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ON $\mathcal{I}$-CONVERGENCE OF DOUBLE SEQUENCES IN THE TOPOLOGY INDUCED BY RANDOM 2-NORMS

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Abstract. In this article we introduce the notion of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchyness of double sequences in the topology induced by random 2-normed spaces and prove some important results.

1. Introduction

Probabilistic metric (PM) spaces were first introduced by Menger [19] as a generalization of ordinary metric spaces and further studied by Schweizer and Sklar [26, 27]. The idea of Menger was to use distribution function instead of non-negative real numbers as values of the metric, which was further developed by several other authors. In this theory, the notion of distance has a probabilistic nature. Namely, the distance between two points x and y is represented by a distribution function F_{xy}; and for $t > 0$, the value $F_{xy}(t)$ is interpreted as the probability that the distance from x to y is less than t. Using this concept, Serstnev [29] introduced the concept of probabilistic normed space, which provides an important method of generalizing the deterministic results of linear normed spaces, also having very useful applications in various fields, among which are continuity properties [1], topological spaces [3], linear operators [7], study of boundedness [8], convergence of random variables [9], statistical and ideal convergence of probabilistic normed space or 2-normed space [14, 21–23, 25, 32] as well as many others.

The concept of 2-normed spaces was initially introduced by Gähler [5, 6] in the 1960's. Since then, many researchers have studied these subjects and obtained various results [10–13, 28, 31].

P. Kostyrko et al. (cf. [17]; a similar concept was invented in [15]) introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization

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of statistical convergence. The notion of statistical convergence of sequences of real numbers was introduced by H. Fast in [2] and H. Steinhaus in [30].

There are many pioneering works in the theory of $\mathcal{I}$-convergence. The aim of this work is to introduce and investigate the idea of $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy of double sequences in a more general setting, i.e., in random 2-normed spaces.

2. Definitions and notations

First we recall some of the basic concepts, which will be used in this paper.

Definition 1. [2, 4] A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\chi_E(k)$ exists. A number sequence $(x_n)_{n\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{n\in\mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0$. If $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to L we write st-lim $x_n = L$, which is necessarily unique.

Definition 2. [16, 17] A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if: (i) $\emptyset \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (iii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ in Y is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, i.e., $\{x\} \in \mathcal{I}$ for each $x \in Y$.

Let $\mathcal{I} \subset P(Y)$ be a non-trivial ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I}\colon M = Y \setminus A\}$, called the filter associated with the ideal $\mathcal{I}$, is a filter on Y.

Definition 3. [17, 18] Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $\xi \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \xi\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Definition 4. [5, 6] Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies: (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent; (ii) $\|x, y\| = \|y, x\|$; (iii) $\|\alpha x, y\| = |\alpha|\,\|x, y\|$, $\alpha \in \mathbb{R}$; (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Also, if x, y and z are linearly dependent, then $\|x, y + z\| = \|x, y\| + \|x, z\|$ or $\|x, y - z\| = \|x, y\| + \|x, z\|$. Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for every $y \in X$.

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [27].

Definition 5. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0,1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & \text{if } t > a, \\ 0 & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 6. A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 7. Let X be a linear space of dimension greater than one, τ is a triangle function, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:

(i) $F(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $F(x, y; t)$ denotes the value of $F(x, y)$ at $t \in \mathbb{R}$,

(ii) $F(x, y; t) \neq H_0(t)$ if x and y are linearly independent,

(iii) $F(x, y; t) = F(y, x; t)$ for all $x, y \in X$,

(iv) $F(\alpha x, y; t) = F(x, y; \frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x, y \in X$,

(v) $F(x + y, z; t) \geq \tau(F(x, z; t), F(y, z; t))$ whenever $x, y, z \in X$, and $t > 0$.

If (v) is replaced by

(vi) $F(x + y, z; t_1 + t_2) \geq F(x, z; t_1) * F(y, z; t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_+$;

then $(X, F, *)$ is called a random 2-normed space (for short, RTN space).

