geq 0$ and $z\in\mathbb{U}$.

THEOREM 1. *A function $f(z)$ of the form* (1.1) *is in the class $S(g,\lambda;\alpha,\beta)$ if*

$$\sum_{k=2}^{\infty}\left\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\right\}b_{k}\left|a_{k}\right|\leq 1-\alpha, \tag{2.1}$$

where $b_{k+1}\geq b_{k}>0$ $(k\geq 2)$.

Proof. Assume that the inequality (2.1) holds true. Then we have

$$\begin{aligned}\beta\left|\frac{z\,(f*g)'(z)}{(1-\lambda)\,(f*g)(z)+\lambda z(f*g)'(z)}-1\right| \\ -\operatorname{Re}\left\{\frac{z\,(f*g)'(z)}{(1-\lambda)\,(f*g)(z)+\lambda z(f*g)'(z)}-1\right\} \\ \leq (1+\beta)\left|\frac{z\,(f*g)'(z)}{(1-\lambda)\,(f*g)(z)+\lambda z(f*g)'(z)}-1\right| \\ \leq \frac{(1+\beta)\sum_{k=2}^{\infty}(1-\lambda)\,(k-1)\,b_k\,|a_k|\,z^{k-1}}{1-\sum_{k=2}^{\infty}[1+\lambda\,(k-1)]\,b_k\,|a_k|\,z^{k-1}} \leq 1-\alpha.\end{aligned}$$

This completes the proof of Theorem 1. ■

THEOREM 2. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.8) *to be in the class $TS(g,\lambda;\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\{k\,(1+\beta)-(\alpha+\beta)\,[1+\lambda\,(k-1)]\}\,a_kb_k \leq 1-\alpha. \tag{2.2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f(z) \in TS(g,\lambda;\alpha,\beta)$ and z is real, then

$$\frac{1-\sum\limits_{k=2}^{\infty}k\;a_k\;b_k\;z^{k-1}}{1-\sum\limits_{k=2}^{\infty}[1+\lambda\,(k-1)]\,a_k\;b_k\;z^{k-1}}-\alpha \geq \beta\left|\frac{\sum\limits_{k=2}^{\infty}(1-\lambda)\,(k-1)\,a_k\;b_k\;z^{k-1}}{1-\sum\limits_{k=2}^{\infty}[1+\lambda\,(k-\lambda)]\,a_k\;b_k\;z^{k-1}}\right|.$$

Letting $z \longrightarrow 1^-$ along the real axis, we obtain

$$\sum_{k=2}^{\infty}\{k\,(1+\beta)-(\alpha+\beta)\,[1+\lambda\,(k-1)]\}\,a_kb_k \leq 1-\alpha.$$

This completes the proof of Theorem 2. ■

COROLLARY 1. *Let the function $f(z)$ defined by* (1.8) *be in the class $TS(g,\lambda;\alpha,\beta)$. Then*

$$a_k \leq \frac{1-\alpha}{\{k\,(1+\beta)-(\alpha+\beta)\,[1+\lambda\,(k-1)]\}\,b_k} \quad (k \geq 2). \tag{2.3}$$

The result is sharp for the function

$$f(z) = z-\frac{1-\alpha}{\{k\,(1+\beta)-(\alpha+\beta)\,[1+\lambda\,(k-1)]\}\,b_k}z^k \quad (k \geq 2). \tag{2.4}$$

By taking $b_k = \Gamma_k(\alpha_1)$, where $\Gamma_k(\alpha_1)$ is defined by (1.10), in Theorem 2, we have:

Corollary 2. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.8) *to be in the class $TS_{q,s}(\alpha_1;\lambda,\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\left\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\right\}\Gamma_k(\alpha_1)a_k\leq 1-\alpha.$$

By taking $b_k=\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m$ $(m\in\mathbb{N}_0,\ \mu,\ell\geq 0)$, in Theorem 2, we have:

Corollary 3. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.8) *to be in the class $TS(m,\mu,\ell,\lambda;\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\left\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\right\}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m a_k\leq 1-\alpha.$$

By taking $b_k=C_k(b,\mu)$, where $C_k(b,\mu)$ defined by (1.11), in Theorem 2, we have:

