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FUNCTIONS OF CLASS $H(\alpha, p)$ AND TAYLOR MEANS

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Abstract. In this paper, we take up Taylor means to study the degree of approximation of $f \in H(\alpha, p)$ space in the generalized Hölder metric and obtain a general theorem which is used to obtain a few more results that improve upon some earlier results obtained by Mohapatra, Holland and Sahney [J. Approx. Theory 45 (1985), 363–374] in L_p-norm, Mohapatra and Chandra [Math. Chronicle 11 (1982), 89–96] in Hölder metric and Chui and Holland [J. Approx. Theory 39 (1983), 24–38] in sup-norm.

1. Definitions and notations

Let f be 2π-periodic and let $f \in L_p[0, 2\pi]$ for $p \geqslant 1$. Let $s_n(f; x)$ be the partial sum of the Fourier series of f at x, i.e.,

$$s_n(f; x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

The space $L_p[0, 2\pi]$ with $p = \infty$ includes the space $C_{2\pi}$ of all 2π-periodic continuous functions over $[0, 2\pi]$. Throughout, all norms are taken with respect to x and we write for $1 \leqslant p \leqslant \infty$,

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right\}^{1/p} \qquad (1 \leqslant p < \infty),$$
$$\|f\|_\infty = \|f\|_c = \sup_{0 \leqslant x \leqslant 2\pi} |f(x)|.$$

For the convenience in the working, we also write $\|f(x)\|_p$ for $\|f\|_p$ $(1 \leqslant p \leqslant \infty)$.

Let $\omega(\delta; f)$, $\omega_p(\delta; f)$ and $\omega_p^{(2)}(\delta; f)$ denote, respectively, the modulus of continuity, integral modulus of continuity and integral modulus of smoothness which are non-negative and non-decreasing (see [15, pp. 42 and 45] and [7, p. 612]). In the case $0 < \alpha \leqslant 1$ and $\omega(\delta; f) = O(\delta^\alpha)$, we write $f \in Lip\,\alpha$ and if $\omega_p(\delta; f) = O(\delta^\alpha)$, we write $f \in Lip(\alpha, p)$. Also, if either

$$\omega_p(\delta; f) = o(\delta) \quad \text{or} \quad \omega(\delta; f) = o(\delta), \quad \text{as } \delta \to 0, \tag{1.1}$$

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holds then the function f turns out to be constant ([15, p. 45]). Further, the class $Lip(\alpha, p)$ with $p = \infty$ will be taken as $Lip\,\alpha$.

In 1996, Das, Ghosh and Ray [4] gave the following generalization of Hölder metric (see [14]).

For $0 < \alpha \leqslant 1$ and a positive constant K, define

$$H(\alpha, p) = \{\, f \in L_p : \|f(x+y) - f(x)\|_p \leqslant K|h|^\alpha \,\}, \quad 1 \leqslant p \leqslant \infty,$$

and introduce the following metric for $\alpha \geqslant 0$:

$$\left.\begin{aligned} &(i) \quad \|f\|_{(\alpha,p)} = \|f\|_p + \sup_{h \neq 0} \frac{\|f(x+h) - f(x)\|_p}{|h|^\alpha}, \quad \alpha > 0, \\ &(ii) \quad \|f\|_{(0,p)} = \|f\|_p, \quad \alpha = 0. \end{aligned}\right\} \tag{1.2}$$

It can be easily verified that (1.2) is a norm for $p \geqslant 1$ and that $H(\alpha, p)$ is a Banach space for $p \geqslant 1$. See also Lasuriya [10].

$H(\alpha, \infty)$ is the familiar H_α-space introduced by Prösdorff [14] and it is a Banach space with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x \neq y} \Delta^a f(x, y),$$

where

$$\Delta^a f(x, y) = \begin{cases} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha}, & \alpha > 0, \\ 0, & \alpha = 0. \end{cases}$$

Let (a_{nk}) be an infinite matrix defined by

$$\frac{(1-r)^{n+1}\theta^n}{(1-r\theta)^{n+1}} = \sum_{k=0}^{\infty} a_{nk}\theta^k, \qquad |r\theta| < 1, \quad n = 0, 1, \ldots \infty.$$

Then the Taylor mean of $(s_n(f; x))$ is given by

$$T_n^r(f; x) = \sum_{k=0}^{\infty} a_{nk} s_n(f; x), \tag{1.3}$$

whenever the series on the right is convergent for each $n = 0, 1, 2, \ldots$. See Miracle [11].