Remark 1. Note that every 2-norm space $(X, \|\cdot,\cdot\|)$ can be made a random 2-normed space in a natural way, by setting

(i) $F(x, y; t) = H_0(t - \|x, y\|)$, for every $x, y \in X$, $t > 0$ and $a * b = \min\{a, b\}$, $a, b \in S$; or

(ii) $F(x, y; t) = \frac{t}{t + \|x, y\|}$ for every $x, y \in X$, $t > 0$ and $a * b = ab$ for $a, b \in S$.

Let $(X, F, *)$ be an RTN space. Since $*$ is a continuous t-norm, the system of (ε, λ)-neighborhoods of θ (the null vector in X) $\{\mathcal{N}_\theta(\varepsilon, \lambda) : \varepsilon > 0,\ \lambda \in (0,1)\}$, where

$$\mathcal{N}_\theta(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\},$$

determines a first countable Hausdorff topology on X, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice-versa.

A double sequence $x = (x_{jk})$ in X is said to be F-convergence to $L \in X$ if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and for each nonzero $z \in X$ there exists a positive integer N such that

$$x_{jk}, z - L \in \mathcal{N}_\theta(\varepsilon, \lambda) \text{ for each } j, k \geq N$$

or, equivalently,

$$x_{jk}, z \in \mathcal{N}_L(\varepsilon, \lambda) \text{ for each } j, k \geq N.$$

In this case we write $F\text{-}\lim x_{jk}, z = L$.

Lemma 1. *Let $(X, \|\cdot,\cdot\|)$ be a real 2-normed space and $(X, F, *)$ be an RTN space induced by the random norm $F_{x,y}(t) = \frac{t}{t+\|x,y\|}$, where $x, y \in X$ and $t > 0$. Then for every double sequence $x = (x_{jk})$ and nonzero y in X*

$$\lim \|x - L, y\| = 0 \Rightarrow F\text{-}\lim (x - L), y = 0.$$

Proof. Suppose that $\lim \|x - L, y\| = 0$. Then for every $t > 0$ and for every $y \in X$ there exists a positive integer $N = N(t)$ such that

$$\|x_{jk} - L, y\| < t \text{ for each } j, k \geq N.$$

We observe that for any given $\varepsilon > 0$,

$$\frac{\varepsilon + \|x_{jk} - L, y\|}{\varepsilon} < \frac{\varepsilon + t}{\varepsilon}$$

which is equivalent to

$$\frac{\varepsilon}{\varepsilon + \|x_{jk} - L, y\|} > \frac{\varepsilon}{\varepsilon + t} = 1 - \frac{t}{\varepsilon + t}.$$

Therefore, by letting $\lambda = \frac{t}{\varepsilon+t} \in (0, 1)$ we have

$$F_{x_{jk}-L,y}(\varepsilon) > 1 - \lambda \text{ for each } j, k \geq N.$$

This implies that $x_{jk}, y \in \mathcal{N}_L(\varepsilon, \lambda)$ for each $j, k \geq N$ as desired. ∎

2. $\mathcal{I}_2^F$ and $\mathcal{I}_2^{F_*}$-convergence for double sequences in RTN spaces

In this section we study the concept of $\mathcal{I}$ and $\mathcal{I}^*$-convergence of a double sequence in $(X, F, *)$ and prove some important results. Throughout the paper we take $\mathcal{I}_2^F$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 8. Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be a proper ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{jk})$ in X is said to be $\mathcal{I}_2^F$-convergent to $L \in X$ ($\mathcal{I}_2^F$-convergent to $L \in X$ with respect to F-topology) if for each $\varepsilon > 0$, $\lambda \in (0, 1)$ and each nonzero $z \in X$,

$$\{(j, k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{I}_2.$$

In this case the vector L is called the $\mathcal{I}_2^F$-limit of the double sequence $x = (x_{jk})$ and we write $\mathcal{I}_2^F\text{-}\lim x, z = L$.