Corollary 4. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.8) *to be in the class $TS(b,\mu,\lambda;\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\left\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\right\}\left|C_k(b,\mu)\right|\left|a_k\right|\leq 1-\alpha.$$

3. Distortion theorem

Theorem 3. *Let the function $f(z)$ of the form* (1.8) *be in the class $TS(g,\lambda;\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$|f(z)|\geq r-\frac{1-\alpha}{\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]b_2}r^2 \tag{3.1}$$

and

$$|f(z)|\leq r+\frac{1-\alpha}{\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]b_2}r^2, \tag{3.2}$$

provided that $b_{k+1}\geq b_k>0$ $(k\geq 2)$. The equalities in (3.1) *and* (3.2) *are attained for the function $f(z)$ given by*

$$f(z)=z-\frac{1-\alpha}{\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]b_2}z^2, \tag{3.3}$$

at $z=r$ and $z=re^{i(2k+1)\pi}$ $(k\in\mathbb{Z})$.

Proof. Since for $k\geq 2$,

$$\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]b_2\leq\left\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\right\}b_k,$$

using Theorem 2, we have

$$
\begin{aligned}
&[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2 \sum_{k=2}^{\infty} a_k \\
&\qquad \leq \sum_{k=2}^{\infty} \{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\}\, a_k b_k \leq 1-\alpha, \qquad (3.4)
\end{aligned}
$$

that is, that

$$
\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2}. \qquad (3.5)
$$

From (1.8) and (3.5), we have

$$
|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - \frac{1-\alpha}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2} r^2
$$

and

$$
|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k \leq r + \frac{1-\alpha}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2} r^2.
$$

This completes the proof of Theorem 3. ■

Theorem 4. *Let the function $f(z)$ of the form* (1.8) *be in the class $TS(g,\lambda;\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$
|f'(z)| \geq r - \frac{2(1-\alpha)}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2} r \qquad (3.6)
$$

and

$$
|f'(z)| \leq r + \frac{2(1-\alpha)}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2} r, \qquad (3.7)
$$

provided that $b_{k+1} \geq b_k > 0 \quad (k \geq 2)$. The result is sharp for the function $f(z)$ given by (3.3).

Proof. From Theorem 2 and (3.5), we have

$$
\sum_{k=2}^{\infty} k a_k \leq \frac{2(1-\alpha)}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\, b_2},
$$

and the remaining part of the proof is similar to the proof of Theorem 3. ■

4. Convex linear combinations

Theorem 5. *Let $\mu_\upsilon \geq 0$ for $\upsilon = 1,2,\ldots,\ell$ and $\sum_{\upsilon=1}^{\ell} \mu_\upsilon \leq 1$. If the functions $F_\upsilon(z)$ defined by*

$$
F_\upsilon(z) = z - \sum_{k=2}^{\infty} a_{k,\upsilon} z^k \quad (a_{k,\upsilon} \geq 0;\ \upsilon = 1,2,\ldots,\ell), \qquad (4.1)
$$

are in the class $TS(g,\lambda;\alpha,\beta)$ for every $\upsilon = 1,2,\ldots,\ell$, then the function $f(z)$ defined by

$$
f(z) = z - \sum_{k=2}^{\infty} \left(\sum_{\upsilon=1}^{\ell} \mu_\upsilon a_{k,\upsilon} \right) z^k
$$

is in the class $TS(g,\lambda;\alpha,\beta)$.

Proof. Since $F_\upsilon(z) \in TS(g,\lambda;\alpha,\beta)$, it follows from Theorem 2 that

$$\sum_{k=2}^{\infty} \{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} a_{k,\upsilon} b_k \le 1-\alpha, \tag{4.2}$$

for every $\upsilon = 1, 2, \ldots, \ell$. Hence

$$\begin{aligned} &\sum_{k=2}^{\infty} \{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} \left(\sum_{\upsilon=1}^{\ell} \mu_\upsilon a_{k,\upsilon}\right) b_k \\ &\quad = \sum_{\upsilon=1}^{\ell} \mu_\upsilon \left(\sum_{k=2}^{\infty} \{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} a_{k,\upsilon} b_k\right) \\ &\le (1-\alpha) \sum_{\upsilon=1}^{\ell} \mu_\upsilon \le 1-\alpha. \end{aligned}$$