In this paper, we shall use the following notations for $0 < r < 1$, $0 < t \leqslant \pi$ and for real x and y:

$$\phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\},$$

$$L_\phi(t) = \phi_x(t) - \phi_{x+y}(t),$$

$$B = \frac{r}{2(1-r)^2}, \quad h = (1-r)\sqrt{1 + 8B\sin^2 \tfrac{1}{2}t}, \tag{1.4}$$

$$1 - r\exp(it) = h\exp(i\theta), \quad \theta = \tan^{-1}\Big\{\frac{r\sin t}{1 - r\cos t}\Big\}, \tag{1.5}$$

$$L(n, r, t, \theta) = \{(1-r)/h\}^{n+1} \sin\{(n + \tfrac{1}{2})t + (n+1)\theta\}, \tag{1.6}$$

$$a_n = \pi \Big/ \Big\{(n + \tfrac{1}{2}) + (n+1)\frac{r}{1-r}\Big\} \quad \text{and} \quad b_n = a_n^\delta, \quad 0 < \delta < \tfrac{1}{2},$$

$$c_n = (1-r)\pi/n \quad \text{and} \quad d_n = \sqrt{\frac{\log n}{An}}, \quad A > 0, \tag{1.7}$$

$$R_n = \int_{c_n}^{d_n} t^{-1} \|\phi_x(t) - \phi_x(t+c_n)\|_p \exp(-Bnt^2)\, dt. \tag{1.8}$$

Similarly, define $\tilde{I}_n$ by R_n with c_n and d_n replaced by a_n and b_n, respectively. We also use the following inequality:

$$t \leqslant \pi \sin \tfrac{1}{2} t, \quad 0 \leqslant t \leqslant \pi. \tag{1.9}$$

2. Introduction and formulation of results

Throughout, we assume $f \in L_p$ $(1 \leqslant p \leqslant \infty)$ is non-constant and hence $\delta^{-1}\omega_p(\delta; f) \nrightarrow 0$ as $\delta \to 0$. Otherwise, by (1.1), f turns out to be a constant function in which case there is nothing to prove. This enables us to write

$$n^{-1} = O(1)\omega_p(n^{-1}; f) \quad (n \to \infty).$$

In 1982, Mohapatra and Chandra [12] used Taylor transform $T_n^r(f; x)$ to approximate $f \in H_\alpha$-space and obtained the following

Theorem A. *Let $0 \leqslant \beta < \alpha \leqslant 1$. Then for $f \in H_\alpha$,*

$$\|T_n^r(f) - f\|_p = O\{n^{-1/2(\alpha-\beta)} \log^{\beta/\alpha}(n+1)\}.$$

The case $\beta = 0$ of Theorem A yield the following

Corollary 1. *Let $f \in C_{2\pi} \cap Lip\,\alpha$, where $0 < \alpha \leqslant 1$. Then $\|T_n^r(f) - f\|_c = O(n^{-\alpha/2})$.*

With a view to obtain the Jackson order for the degree of approximation of f by Taylor transform $T_n^r(f; x)$, Chui and Holland [3] proved the following

Theorem B. *Let $f \in C_{2\pi} \cap Lip\,\alpha$ $(0 < \alpha < 1)$ and let*

$$\int_{a_n}^{b_n} \frac{\|\phi_x(t) - \phi_x(t+a_n)\|_c}{t} \exp(-Bnt^2)\, dt = O(n^{-\alpha}), \tag{2.1}$$

where $(1+\alpha)/(3+\alpha) \leqslant \delta < 1/2$. Then $\|T_n^r(f) - f\|_c = O(n^{-\alpha})$.

They further remarked that since the Lebesgue constants for the Taylor method diverge as $n \to \infty$; therefore, in order to get the degree of convergence of Jackson order $O(n^{-\alpha})$, $f \in Lip\,\alpha$ alone is not adequate. Also, we observe that the restriction on δ does not allow them to consider $\alpha = 1$ in (2.1).

By using the Taylor transform of $s_n(f; x)$, a study has been made to obtain the rate of convergence to f in L_p-norm [8, p. 371]. In 1985, Mohapatra, Holland and Sahney [13] obtained a number of results by using Taylor transform. We mention here the following results for the subspaces of L_p space $(p > 1)$.