Lemma 2. *Let $(X, F, *)$ be an RTN space. If a double sequence $x = (x_{jk})$ is $\mathcal{I}_2^F$-convergent with respect to the random 2-norm F, then $\mathcal{I}_2^F$-limit is unique.*

Proof. Let us assume that $\mathcal{I}_2^F\text{-}\lim x, z = L_1$ and $\mathcal{I}_2^F\text{-}\lim x, z = L_2$ where $L_1 \neq L_2$. Since $L_1 \neq L_2$, select $\varepsilon > 0$, $\lambda \in (0,1)$ and each nonzero $z \in X$ such that $\mathcal{N}_{L_1}(\varepsilon, \lambda)$ and $\mathcal{N}_{L_2}(\varepsilon, \lambda)$ are disjoint neighborhoods of L_1 and L_2. Since L_1 and L_2 both are $\mathcal{I}_2^F$-limit of the sequence (x_{jk}), we have

$$A = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_{L_1}(\varepsilon, \lambda)\}$$

and

$$B = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_{L_2}(\varepsilon, \lambda)\}$$

both belong to $\mathcal{I}_2^F$. This implies that the sets

$$A^c = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \in \mathcal{N}_{L_1}(\varepsilon, \lambda)\}$$

and $B^c = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \in \mathcal{N}_{L_2}(\varepsilon, \lambda)\}$ belong to $\mathcal{F}(\mathcal{I}_2)$. In this way we obtain a contradiction to the fact that the neighborhoods $\mathcal{N}_{L_1}(\varepsilon, \lambda)$ and $\mathcal{N}_{L_2}(\varepsilon, \lambda)$ of L_1 and L_2 are disjoint. Hence we have $L_1 = L_2$. This completes the proof. ■

Lemma 3. *Let $(X, F, *)$ be an RTN space. Then we have*

(i) *$F\text{-}\lim x_{jk}, z = L$, then $\mathcal{I}_2^F\text{-}\lim x_{jk}, z = L$.*

(ii) *If $\mathcal{I}_2^F\text{-}\lim x_{jk}, z = L_1$ and $\mathcal{I}_2^F\text{-}\lim y_{jk}, z = L_2$, then $\mathcal{I}_2^F\text{-}\lim (x_{jk} + y_{jk}), z = L_1 + L_2$.*

(iii) *If $\mathcal{I}_2^F\text{-}\lim x_{jk}, z = L$ and $\alpha \in \mathbb{R}$, then $\mathcal{I}_2^F\text{-}\lim \alpha x_{jk}, z = \alpha L$.*

(iv) *If $\mathcal{I}_2^F\text{-}\lim x_{jk}, z = L_1$ and $\mathcal{I}_2^F\text{-}\lim y_{jk}, z = L_2$, then $\mathcal{I}_2^F\text{-}\lim (x_{jk} - y_{jk}), z = L_1 - L_2$.*

Proof. (i) Suppose that $F\text{-}\lim x_{jk}, z = L$. Let $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Then there exists a positive integer N such that $x_{jk}, z \in \mathcal{N}_L(\varepsilon, \lambda)$ for each $j, k > N$. Since the set

$$A = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \subseteq \{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, N-1\}$$

and the ideal a $\mathcal{I}_2^F$ is admissible, we have $A \in \mathcal{I}_2^F$. This shows that $\mathcal{I}_2^F\text{-}\lim x_{jk}, z = L$.

(ii) Let $\varepsilon > 0, \lambda \in (0,1)$ and nonzero $z \in X$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > (1-\lambda)$. Since $\mathcal{I}_2^F\text{-}\lim x_{jk}, z = L_1$ and $\mathcal{I}_2^F\text{-}\lim y_{jk}, z = L_2$, the sets

$$A = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_{L_1}\left(\frac{\varepsilon}{2}, \lambda\right)\right\}$$

and

$$B = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : y_{jk}, z \notin \mathcal{N}_{L_2}\left(\frac{\varepsilon}{2}, \lambda\right)\right\}$$

belong to $\mathcal{I}_2^F$. Let $C = \{(j,k) \in \mathbb{N} \times \mathbb{N} : (x_{jk} + y_{jk}), z \notin \mathcal{N}_{L_1+L_2}(\varepsilon, \lambda)\}$. Since $\mathcal{I}_2^F$ is an ideal, it is sufficient to show that $C \subset A \cup B$. This is equivalent to show that $C^c \supset A^c \cap B^c$ where A^c and B^c belong to $\mathcal{F}(\mathcal{I}_2)$. Let $(j,k) \in A^c \cap B^c$, i.e., $(j,k) \in A^c$ and $(j,k) \in B^c$, and we have

$$F_{(x_{jk}+y_{jk})-(L_1+L_2),z}(\varepsilon) \geq F_{x_{jk}-L_1,z}\left(\frac{\varepsilon}{2}\right) * F_{y_{jk}-L_2,z}\left(\frac{\varepsilon}{2}\right)$$
$$> (1-\eta) * (1-\eta) > (1-\lambda).$$