By Theorem 2, it follows that $f(z) \in TS(g,\lambda;\alpha,\beta)$. ■

Corollary 5. *The class $TS(g,\lambda;\alpha,\beta)$ is closed under convex linear combinations.*

Theorem 6. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k} z^k \quad (k \ge 2). \tag{4.3}$$

Then $f(z)$ is in the class $TS(g,\lambda;\alpha,\beta)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{4.4}$$

where $\mu_k \ge 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k} \mu_k z^k. \tag{4.5}$$

Then it follows that

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\}b_k}{1-\alpha} \cdot \frac{1-\alpha}{\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\}b_k} \mu_k \\ &\qquad = \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \le 1. \end{aligned} \tag{4.6}$$

So, by Theorem 2, $f(z) \in TS(g,\lambda;\alpha,\beta)$.

Conversely, assume that the function $f(z)$ defined by (1.8) belongs to the class $TS(g,\lambda;\alpha,\beta)$. Then

$$a_k \le \frac{1-\alpha}{\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k} \quad (k \ge 2). \tag{4.7}$$

Setting

$$\mu_k = \frac{\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} a_k b_k}{1-\alpha} \quad (k \geq 2), \tag{4.8}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \tag{4.9}$$

we can see that $f(z)$ can be expressed in the form (4.4). This completes the proof of Theorem 6. ■

Corollary 6. *The extreme points of the class $TS(g,\lambda;\alpha,\beta)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k} z^k \ (k \geq 2). \tag{4.10}$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 7. *Let the function $f(z)$ defined by (1.8) be in the class $TS(g,\lambda;\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = \inf_{k \geq 2} \left\{ \frac{(1-\rho)\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k}{k(1-\alpha)} \right\}^{\frac{1}{k-1}}. \tag{5.1}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho \ \text{ for } |z| < r_1,$$

where r_1 is given by (5.1). Indeed we find from (1.8) that

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \rho$, if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k |z|^{k-1} \leq 1. \tag{5.2}$$

But, by Theorem 2, (5.2) will be true if

$$\left(\frac{k}{1-\rho} \right) |z|^{k-1} \leq \frac{\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k}{1-\alpha},$$

that is, if

$$|z| \leq \left\{ \frac{(1-\rho)\{k(1+\beta) - (\alpha+\beta)[1+\lambda(k-1)]\} b_k}{k(1-\alpha)} \right\}^{\frac{1}{k-1}} \quad (k \geq 2). \tag{5.3}$$

Theorem 7 follows easily from (5.3). ■

Theorem 8. *Let the function $f(z)$ defined by (1.8) be in the class $TS(g,\lambda;\alpha,\beta)$. Then $f(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_{k\ge 2}\left\{\frac{(1-\rho)\left\{k(1+\beta)-(\alpha+\beta)\left[1+\lambda(k-1)\right]\right\}b_k}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{5.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho \quad \text{for } |z|<r_2,$$

where r_2 is given by (5.4). Indeed we find from (1.8) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho$, if

$$\sum_{k=2}^{\infty}\frac{(k-\rho)a_k|z|^{k-1}}{1-\rho} \le 1. \tag{5.5}$$

But, by Theorem 2, (5.5) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{1-\rho} \le \frac{\left\{k(1+\beta)-(\alpha+\beta)\left[1+\lambda(k-1)\right]\right\}b_k}{1-\alpha},$$

that is, if

$$|z| \le \left\{\frac{(1-\rho)\left\{k(1+\beta)-(\alpha+\beta)\left[1+\lambda(k-1)\right]\right\}b_k}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}} \quad (k\ge 2). \tag{5.6}$$

Theorem 8 follows easily from (5.6). ∎

Corollary 7. *Let the function $f(z)$ defined by (1.8) be in the class $TS(g,\lambda;\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_{k\ge 2}\left\{\frac{(1-\rho)\left\{k(1+\beta)-(\alpha+\beta)\left[1+\lambda(k-1)\right]\right\}b_k}{k(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