THEOREM C. *Let $f \in Lip(\alpha, p)$, where $0 < \alpha \leqslant 1$ and $p > 1$. Then*

$$\|T_n^r(f) - f\|_p = O(n^{-\alpha\delta}) \quad (0 < \delta < \tfrac{1}{2}). \tag{2.2}$$

THEOREM D. *Let $f \in Lip(\alpha, p)$, $0 < \alpha < 1$, $p > 1$ and let*

$$\tilde{I}_n = O(n^{-\alpha}), \tag{2.3}$$

where $(1 + \alpha)/(3 + \alpha) \leqslant \delta < 1/2$. Then $\|T_n^r(f) - f\| = O(n^{-\alpha})$.

Analogous to a result of Izumi [9], they proved in [6] the following

THEOREM E. *If $f \in Lip(\alpha, p)$, $0 < \alpha \leqslant 1$, $p > 1$ and $\alpha p > 1$, then*

$$T_n^r(f; x) - f(x) = O(n^{-(\alpha - 1/p)\delta}) \quad (0 < \delta < \tfrac{1}{2}),$$

uniformly in x almost everywhere.

Motivated by the results obtained in [1], we have recently studied in [2] the degree of approximation of functions of L_p-space and obtained a few results in L_p-norm.

In this paper, we study the degree of approximation of $f \in H(\alpha, p)$ $(0 < \alpha \leqslant 1,$ $1 \leqslant p \leqslant \infty)$ by Taylor transform $T_n^r(f; x)$ of its Fourier series in the generalized Hölder metric which is defined by

$$\|T_n^r(f) - f\|_{(\beta,p)} = \|H_n^r\|_p + \sup_{y \neq 0} \frac{\|H_n^r(x + y) - H_n^r(x)\|_p}{|y|^\beta}, \tag{2.4}$$

where $H_n^r(x) = T_n^r(f; x) - f(x)$ and $0 \leqslant \beta < \alpha \leqslant 1$.

Our Theorem 1, as special cases, yield some interesting and new results for $C_{2\pi}$, H_α and $Lip(\alpha, p)$ $(1 \leqslant p < \infty)$ spaces; some of them provide improved versions of known results obtained earlier. More precisely, we prove the following

THEOREM 1. *Let $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$. Then for $0 \leqslant \beta < \alpha \leqslant 1$,*

$$\|T_n^r(f) - f\|_{(\beta,p)} = O(1) R_n^{1-\beta/\alpha} \log^{\beta/\alpha}(n + 1) + O(g_n^\alpha(\beta)), \tag{2.5}$$

where

$$g_n^\alpha(\beta) = \begin{cases} n^{\beta-\alpha} \log^{\beta/\alpha}(n + 1), & 0 < \alpha < 1, \\ n^{\beta-1} \log(n + 1), & \alpha = 1. \end{cases} \tag{2.6}$$

We now deduce the following from Theorem 1.

THEOREM 2. *Let $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$. Then for $0 \leqslant \beta < \alpha \leqslant 1$,*

$$\|T_n^r(f) - f\|_{(\beta,p)} = O(1) n^{-\alpha+\beta} \log(n + 1). \tag{2.7}$$

We observe that for $0 < \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant \alpha/2$,

$$n^{-\alpha\delta} > n^{-\alpha+\beta} \log(n + 1) \quad (0 < \delta < 1/2) \tag{2.8}$$

and hence the estimate (2.7) of Theorem 2 for $0 \leqslant \beta \leqslant \alpha/2$ is sharper than the one obtained in (2.2) of Theorem C. For a subclass of $H(\alpha, p)$ space, we state the following theorem which, in particular, gives Jackson order.

THEOREM 3. *Let $f \in H(\alpha, p)$ for $0 < \alpha < 1$ and $1 \leqslant p \leqslant \infty$ and suppose that*

$$R_n = O(n^{-\alpha}) \quad (0 < \alpha < 1). \tag{2.9}$$

Then for $0 \leqslant \beta < \alpha < 1$,

$$\|T_n^r(f) - f\|_{(\beta,p)} = O\{n^{\beta-\alpha} \log^{\beta/\alpha}(n+1)\}.$$

REMARK. We observe that $a_n < c_n < d_n < b_n$ and

$$R_n = \int_{c_n}^{d_n} \frac{\|\phi_x(t) - \phi_x(t+c_n)\|_p}{t} \exp(-Bnt^2)\, dt + O(n^{-2}\log n). \tag{2.10}$$

Further, the integral on right of (2.10) is $\leqslant \tilde{I}_n$. Therefore, the condition (2.9) is stronger than (2.3). Thus the case $\beta = 0$ of Theorem 3, which gives Jackson order, may be compared with Theorem D.

