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A CLASS OF CAYLEY GRAPHS INDUCED BY RIGHT SOLVABLE WARD GROUPOIDS

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Abstract. In this paper, we introduce a class of Cayley graphs induced by right solvable Ward groupoids. Thisi class of Cayley graphs can be considered as a generalization of Cayley graphs induced by groups. Also, many graph properties are expressed in terms of algebraic properties. This did not attract much attention in the literature.

1. Introduction

A relation R on a set G is a subset of the cartesian product $G \times G$. A relation R on a set G is said to be *reflexive* if $(a, a) \in R$ for all $a \in S$ and *symmetric* if, $(a, b) \in R$ implies $(b, a) \in R$. Graphs come in to two principal types: directed graphs and non directed graphs. We shall refer to directed graphs as digraphs and use the term graph to refer to undirected graphs. The following is a list of formal definitions.

A *digraph* $\mathcal{G}$ is an ordered pair (G, R), where G is a nonempty set and R is a relation on G. The elements of G are called *vertices* and the elements of R are called *edges*. If $(x, y) \in R$, then the edge (x, y) is said to join x and y, and x is *adjacent* to y. An edge of the form (x, x) is called a *loop*. A *graph* is a digraph with no edges of the form (x, x) and with the property that $(x, y) \in R$ implies $(y, x) \in R$. That is, a digraph (G, R) is said to be a graph if the relation R is symmetric and non reflexive. A graph (G, R) is said to be vertex-transitive if its automorphism group acts transitively upon its vertices [1, 2, 3, 7, 8, 9]. A graph (G, R) is called a *Hasse-diagram* if for every positive integer $n \geq 2$ and every $x_0, x_1, \ldots, x_n$ of G, $(x_i, x_{i+1}) \in R$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(x_0, x_n) \notin R$ [13].

Let G be a group and let D be a subset of G. The *Cayley digraph* $\vec{X} = \vec{C}(G, D)$ is the digraph with vertex set G, and the vertex x is adjacent to the vertex y if and only if $x^{-1}y \in D$ [13]. The subset D is called the connection set of $\vec{X}$. That is,

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Cayley digraph $\vec{C}(G;D)$ has as its vertex-set and edge-set, respectively, G and

$$\begin{aligned} R &= \{x_z : x \in G \text{ and } z \in D\} = \{(x, xz) : z \in D\} \\ &= \{(x,y) : y = xz \text{ for some } z \in D\}. \end{aligned}$$

An algebraic structure $(G, .)$ is called a *quasigroup*, if for every $a, b \in G$, the equations, $ax = b$ and $ya = b$ are uniquely solvable in G. This implies both right and left cancelation laws [6]. In the terminology of [10], an algebraic structure $(G, .)$ is said to be a right solvable Ward groupoid if and only if for any $a, b, c \in G$ there is an element $x \in G$ such that $ax = b$ and the following identity holds:

$$(ac)(bc) = ab.$$

In [11], it is proved that a right solvable Ward groupoid is a quasigroup. Let A be a subset of a right solvable Ward groupoid G. We say that A is $\mathcal{R}$ associative (or right associative), if

$$(xy)A = x(yA) \quad \text{for every } x, y \in G.$$

This means, if $x, y \in G$ and $a \in A$, then $(xy)a = x(ya')$ for some $a' \in A$. Observe that the $\mathcal{R}$ associative law not only allows to interchange the positions of parenthesis, the left most two elements should be in G and they will be same on both sides, the rightmost element in the left hand side is in A and is changed to another element $a' \in A$ as the right most element in the right side.

Here we have the following

Lemma 1.1. [11] *In any right solvable Ward groupoid G there is the unique determined element o such that the following identities hold:*

$$aa = o \qquad ao = a.$$

2. Main theorem

In this section, we prove that a bigger class of Cayley graphs can be induced by Ward groupoids. These graphs can be considered as the generalization of Cayley digraphs induced by groups. Moreover interesting results are obtained between the properties of graphs and those of Ward groupoids.

Theorem 2.1. *Let G be a right solvable Ward groupoid and let Δ be a $\mathcal{R}$ associative subset of G. Let*

$$R_\Delta = \{(x,y) \in G \times G : z \in \Delta\}$$

where z denotes the solution of the equation $y = xz$ in G. Then we have the following:

(a) *(G, R_Δ) is a vertex-transitive graph.*

(b) (G, R_Δ) *is an undirected graph.*

(c) (G, R_Δ) *is a regular graph of degree* $|\Delta|$.

Proof. (a) First, we will show that the graph (G, R_Δ) is a vertex-transitive graph.