Since $(j,k) \in C^c \supset A^c \cap B^c \in \mathcal{F}(\mathcal{I}_2)$, we have $C \subset A \cup B \in \mathcal{I}_2^F$.

(iii) It is trivial for $\alpha = 0$. Now let $\alpha \neq 0$, $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Since $\mathcal{I}_2^F$-$\lim x_{jk}, z = L$, we have

$$A = \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{I}_2$$

This implies that

$$A^c = \{(j,k) \in : x_{jk}, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{F}(\mathcal{I}_2).$$

Let $(j,k) \in A^c$. Then we have

$$F_{\alpha x_{jk}-\alpha L,z}(\varepsilon) = F_{x_{jk}-L,z}\left(\frac{\varepsilon}{|\alpha|}\right)$$
$$\geq F_{x_{jk}-L,z}(\varepsilon) * F_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right)$$
$$> (1-\lambda) * 1 = (1-\lambda).$$

So $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \alpha x_{jk}, z \notin \mathcal{N}_{\alpha L}(\varepsilon, \lambda)\} \in \mathcal{I}_2$. Hence $\mathcal{I}_2^F$ -$\lim \alpha x_{jk}, z = \alpha L$.

(iv) The result follows from (ii) and (iii). ■

We introduce the concept of $\mathcal{I}_2^{F*}$-convergence closely related to $\mathcal{I}_2^F$-convergence of double sequences in random 2-normed space and show that $\mathcal{I}_2^{F*}$-convergence implies $\mathcal{I}_2^F$-convergence but not conversely.

Definition 9. Let $(X, F, *)$ be an RTN space. We say that a sequence $x = (x_{jk})$ in X is $\mathcal{I}_2^{F*}$-convergent to $L \in X$ with respect to the random 2-norm F if there exists a subset

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots;\ k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus K \in \mathcal{I}_2$) and F-$\lim_m x_{j_m,k_m}, z = L$ for each nonzero $z \in X$.

In this case we write $\mathcal{I}_2^{F*}$-$\lim x, z = L$ and L is called the $\mathcal{I}_2^{F*}$-limit of the double sequence $x = (x_{jk})$.

Theorem 1. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}_2$ be an admissible ideal. If $\mathcal{I}_2^{F*}$-$\lim x, z = L$, then $\mathcal{I}_2^F$-$\lim x, z = L$.*

Proof. Suppose that $\mathcal{I}_2^{F*}$-$\lim x, z = L$. Then by definition, there exists

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots;\ k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_2)$$

such that $F\text{-}\lim_m x_{j_m,k_m}, z = L$. Let $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ be given. Since $F\text{-}\lim_m x_{j_m k_m}, z = L$, there exists $N \in \mathbb{N}$ such that $x_{j_m k_m}, z \in \mathcal{N}_L(\varepsilon, \lambda)$ for every $m \geq N$. Since

$$A = \{(j_m, k_m) \in K : x_{j_m k_m}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\}$$

is contained in

$$B = \{j_1, j_2, \ldots, j_{N-1};\ k_1, k_2, \ldots, k_{N-1}\}$$

and the ideal $\mathcal{I}_2$ is admissible, we have $A \in \mathcal{I}_2$. Hence

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \subseteq K \cup B \in \mathcal{I}_2$$

for $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Therefore, we conclude that $\mathcal{I}_2^F\text{-}\lim x, z = L$. ■

The following example shows that the converse of Theorem 1 need not be true.