We now give the following results for the Hölder space $H_\alpha = H(\alpha, \infty)$ defined by Prösdorff [14] in the Hölder metric.

THEOREM 4. *Let $0 \leqslant \beta < \alpha \leqslant 1$. Then for $f \in H_\alpha$,*

$$\|T_n^r(f) - f\|_\beta = O(1) n^{\beta-\alpha} \log(n+1).$$

This theorem provides sharper estimate than the one obtained in Theorem A. The case $\beta = 0$ yields the following result in sup-norm which may be compared with Corollary 1.

COROLLARY 2. *Let $f \in C_{2\pi} \cap Lip\,\alpha$ $(0 < \alpha \leqslant 1)$. Then $\|T_n^r(f) - f\|_c = O(1)n^{-\alpha}\log(n+1)$.*

Finally, we give the following result for H_α-space $(0 < \alpha < 1)$.

THEOREM 5. *Let $f \in H_\alpha$, $0 < \alpha < 1$ and let (2.9) hold with $p = \infty$. Then for $0 \leqslant \beta < \alpha \leqslant 1$,*

$$\|T_n^r(f) - f\|_\beta = O(1) n^{\beta-\alpha} \log^{\beta/\alpha}(n+1).$$

The case $\beta = 0$ of this theorem may be compared with Theorem B which holds for $0 < \alpha < 1$.

THEOREM 6. *If $f \in Lip(\alpha, p)$, $0 < \alpha \leqslant 1$, $p > 1$ and $\alpha p > 1$, then*

$$T_n^r(f; x) - f(x) = O(n^{-(\alpha - 1/p)} \log(n+1)),$$

uniformly in x almost everywhere.

Inequality (2.8) for $0 \leqslant 2\beta \leqslant \alpha - 1/p$ suggests that the above theorem provides sharper estimates than the one obtained in Theorem E.

3. Lemmas

We require the following lemmas for the proof of the theorems.

LEMMA 1. *Let $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$. Then,*

$$\|L_\phi(t)\|_p \leqslant 2\omega_p^{(2)}(t; f) = O(|t|^\alpha), \tag{3.1}$$

$$\|L_\phi(t)\|_p \leqslant 4\|f(x) - f(x+y)\|_p \leqslant 4K|y|^\alpha. \tag{3.2}$$

For its proof, one may proceed as in Lemma 1 of Das, Ghosh and Ray [4].

LEMMA 2. [5]

$$((1-r)/h)^n \leqslant \exp(-Ant^2), \quad A > 0, \ 0 \leqslant t \leqslant \pi/2, \tag{3.3}$$

$$|((1-r)/h)^n - \exp(-Bnt^2)| \leqslant Knt^4 \quad (t > 0). \tag{3.4}$$

LEMMA 3. [11] *For $0 \leqslant t \leqslant \pi/2$, $|\theta - rt/(1-r)| \leqslant Kt^3$.*

LEMMA 4. *For $0 \leqslant t \leqslant \pi/2$ and $0 < r < 1$,*

$$\left|\sin\left\{\left(n+\frac{1}{2}\right)t + (n+1)\theta\right\}\right| \leqslant \left(n+\frac{1}{2}\right)t + K(n+1)t^3 + \frac{(n+1)rt}{1-r}.$$

This is an easy consequence of Lemma 3.

LEMMA 5. [6, Theorem 5(ii), p. 627] *Suppose that $f \in Lip(\alpha, p)$, where $p \geqslant 1$, $0 < \alpha \leqslant 1$ and $\alpha p > 1$. Then f is equal to a function $g \in Lip(\alpha - 1/p)$ almost everywhere.*

LEMMA 6. [Generalized Minkowski inequality, see, e.g., Zygmund [15, p. 19].] *Let $h(x, y)$ be a function defined for $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. Then the following inequality holds*

$$\left\{\int_a^b \left|\int_c^d h(x,y)\,dy\right|^r dx\right\}^{1/r} \leqslant \int_c^d \left\{\int_a^b |h(x,y)|^r\,dx\right\}^{1/r} dy \quad (r \geqslant 1).$$