6. A family of integral operators

In view of Theorem 2, we see that $z-\sum_{k=2}^{\infty} d_k z^k$ is in the class $TS(g,\lambda;\alpha,\beta)$ as long as $0 \le d_k \le a_k$ for all k. In particular, we have:

Theorem 9. *Let the function $f(z)$ defined by (1.8) be in the class $TS(g,\lambda;\alpha,\beta)$ and c be a real number such that $c>-1$. Then the function $F(z)$ defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \quad (c>-1)\,, \tag{6.1}$$

also belongs to the class $TS(g,\lambda;\alpha,\beta)$.

Proof. From the representation (6.1) of $F(z)$, it follows that

$$F(z)=z-\sum_{k=2}^{\infty} d_k z^k,$$

where

$$d_k=\left(\frac{c+1}{c+k}\right)a_k \le a_k \quad (k\ge 2)\,.$$

On the other hand, the converse is not true. This leads to a radius of univalence result. ∎

Theorem 10. *Let the function $F(z)=z-\sum_{k=2}^{\infty} a_k z^k$ $(a_k\ge 0)$ be in the class $TS(g,\lambda;\alpha,\beta)$, and let c be a real number such that $c>-1$. Then the function $f(z)$ given by* (6.1) *is univalent in $|z|<R^*$, where*

$$R^*=\inf_{k\ge 2}\left\{\frac{(c+1)\left\{k\,(1+\beta)-(\alpha+\beta)\left[1+\lambda\,(k-1)\right]\right\}b_k}{k\,(c+k)\,(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{6.2}$$

The result is sharp.

Proof. From (6.1), we have

$$f(z)=\frac{z^{1-c}\left|z^c F(z)\right|'}{c+1}=z-\sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right)a_k z^k.$$

In order to obtain the required result, it suffices to show that

$$|f'(z)-1|<1 \quad \text{wherever} \quad |z|<R^*,$$

where R^* is given by (6.2). Now

$$|f'(z)-1|\le\sum_{k=2}^{\infty}\frac{k\,(c+k)}{(c+1)}a_k|z|^{k-1}.$$

Thus $|f'(z)-1|<1$ if

$$\sum_{k=2}^{\infty}\frac{k\,(c+k)}{(c+1)}a_k|z|^{k-1}<1. \tag{6.3}$$

But Theorem 2 confirms that

$$\sum_{k=2}^{\infty} \frac{\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\} a_k b_k}{1-\alpha} \leq 1. \tag{6.4}$$

Hence (6.3) will be satisfied if

$$\frac{k(c+k)}{(c+1)}|z|^{k-1} < \frac{\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\} b_k}{1-\alpha},$$

that is, if

$$|z| < \left\{\frac{(c+1)\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\} b_k}{k(c+k)(1-\alpha)}\right\}^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.5}$$

Therefore, the function $f(z)$ given by (6.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(c+k)(1-\alpha)}{(c+1)\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\} b_k} z^k \quad (k \geq 2). \quad \blacksquare \tag{6.6}$$

7. Partial sums

Following the earlier works by Silverman [16] and Siliva [17] on partial sums of analytic functions, we consider in this section partial sums of functions in the class $TS(g,\lambda;\alpha,\beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f'_n(z)$.

Theorem 11. *Define the partial sums $f_1(z)$ and $f_n(z)$ by*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z + \sum_{k=2}^{n} a_k z^k, \quad (n \in \mathbb{N} \setminus \{1\}).$$

Let $f(z) \in TS(g,\lambda;\alpha,\beta)$ be given by (1.8) and satisfy condition (2.2) and

$$c_k \geq \begin{cases} 1, & k = 2,3,\ldots,n, \\ c_{n+1}, & k = n+1, n+2, \ldots, \end{cases} \tag{7.1}$$

where, for convenience,

$$c_k = \frac{\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\} b_k}{1-\alpha}. \tag{7.2}$$

Then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{c_{n+1}} \quad (z \in \mathbb{U}; n \in \mathbb{N}), \tag{7.3}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{n+1}}{1+c_{n+1}}. \tag{7.4}$$