Let a and b be any two arbitrary elements in G. Define a mapping $\theta : G \longrightarrow G$ by

$$\theta(x) = b/a(x)$$

where b/a denote the solution z of the equation $b = za$. One can easily verify that the map θ is bijective. Furthermore, for every $x, y \in G$ with $y = xz$, if $(x, y) \in R_\Delta$, then $z \in \Delta$. This implies that

$$(b/a)y = ((b/a)x)z' \text{ for some } z' \in \Delta.$$

This equation tells us that $(\theta(x), \theta(y)) \in R_\Delta$. Conversely, if $(\theta(x), \theta(y)) \in R_\Delta$, one can prove that $(x, y) \in R_\Delta$. Hence, the map θ is a graph automorphism of G. Finally,

$$\theta(a) = (b/a)(a) = b$$

(b) Let $y = xz$. If $(x, y) \in R_\Delta$, then $z \in \Delta$. Consider the equation $x = yz'$. We will show that $z' \in \Delta$. Observe that $y = (yz')z$. Since Δ is $\mathcal{R}$ associative, there exits some $z'' \in \Delta$ such that $y = y(z'z'')$. From Lemma 1.1, it follows that $yo = y(z'z'')$. By left cancelation law, we obtain $o = z'z''$. Therefore, by Lemma 1.1, $z' = z''$. This tells us that $(y, x) \in R_\Delta$. Hence the graph (G, R_Δ) is an undirected graph.

(c) Since the graph is vertex-transitive, it suffices to consider the degree of the vertex o. Let

$$\begin{aligned} \rho(o) &= \{x \in G : \{o, x\} \in R_\Delta\} \\ x \in \rho(0) &\Leftrightarrow \{o, x\} \in R_\Delta \\ &\Leftrightarrow x = z \text{ for some } z \in \Delta. \end{aligned}$$

This implies that $|\rho(0)| = |\Delta|$ ■

Corollary 2.2. (G, R_Δ) *is empty* $\Leftrightarrow R_\Delta = \emptyset$.

Corollary 2.3. (G, R_Δ) *is a reflexive graph (each vertex has a loop)* $\Leftrightarrow o \in \Delta$.

Corollary 2.4. (G, R_Δ) *is a complete graph* $(R_\Delta = G \times G) \Leftrightarrow G = \Delta$.

Corollary 2.5. (G, R_Δ) *is a transitive graph* $(R_\Delta \circ R_\Delta \subseteq R_\Delta) \Leftrightarrow \Delta^2 \subseteq \Delta$.

Corollary 2.6. (G, R_Δ) *is a connected graph if and only if* $G = [\Delta]$, *where* $[\Delta]$ *denotes the set of all finite products* $z_1 z_2 z_3 \dots z_n$ *of elements of* Δ.

Corollary 2.7. (G, R_Δ) *is a Hasse-diagram* $\Leftrightarrow \Delta^n \cap \Delta = \emptyset$ *for all* $n \geq 2$.

Proof. Assume that (G, R_Δ) is a Hasse-diagram. Then for every $x_0, x_1, \dots, x_n \in G$ with $(x_i, x_{i+1}) \in R_\Delta$ for $i = 0, 1, 2, \dots, n-1$ implies $(x_0, x_n) \notin R_\Delta$. Then by the definition of R_Δ, we have

$$x_1 = x_0 z_1,\ x_2 = x_1 z_2,\ \dots,\ x_n = x_{n-1} z_n$$

for some $z_i \in \Delta$, $i = 1, 2, \dots, n$. That is,

$$x_n = x_0 z_1' z_2' \dots z_n'$$

for some $z_i' \in \Delta$, $i = 1, 2, \dots, n$. That is, $x_n = x_0 z$, where $z = z_1' z_2' \dots z_n' \in (\Delta)^n$. Since, $(x_0, x_n) \notin R_A$, therefore, $\Delta^n \cap \Delta = \emptyset$.

Conversely, assume that $\Delta^n \cap \Delta = \emptyset$, for $n \geq 2$. We will show that (G, R_Δ) is a Hasse-diagram. Let $x_0, x_1, \dots, x_n$ any $(n+1)$ elements of G with $n \geq 2$ and $(x_i, x_{i+1}) \in R_\Delta$ for $i = 0, 1, \dots, n-1$. Then we have

$$x_n = x_0 z_1 z_2 \dots z_n$$

for some $z_i \in \Delta$, $i = 1, 2, \dots, n$. Since, $\Delta^n \cap \Delta = \emptyset$, therefore, $(x_0, x_n) \notin R_\Delta$. Hence (G, R_Δ) is a Hasse-diagram. ■

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