Example 1. Consider $X = \mathbb{R}^2$ with $\|x, y\| := |x_1 y_2 - x_2 y_1|$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ and let $a * b = ab$ for all $a, b \in S$. For all $(x, y) \in \mathbb{R}^2$ and $t > 0$, consider

$$F_{x,y}(t) = \frac{t}{t + \|x, y\|}.$$

Then $(\mathbb{R}^2, F, *)$ is an RTN space. Consider a decomposition of $\mathbb{N} \times \mathbb{N}$ as $\mathbb{N} \times \mathbb{N} = \bigcup_{i,j} \Delta_{ij}$ such that for any $(m,n) \in \mathbb{N} \times \mathbb{N}$ each Δ_{ij} contains infinitely many (i,j)'s where $i \geq m$, $j \geq n$ and $\Delta_{ij} \cap \Delta_{mn} = \emptyset$ for $(i,j) \neq (m,n)$. Let $\mathcal{I}_2$ be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ which intersect at most a finite number of Δ_{ij}'s. Then $\mathcal{I}_2$ is an admissible ideal. We define a double sequence (x_{mn}) as follows: $x_{mn} = \left(\frac{1}{ij}, 0\right) \in \mathbb{R}^2$ if $(m,n) \in \Delta_{ij}$. Then for nonzero $z \in X$, we have

$$F_{x_{mn},z}(t) = \frac{t}{t + \|x_{mn}, z\|} \to 1$$

as $m, n \to \infty$. Hence $\mathcal{I}_2^F\text{-}\lim_{m,n} x_{mn}, z = 0$.

Now, we show that $\mathcal{I}_2^{F_*}\text{-}\lim_{m,n} x_{mn}, z \neq 0$. Suppose that $\mathcal{I}_2^{F_*}\text{-}\lim_{m,n} x_{mn}, z = 0$. Then by definition, there exists a subset

$$K = \{(m_j, n_j) : m_1 < m_2 < \cdots;\ n_1 < n_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $F\text{-}\lim_j x_{m_j n_j}, z = 0$. Since $K \in \mathcal{F}(\mathcal{I}_2)$, there exists $H \in \mathcal{I}_2$ such that $K = \mathbb{N} \times \mathbb{N} \setminus H$. Then there exists positive integers p and q such that

$$H \subset \left(\bigcup_{m=1}^{p}\left(\bigcup_{n=1}^{\infty} \Delta_{mn}\right)\right) \cup \left(\bigcup_{n=1}^{q}\left(\bigcup_{m=1}^{\infty} \Delta_{mn}\right)\right).$$

Thus $\Delta_{p+1,q+1} \subset K$ and so $x_{m_j n_j} = \frac{1}{(p+1)(q+1)} > 0$ for infinitely many values (m_j, n_j)'s in K. This contradicts the assumption that $F\text{-}\lim_j x_{m_j n_j}, z = 0$. Hence $\mathcal{I}_2^{F_*}\text{-}\lim_{m,n} x_{mn}, z \neq 0$.

Hence the converse of Theorem 1 need not be true.

The following theorem shows that the converse holds if the ideal $\mathcal{I}_2$ satisfies condition (AP).

Definition 10. [23] An admissible ideal $\mathcal{I}_2 \subset P(\mathbb{N}\times\mathbb{N})$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}_2$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}_2$.

Theorem 2. *Let $(X, F, *)$ be an RTN space and the ideal $\mathcal{I}_2$ satisfy the condition (AP). If $x = (x_{jk})$ is a double sequence in X such that $\mathcal{I}_2^F$-$\lim x, z = L$, then $\mathcal{I}_2^{F_*}$-$\lim x, z = L$.*

Proof. Since $\mathcal{I}_2^F$-$\lim x, z = L$, so for every $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, the set

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : x_{jk}, z \notin \mathcal{N}_L(\varepsilon,\lambda)\} \in \mathcal{I}_2.$$

We define the set A_p for $p \in \mathbb{N}$ as

$$A_p = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : 1 - \frac{1}{p} \le F_{x_{jk}, z-L} < 1 - \frac{1}{p+1}\right\}.$$