4. Proof of the theorems

4.1. *Proof of Theorem 1.* We have [13]

$$H_n^r(x) = T_n^r(f; x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{1}{2}t} L(n, r, t, \theta)\,dt,$$

by using (1.3)–(1.6). Now we write

$$\begin{aligned} H_n^r(x) - H_n^r(x+y) &= \frac{1}{\pi}\int_0^{\pi} \frac{L_\phi(t)}{\sin\frac{1}{2}t}\, L(n,r,t,\theta)\, dt \\ &= \frac{1}{\pi}\Big\{\int_0^{d_n} + \int_{d_n}^{\pi}\Big\}\left(\frac{L_\phi(t)}{\sin\frac{1}{2}t}\, L(n,r,t,\theta)\, dt\right) = I_1 + I_2, \text{ say }, \end{aligned}$$

where constant A involved in d_n, defined in (1.7), is the same as in (3.3) of Lemma 2. Then by the generalized Minkowski inequality,

$$\|H_n^r(x) - H_n^r(x+y)\|_p \leqslant \|I_1\|_p + \|I_2\|_p. \tag{4.1.1}$$

Now, splitting up the integral I_2 into $I_{2,1} = \int_{d_n}^{\pi/2}$ and $I_{2,2} = \int_{\pi/2}^{\pi}$ and by using the generalized Minkowski inequality, (3.1), (1.6), (1.9), (3.3) and proceeding as in (4.1.5) of [2], we get $\|I_{2,1}\|_p = O(n^{-1})$. And once again by the generalized Minkowski inequality, (3.1) and (1.4) and proceeding as in (4.1.2) of [2], we get $\|I_{2,2}\|_p = O(n^{-1})$. Thus, combining the obtained estimates, we get

$$\|I_2\|_p = O(n^{-1}). \tag{4.1.2}$$

Now, for c_n and d_n, defined in (1.7), we split up the integral I_1 into $I_{1,1} = \int_0^{c_n}$ and $I_{1,2} = \int_{c_n}^{d_n}$, to get

$$\|I_1\|_p \leqslant \|I_{1,1}\|_p + \|I_{1,2}\|_p. \tag{4.1.3}$$

By using once again the generalized Minkowski inequality, (1.6), (1.9), (3.1) and Lemma 4, we get $\|I_{1,1}\|_p = O(n^{-\alpha})$ and, by (1.6),

$$\begin{aligned} I_{1,2} = \frac{1}{\pi}\int_{c_n}^{d_n} \frac{L_\phi(t)}{\sin\frac{1}{2}t}\Big[\Big(\frac{1-r}{h}\Big)^{n+1} - \exp(-B(n+1)t^2)\Big] \sin\{(n+\tfrac{1}{2})t + (n+1)\theta\}\, dt \\ + \frac{1}{\pi}\int_{c_n}^{d_n} \frac{L_\phi(t)}{\sin\frac{1}{2}t} \exp(-B(n+1)t^2)\sin\{(n+\tfrac{1}{2})t + (n+1)\theta\}\, dt \\ = I_{1,2,1} + I_{1,2,2}, \text{ say}. \end{aligned}$$

Then, by the generalized Minkowski inequality, $\|I_{1,2}\|_p \leqslant \|I_{1,2,1}\|_p + \|I_{1,2,2}\|_p$.

Now, proceeding as above and using (3.4) of Lemma 2, we get

$$\|I_{1,2,1}\|_p \leqslant Kn\int_0^{d_n} t^\alpha \cdot t^3\, dt = O(n^{-1})$$

and

$$\begin{aligned} I_{1,2,2} = \frac{1}{\pi}\int_{c_n}^{d_n} \frac{L_\phi(t)}{\sin\frac{1}{2}t} \exp(-B(n+1)t^2)\sin\{(n+1)(t+\theta)\}\, dt \\ + O(1)\int_{c_n}^{d_n} |L_\phi(t)|\exp(-B(n+1)t^2)\, dt = R_1 + R_2, \text{ say}. \end{aligned}$$

Arguing as above and using (3.1) of Lemma 1, $\|R_2\|_p = O(n^{-\alpha})$ and

$$R_1 = \frac{1}{\pi}\int_{c_n}^{d_n} \frac{L_\phi(t)}{\sin\frac{1}{2}t} \exp(-Bnt^2)\sin\{n(t+\theta)\}\, dt + O(n^{-1}) = R_1' + O(n^{-1}), \text{ say}.$$

Therefore, by the generalized Minkowski inequality, $\|I_{1,2,2}\|_p = \|R_1'\|_p + O(n^{-a})$ and for $1/(1-r) = q$, we have by Lemma 3,

$$|\sin n(t+\theta) - \sin nqt| \leqslant n|\theta - rqt| \leqslant Knt^3. \tag{4.1.4}$$