Proof. For the coefficients c_k given by (7.2) it is not difficult to verify that

$$c_{k+1} > c_k > 1. \tag{7.5}$$

Therefore we have

$$\sum_{k=2}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=2}^{\infty} c_k a_k \leq 1. \tag{7.6}$$

By setting

$$g_1(z) = c_{n+1} \left\{ \frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right) \right\} = 1 + \frac{c_{n+1} \sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{n} a_k z^{k-1}}, \tag{7.7}$$

and applying (7.6), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{c_{n+1} \sum\limits_{k=n+1}^{\infty} a_k}{2 - 2\sum\limits_{k=2}^{n} a_k - c_{n+1} \sum\limits_{k=n+1}^{\infty} a_k}. \tag{7.8}$$

Now $\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1$ if

$$\sum_{k=2}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq 1.$$

From condition (2.2), it is sufficient to show that

$$\sum_{k=2}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=2}^{\infty} c_k a_k$$

which is equivalent to

$$\sum_{k=2}^{n} (c_k - 1)\, a_k + \sum_{k=n+1}^{\infty} (c_k - c_{n+1})\, a_k \geq 0 \tag{7.9}$$

which readily yields the assertion (7.3) of Theorem 11. In order to see that

$$f(z) = z + \frac{z^{n+1}}{c_{n+1}} \tag{7.10}$$

gives sharp result, we observe that for $z = r\, e^{\frac{i\pi}{n}}$ that $\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{c_{n+1}} \to 1 - \frac{1}{c_{n+1}}$ as $z \to 1^-$. Similarly, if we take

$$g_2(z) = (1 + c_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1 + c_{n+1}} \right\} = 1 - \frac{(1 + c_{n+1}) \sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{\infty} a_k z^{k-1}}, \tag{7.11}$$

and making use of (7.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+c_{n+1})\sum\limits_{k=n+1}^{\infty} a_k}{2-2\sum\limits_{k=2}^{n} a_k-(1-c_{n+1})\sum\limits_{k=n+1}^{\infty} a_k} \tag{7.12}$$

which leads us immediately to the assertion (7.4) of Theorem 11.

The bound in (7.4) is sharp for each $n \in \mathbb{N}$ with the extremal function $f(z)$ given by (7.10). The proof of Theorem 11 is thus completed. ∎

THEOREM 12. *If $f(z)$ of the form* (1.8) *satisfies condition* (2.2), *then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq 1-\frac{n+1}{c_{n+1}}, \tag{7.13}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{n+1+c_{n+1}} \tag{7.14}$$

where c_k is defined by (7.2) *and satisfies the condition*

$$c_k \geq \begin{cases} k, & k=2,3,\ldots,n, \\ \frac{kc_{n+1}}{n+1}, & k=n+1,n+2,\ldots \end{cases} \tag{7.15}$$

The results are sharp with the function $f(z)$ given by (7.10).

Proof. By setting

$$\begin{aligned} g(z) &= \frac{c_{n+1}}{n+1}\left\{\frac{f'(z)}{f_n'(z)}-\left(1-\frac{n+1}{c_{n+1}}\right)\right\} \\ &= 1+\frac{1+\frac{c_{n+1}}{n+1}\sum\limits_{k=n+1}^{\infty} ka_k z^{k-1}+\sum\limits_{k=2}^{n} ka_k z^{k-1}}{1+\sum\limits_{k=2}^{n} ka_k z^{k-1}}, \\ &= 1+\frac{\frac{c_{n+1}}{n+1}\sum\limits_{k=n+1}^{\infty} ka_k z^{k-1}}{1+\sum\limits_{k=2}^{n} ka_k z^{k-1}}, \end{aligned} \tag{7.16}$$

we obtain

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{c_{n+1}}{n+1}\sum\limits_{k=n+1}^{\infty} ka_k}{2-2\sum\limits_{k=2}^{n} ka_k-\frac{c_{n+1}}{n+1}\sum\limits_{k=n+1}^{\infty} ka_k}. \tag{7.17}$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$ if

$$\sum_{k=2}^{n} ka_k+\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} ka_k \leq 1, \tag{7.18}$$

since the left-hand side of (7.18) is bounded above by $\sum_{k=2}^{\infty} c_k a_k$ if

$$\sum_{k=2}^{n} (c_k - k)\, a_k + \sum_{k=n+1}^{\infty} \left(c_k - \tfrac{c_{n+1}}{n+1} k\right) a_k \geq 0 \tag{7.19}$$

and the proof of (7.13) is completed.