Then it is clear that $\{A_1, A_2, \dots\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}_2$ and so by the condition (AP) there is a countable family of sets $\{B_1, B_2, \dots\} \in \mathcal{I}_2$ such that the symmetric difference $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}_2$. Since $B \in \mathcal{I}_2$, there is a set $K \in F(\mathcal{I}_2)$ such that $K = \mathbb{N}\times\mathbb{N} \setminus B$. Now we prove that the subsequence $(x_{jk})_{(j,k)\in K}$ is convergent to L with respect to the random 2-norm F. Let $\eta \in (0,1)$, $\varepsilon > 0$ and nonzero $z \in X$. Choose a positive q such that $q^{-1} < \eta$. Then

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : x_{jk}, z \notin \mathcal{N}_L(\varepsilon,\eta)\}$$
$$\subset \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : x_{jk}, z \notin \mathcal{N}_L\left(\varepsilon, \frac{1}{q}\right)\right\} \subset \bigcup_{i=1}^{q-1} A_i.$$

Since $A_i \Delta B_i$ is a finite set for each $i = 1, 2, \dots, q-1$, there exists $(j_0, k_0) \in \mathbb{N}\times\mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{q-1} B_i\right) \cap \{(j,k) \in \mathbb{N}\times\mathbb{N} : j \ge j_0 \text{ and } k \ge k_0\}$$
$$= \left(\bigcup_{i=1}^{q-1} A_i\right) \cap \{(j,k) \in \mathbb{N}\times\mathbb{N} : j \ge j_0 \text{ and } k \ge k_0\}.$$

If $j \ge j_0$, $k \ge k_0$ and $(j,k) \in K$, then $(j,k) \notin \bigcup_{i=1}^{q-1} B_i$ and $(j,k) \notin \bigcup_{i=1}^{q-1} A_i$. Hence for every $j \ge j_0$, $k \ge k_0$ and $(j,k) \in K$ we have

$$x_{jk}, z \notin \mathcal{N}_L(\varepsilon,\eta).$$

Since this holds for every $\varepsilon > 0$, $\eta \in (0,1)$ and nonzero $z \in X$, so we have $\mathcal{I}_2^{F_*}$-$\lim x, z = L$. This completes the proof of the theorem. ∎

4. $\mathcal{I}_2^F$ and $\mathcal{I}_2^{F_*}$-double Cauchy sequences in RTN spaces

In this section we study the concepts of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy double sequences in $(X, F, *)$. Also, we will study the relations between these concepts.

Definition 11. Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. Then a double sequence $x = (x_{jk})$ of elements in X is called a $\mathcal{I}_2^F$-Cauchy sequence in X if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, there exists $s = s(\varepsilon)$, $t = t(\varepsilon)$ such that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} - x_{st}, z \notin \mathcal{N}_\theta(\varepsilon, \lambda)\} \in \mathcal{I}_2.$$

Definition 12. Let $(X, F, *)$ be a RTN space and $\mathcal{I}$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. We say that a double sequence $x = (x_{jk})$ of elements in X is a $\mathcal{I}_2^{F_*}$-Cauchy sequence in X if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, there exists a set

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ;\ k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in F(\mathcal{I}_2)$ and (x_{j_m, k_m}) is an ordinary F-Cauchy in X.

The next theorem gives that each $\mathcal{I}_2^{F_*}$-double Cauchy sequence is a $\mathcal{I}_2^F$-double Cauchy sequence.

Theorem 3. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be a nontrivial ideal of $\mathbb{N} \times \mathbb{N}$. If $x = (x_{jk})$ is a $\mathcal{I}_2^{F*}$-double Cauchy sequence, then $x = (x_{jk})$ is a $\mathcal{I}_2^F$-double Cauchy sequence, too.*

Proof. Let (x_{jk}) be a $\mathcal{I}_2^{F*}$-Cauchy sequence. Then for $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, there exists

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ;\ k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_2)$$

and a number $N \in \mathbb{N}$ such that

$$x_{j_m k_m} - x_{j_p k_p}, z \in \mathcal{N}_\theta(\varepsilon, \lambda)$$

for every $m, p \geq N$. Now, fix $p = j_{N+1}$, $r = k_{N+1}$. Then for every $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, we have

$$x_{j_m k_m} - x_{pr}, z \in \mathcal{N}_\theta(\varepsilon, \lambda) \text{ for every } m \geq N.$$