Thus, arguing as above and using (4.1.4) and (3.1), we have

$$\|R_1'\|_p = O(1)d_n^{1+\alpha} + \|J\|_p, \text{ say,}$$

where

$$J = \frac{1}{\pi}\int_{c_n}^{d_n} L_\phi(t)\Big\{\operatorname{cosec}\frac{t}{2} - \frac{2}{t}\Big\}\exp(-Bnt^2)\sin nqt\,dt$$
$$+ \frac{2}{\pi}\int_{c_n}^{d_n} t^{-1}L_\phi(t)\exp(-Bnt^2)\sin nqt\,dt = J_1 + J_2, \text{ say.}$$

Now, proceeding as above and using (3.1) and $\operatorname{cosec}\frac{t}{2} - \frac{2}{t} = O(t)$ in J_1, we get

$$\|J\|_p = O(1)\int_{c_n}^{d_n} t^{1+\alpha}\exp(-Bnt^2)\,dt + \|J_2\|_p = O(1)n^{-\alpha} + \|J_2\|_p.$$

An by using transformation $t \mapsto t + c_n$, we get $\sin nq(t+c_n) = -\sin nqt$ and

$$\pi J_2 = \int_{c_n}^{d_n} \frac{L_\phi(t) - L_\phi(t+c_n)}{t}\exp(-Bnt^2)\sin nqt\,dt$$
$$+ \int_{c_n}^{d_n} \frac{L_\phi(t+c_n)}{t}\exp(-Bnt^2)\sin nqt\,dt$$
$$- \int_0^{d_n-c_n} \frac{L_\phi(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt$$
$$= \pi(J_{2,1} + J_{2,2} + J_{2,3}), \text{ say.}$$

Then by using the generalized Minkowski inequality, (1.8) and 2π-periodicity of f, we get

$$\|J_2\|_p \leqslant 2R_n + \|J_{2,2} + J_{2,3}\|_p$$

and

$$\pi(J_{2,2} + J_{2,3}) = \int_{c_n}^{d_n} \frac{L_\phi(t+c_n)}{t}\{\exp(-Bnt^2) - \exp(-Bn(t+c_n)^2)\}\sin nqt\,dt$$
$$+ c_n\int_{c_n}^{d_n} \frac{L_\phi(t+c_n)}{t(t+c_n)}\exp(-Bn(t+c_n)^2)\sin nqt\,dt$$
$$+ \int_{d_n-c_n}^{d_n} \frac{L_\phi(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt$$
$$- \int_0^{c_n} \frac{L_\phi(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt$$
$$= \pi(L_1 + L_2 + L_3 + L_4), \text{ say.}$$

Therefore, by the generalized Minkowski inequality,

$$\|J_{2,2} + J_{2,3}\|_p \leqslant \|L_1\|_p + \|L_2\|_p + \|L_3\|_p + \|L_4\|_p.$$

Now, proceeding as in [2] and using (3.1), we get for $0 < \alpha \leqslant 1$,

$$\|L_1\|_p = O(1) \int_{c_n}^{d_n} \omega_p^{(2)}(t + c_n; f) \exp(-Bnt^2)\, dt = O(n^{-\alpha}),$$

$\|L_2\|_p = O(1) g_n^\alpha(0)$, $\|L_3\|_p = O(1) d_n^{1+\alpha}$ and $\|L_4\|_p = O(1)\omega_p^{(2)}(n^{-1}; f) = O(n^{-\alpha})$. Collecting the obtained estimates, we get for $0 < \alpha \leqslant 1$,

$$\|I_1\|_p = O(1)[R_n + n^{-\alpha} + d_n^{1+\alpha} + g_n^\alpha(0)],$$

where $g_n^\alpha(\beta)$ for $0 \leqslant \beta < \alpha \leqslant 1$ is defined by (2.6). However, there exists a positive integer n_0 such that for $n \geqslant n_0$,

$$(i)\ d_n^{1+\alpha} \leqslant n^{-\alpha} = K_1 g_n^\alpha(0) \quad (0 < \alpha < 1),$$
$$(ii)\ n^{-\alpha} \leqslant d_n^{1+\alpha} = K_2 g_n^\alpha(0) \quad (\alpha = 1).$$

Hence,

$$\|I_1\|_p = O(1)[R_n + g_n^\alpha(0)]. \tag{4.1.5}$$

We now calculate $\|I_1\|_p$ and $\|I_2\|_p$ of (4.1.1) by using (3.2) in place of (3.1) of Lemma 1.