To prove result (7.14), define the function $g(z)$ by

$$g(z) = \left(\frac{n+1+c_{n+1}}{n+1}\right) \left\{ \frac{f_n'(z)}{f'(z)} - \frac{c_{n+1}}{n+1+c_{n+1}} \right\} = 1 - \frac{\left(1 + \tfrac{c_{n+1}}{n+1}\right) \sum\limits_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum\limits_{k=2}^{\infty} k a_k z^{k-1}},$$

and making use of (7.19), we deduce that

$$\left| \frac{g(z)-1}{g(z)+1} \right| \leq \frac{\left(1 + \tfrac{c_{n+1}}{n+1}\right) \sum\limits_{k=n+1}^{\infty} k a_k}{2 - 2\sum\limits_{k=2}^{n} k a_k - \left(1 - \tfrac{c_{n+1}}{n+1}\right) \sum\limits_{k=n+1}^{\infty} k a_k} \leq 1,$$

which leads us immediately to the assertion (7.14) of Theorem 12. ■

Acknowledgement. The authors would like to thank the referees of the paper for their helpful suggestions.

REFERENCES

[**1**] M.K. Aouf, A.O. Mostafa, *Some properties of a subclass of uniformly convex functions with negative coefficients*, Demonstratio Math. **2** (2008), 353–370.

[**2**] R. Bharati, R. Parvatham, A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamakang J. Math. **28** (1997), 17–32.

[**3**] A. Cătaş, G.I. Oros, G. Oros, *Differential subordinations associated with multiplier transformations*, Abstract Appl. Anal. (2008), Article ID845724, 11 pages.

[**4**] J. Dziok, H.M. Srivastava, *Classes of analytic functions with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[**5**] J. Dziok, H.M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct. **14** (2003), 7–18.

[**6**] A.W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), 87–92.

[**7**] A.W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155** (1991), 364–370.

[**8**] G. Murugusundaramoorthy, N. Magesh, *A new subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient*, J. Inequal. Pure Appl. Math. **5:4** (2004), Art. 85, 1–10.

[**9**] G. Murugusundaramoorthy, N. Magesh, *Linear operators associated with a subclass of uniformly convex functions*, Internat. J. Pure Appl. Math. Sci. **3** (2006), 113–125.

[**10**] G. Murugusundaramoorthy, N. Magesh, *On certain subclasses of analytic functions associated with hypergeometric functions*, Appl. Math Lett. **24** (2011), 494–500.

[**11**] F. Ronning, *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae-Curie-Sklodowska, Sect. A **45** (1991), 117–122.

[**12**] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 189–196.

[**13**] T. Rosy, G. Murugusundaramoorthy, *Fractional calculus and their applications to certain subclass of uniformly convex functions*, Far East J. Math. Sci. **15** (2004), 231–242.

[**14**] S. Shams, S.R. Kulkarni, J.M. Jahangiri, *Classes of uniformly starlike and convex functions*, Internat. J. Math. Math. Sci. 55 (2004), 2959–2961.

[**15**] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[**16**] H. Silverman, *Partial sums of starlike and convex functions* J. Math. Anal. Appl. **209** (1997), 221–227.

[**17**] E.M. Silvia, *On partial sums of convex functions of order* α, Houston J. Math. **11** (1985), 397–404.

[**18**] H.M. Srivastava, A.A. Attiya, *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct. **18** (2007), 207–216.

(received 13.03.2011; in revised form 01.04.2012; available online 01.05.2012)

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail: mkaouf127@yahoo.com, aashamandy@hotmail.com, adelaeg254@yahoo.com, awagdyfos@yahoo.com