Let $H = \mathbb{N} \times \mathbb{N} \setminus K$. It is obvious that $H \in \mathcal{I}_2$ and

$$\begin{aligned} A(\varepsilon, \lambda) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} - x_{pr}, z \notin \mathcal{N}_\theta(\varepsilon, \lambda)\} \\ &\subset H \cup \{j_1 < j_2 < \cdots < j_N;\ k_1 < k_2 < \cdots < k_N\} \in \mathcal{I}_2. \end{aligned}$$

Therefore, for every $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, we can find $(p, r) \in \mathbb{N} \times \mathbb{N}$ such that $A(\varepsilon, \lambda) \in \mathcal{I}_2$, i.e., (x_{jk}) is a $\mathcal{I}_2^F$-double Cauchy sequence. ■

Now we will prove that $\mathcal{I}_2^{F_*}$-convergence implies $\mathcal{I}_2^F$-Cauchy condition in a 2-normed space.

Theorem 4. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If a sequence $x = (x_{jk})$ is $\mathcal{I}_2^{F*}$-convergent, then it is a $\mathcal{I}_2^F$-double Cauchy sequence.*

Proof. By assumption there exists a set

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ;\ k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $F\text{-}\lim_m x_{j_m,k_m}, z = L$ for each nonzero z in X, i.e., there exists $N \in \mathbb{N}$ such that $x_{j_m k_m}, z \in \mathcal{N}_L(\varepsilon, \lambda)$ for every $\varepsilon > 0$, $\lambda \in (0,1)$, each nonzero z in X and $m > N$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > (1-\lambda)$. Since

$$\begin{aligned} F_{x_{j_m k_m} - x_{j_p k_p}, z}(\varepsilon) &\geq F_{x_{j_m k_m} - L, z}\left(\frac{\varepsilon}{2}\right) * F_{x_{j_p k_p} - L, z}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\eta) * (1-\eta) > 1 - \lambda \end{aligned}$$

for every $\varepsilon > 0$, $\lambda \in (0,1)$, each nonzero z in X and $m > N$, $p > N$, we have $x_{j_m k_m} - x_{j_p k_p}, z \notin \mathcal{N}_L(\varepsilon, \lambda)$ for every $m, p > N$ and each nonzero $z \in X$, i.e., (x_{jk}) in X is an $\mathcal{I}_2^{F*}$-double Cauchy sequence in X. Then by Theorem 3 (x_{jk}) is a $\mathcal{I}_2^F$-double Cauchy sequence in the RTN space. ■

THEOREM 5. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If a sequence $x = (x_{jk})$ of elements in X is $\mathcal{I}_2^F$-convergent, then it is a $\mathcal{I}_2^F$-double Cauchy sequence.*

Proof. Suppose that (x_{jk}) is $\mathcal{I}_2^F$-convergent to $L \in X$. Let $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ be given. Then we have

$$A = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \notin \mathcal{N}_L\left(\frac{\varepsilon}{2}, \lambda\right)\right\} \in \mathcal{I}_2$$

This implies that

$$A^c = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk}, z \in \mathcal{N}_L\left(\frac{\varepsilon}{2}, \lambda\right)\right\} \in \mathcal{F}(\mathcal{I}_2)$$

Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > (1-\lambda)$. Then for every $(j,k), (s,t) \in A^c$,

$$F_{x_{jk} - x_{st}, z}(\varepsilon) \geq F_{x_{jk} - L, z}\left(\frac{\varepsilon}{2}\right) * F_{x_{st} - L, z}\left(\frac{\varepsilon}{2}\right) > (1-\eta) * (1-\eta) > (1-\lambda).$$

Hence $\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} - x_{st}, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \in \mathcal{F}(\mathcal{I}_2)$ for nonzero $z \in X$. This implies that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} - x_{st}, z \notin \mathcal{N}_\theta(\varepsilon, \lambda)\} \in \mathcal{I}_2,$$

i.e., (x_{jk}) is a $\mathcal{I}_2^F$-double Cauchy sequence. ■

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