Proceeding as in $\|I_{1,1}\|_p$ of (4.1.3) and using (3.2) for (3.1) of Lemma 1, we get $\|I_{1,1}\|_p = O(|y|^\alpha)$. And by the generalized Minkowski inequality, (1.4), (1.6), (1.9) and (3.2), we get $\|I_{1,2}\|_p = O(1)|y|^\alpha \log(n + 1)$. Using these estimates in (4.1.3), we get

$$\|I_1\|_p = O(1)|y|^\alpha \log(n + 1). \tag{4.1.6}$$

Also proceeding as earlier for $\|I_2\|_p$ and using (3.2) for (3.1) of Lemma 1, we get

$$\|I_2\|_p = O(1)|y|^\alpha \frac{\log(n + 1)}{n}. \tag{4.1.7}$$

Now, for $k = 1, 2$ we write for $0 \leqslant \beta < \alpha \leqslant 1$

$$\|I_k\|_p = \|I_k\|_p^{1-\beta/\alpha} \|I_k\|_p^{\beta/\alpha} \tag{4.1.8}$$

and for $k = 1$ use (4.1.5) and (4.1.6), respectively in the first and the second factor on the right of identity (4.1.8), we get by using (2.6) that

$$\begin{aligned}\|I_1\|_p &= O(|y|^\beta)(R_n + g_n^\alpha(0))^{1-\beta/\alpha} \log^{\beta/\alpha}(n + 1)\\ &= O(|y|^\beta)(R_n^{1-\beta/\alpha} \log^{\beta/\alpha}(n + 1) + g_n^\alpha(\beta)),\end{aligned} \tag{4.1.9}$$

and for $k = 2$, use (4.1.2) and (4.1.7), respectively in the first and the second factor on the right of identity (4.1.8), we get

$$\|I_2\|_p = O(|y|^\beta) n^{-1} \log^{\beta/\alpha}(n + 1). \tag{4.1.10}$$

Hence, by using (4.1.9) and (4.1.10) in (4.1.1), we get

$$\begin{aligned}&\sup_{y\neq 0} \frac{\|H_n^r(x + y) - H_n^r(x)\|_p}{|y|^\beta}\\ &\quad = O(1)(R_n^{1-\beta/\alpha} + n^{-1}) \log^{\beta/\alpha}(n + 1) + O(1) g_n^\alpha(\beta)\\ &\quad = O(1) R_n^{1-\beta/\alpha} \log^{\beta/\alpha}(n + 1) + O(1) g_n^\alpha(\beta).\end{aligned} \tag{4.1.11}$$

Now, for estimation of $\|H_n^r\|_p$, we proceed as in the case of (4.1.1) and replace $L_\phi(t)$ by $\phi_x(t)$ and use the fact that $\|\phi_x(t)\|_p \leqslant \omega_p^{(2)}(t; f)$ to get

$$\|H_n^r\|_p = O(R_n + g_n^\alpha(0)) + O(n^{-1}) = O(R_n) + O(g_n^\alpha(0)). \tag{4.1.12}$$

Using (4.1.11) and (4.1.12) in (2.4) we get the required result (2.5).

This completes the proof of Theorem 1. ■

4.2. *Proof of Theorem 2.* The proof of Theorem 2 is to be obtained from Theorem 1 by estimating R_n involved in the statement of Theorem 1. We first observe that

$$|\phi_x(t) - \phi_x(t + c_n)| \leqslant |f(x + t + c_n) - f(x + t)| + |f(x - t - c_n) - f(x - t)|.$$

Hence, by using 2π-periodicity of f, we get

$$\|\phi_x(t) - \phi_x(t + c_n)\|_p \leqslant 2\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x + c_n) - f(x)|^p \, dx\right\}^{1/p} = O(n^{-\alpha}), \tag{4.2.1}$$

since $f \in H(\alpha, p)$ for $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$. Now using (4.2.1), we get

$$R_n = \int_{c_n}^{d_n} t^{-1}\|\phi_x(t) - \phi_x(t + c_n)\|_p \exp(-Bnt^2)\, dt = O(n^{-\alpha})\log(n + 1). \tag{4.2.2}$$

By using (4.2.2) in (2.5), we get

$$\begin{aligned}\|T_n^r(f) - f\|_{(\beta,p)} &= O(1)n^{-\alpha+\beta}\log(n + 1) + g_n^\alpha(\beta)\\ &= O(n^{-\alpha+\beta}\log(n + 1)). \quad ■\end{aligned}$$

4.3. *Proof of Theorem 3.* By using (2.9) in (2.5) we get the required result. ■

4.4. *Proof of Theorem 4.* By letting $p = \infty$ in Theorem 2, we get the required result. ■

4.5. *Proof of Theorem 5.* We get the required result by putting $p = \infty$ in Theorem 3. ■

4.6. *Proof of Theorem 6.* From Theorem 1 we get

$$\begin{aligned}T_n^r(f; x) - f(x) &= \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{1}{2}t} L(n, r, t, \theta)\, dt\\ &= \frac{1}{\pi}\left(\int_0^{d_n} + \int_{d_n}^\pi\right)\left(\frac{\phi_x(t)}{\sin\frac{1}{2}t} L(n, r, t, \theta)\, dt\right)\\ &= J_1 + J_2, \text{ say.}\end{aligned}$$

In view of Lemma 5, the hypothesis $f \in Lip(\alpha, p)$ implies that there exists a function $g \in Lip(\alpha - 1/p)$ such that $f = g$ almost everywhere. Hence, for $f \in H(\alpha, p)$, we conclude that for $0 < \alpha \leqslant 1$, $p > 1$ and $\alpha p > 1$,

$$\phi_x(t) = O(t^{\alpha-1/p}) \text{ almost everywhere.} \tag{4.6.1}$$

By using (4.6.1), (1.4), (1.6), Lemma 2 and $|\sin\theta| \leqslant 1$, we get

$$\begin{aligned}J_2 &= O(1)\int_{d_n}^{\pi} t^{(\alpha-1/p)-1}\left(\frac{1-r}{h}\right)^{n+1} dt\\ &= O(1)\int_{d_n}^{\pi/2} t^{(\alpha-1/p)-1}\exp(-Ant^2)\,dt\\ &\qquad + O(1)\int_{\pi/2}^{\pi} t^{(\alpha-1/p)-1}(1+8\sin^2 \tfrac{t}{2})^{-\frac{n+1}{2}}\,dt = O(n^{-(\alpha-1/p)}).\end{aligned}$$

Now splitting up the integral J_1 into $J_{1,1} = \int_0^{c_n}$ and $J_{1,2} = \int_{c_n}^{d_n}$ and using (4.6.1) and Lemma 4, we get $J_{1,1} = O(n^{-(\alpha-1/p)})$ and proceeding as in $I_{1,2}$ of Theorem 1, we get

$$\begin{aligned}J_{1,2} &= \frac{1}{\pi}\int_{c_n}^{d_n}\frac{\phi_x(t)}{\sin\frac{1}{2}t}\left[\left(\frac{1-r}{h}\right)^{n+1} - \exp(-B(n+1)t^2)\right]\sin\{(n+1)(t+\theta)\}\,dt\\ &\quad + \frac{1}{\pi}\int_{c_n}^{d_n}\frac{\phi_x(t)}{\sin\frac{1}{2}t}\exp(-B(n+1)t^2)\sin\{(n+1)(t+\theta)\}\,dt + O(n^{-(\alpha-1/p)})\\ &= J_{1,2,1} + J_{1,2,2} + O(n^{-(\alpha-1/p)}).\end{aligned}$$

By using (3.4) and (4.6.1), we get $J_{1,2,1} = O(n^{-(\alpha-1/p)})$. Proceeding as in $I_{1,2,2}$ and using (4.6.1), we get

$$J_{1,2,2} = Q + O(n^{-(\alpha-1/p)}),\ \text{say},$$

where

$$\begin{aligned}\pi Q &= \int_{c_n}^{d_n}\frac{\phi_x(t)-\phi_x(t+c_n)}{t}\exp(-Bnt^2)\sin nqt\,dt\\ &\quad + \int_{c_n}^{d_n}\frac{\phi_x(t+c_n)}{t}\exp(-Bnt^2)\sin nqt\,dt\\ &\quad - \int_{0}^{d_n-c_n}\frac{\phi_x(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt\\ &= \pi(Q_1+Q_2+Q_3),\ \text{say}.\end{aligned}$$

However, we observe that $|\phi_x(t) - \phi_x(t+c_n)| = O(c_n^{\alpha-1/p})$ and hence

$$Q_1 = O(n^{-(\alpha-1/p)})\log(n+1).$$

Now, proceeding as for $J_{2,2} + J_{2,3}$ of Theorem 1 and using (4.6.1), we get

$$Q_2 + Q_3 = O(n^{-(\alpha-1/p)}).$$

Combining the obtained estimates, we get the required result. ■

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