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A NEW FASTER ITERATION PROCESS APPLIED TO CONSTRAINED MINIMIZATION AND FEASIBILITY PROBLEMS

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Abstract. We introduce a new iteration process and prove that it is faster than all of Picard, Mann and Agarwal et al. processes. We support analytic proof by a numerical example. Our process is independent of all three processes just mentioned. We also prove some weak and strong convergence theorems for two nonexpansive mappings. Moreover, we apply our results to find solutions of constrained minimization problems and feasibility problems.

1. Introduction and preliminaries

Most of the physical problems of applied sciences and engineering are usually formulated as functional equations. Such equations can be written in the form of fixed point equations in an easy manner. It is always desired to develop an iterative process which approximate the solution of these equations in fewer number of steps. The study of variational inequality and complementarity problems of mappings satisfying certain constraints has been at the center of rigorous research activity. Given the fact, complementarity and variational inequality problems which are extremely useful in optimization theory that can be found by solving an equation with some special form of nonlinear function f, it is very important to develop some faster iterative process to find the approximate solution. We introduce a new iteration process and prove that it is faster than all of Picard, Mann and Agarwal et al. [3] processes. We support analytic proof by numerical examples. Our process is independent of all three processes just mentioned. We also prove some weak and strong convergence theorems for the nonexpansive mappings. Moreover, we apply our results to find solutions of constrained minimization problems and feasibility problems.

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let E be a real Banach space and C a nonempty subset of E. Let $T\colon C \to C$ be a mapping. Then we denote the set of all fixed points of T by $F(T)$. T is called L-Lipschitzian

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if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. An L-Lipschitzian is called contraction if $L \in (0,1)$, and nonexpansive if $L = 1$.

We know that the Picard, Mann and Ishikawa iteration processes are defined respectively as:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{1}$$

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \ n \in \mathbb{N} \end{cases} \tag{2}$$

and

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

The following definitions about the rate of convergence are due to Berinde [4]. See also [10].

Definition 1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, respectively b. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

then it is said that the sequence $\{a_n\}$ converges to a faster than $\{b_n\}$ to b.

Definition 2. Suppose that for two fixed point iteration processes $\{u_n\}$ and $\{v_n\}$ both converging to the same fixed point p, the following error estimates

$$\|u_n - p\| \leq a_n \quad \text{for all } n \in \mathbb{N},$$
$$\|v_n - p\| \leq b_n \quad \text{for all } n \in \mathbb{N},$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to p.

In the sequel, whenever we talk about the rate of convergence, we refer to given by the above definitions.

Recently, Agarwal et al. [1] posed the following question:

Question 1. Is it possible to develop an iteration process whose rate of convergence is faster than the Picard iteration?

As an answer, they introduced the following iteration process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)\, x_n + \beta_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They showed that this process converges at a rate same as that of Picard iteration and faster than Mann iteration for contractions. Continuing with the same question, Sahu [10] recently proved that this process converges at a rate faster than both Picard and Mann iterations for contractions, by giving a numerical example in support of his claim.

Having this in mind, we pose the following question:

Question 2. Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration (4)?

As an answer we introduce the following iteration process.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n T z_n, \\ y_n = (1-\beta_n)\,Tx_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)\,x_n + \gamma_n T x_n,\ n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$.

In this paper, we prove some weak and strong convergence theorems for nonexpansive mappings using (5). We prove that our process converges faster than (4). We also give a numerical example in support of our claim.

We recall the following. Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functional f on E. The space E has:

(i) *Gâteaux differentiable norm* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t},$$

exists for each x and y in S;

(ii) *Fréchet differentiable norm* (see e.g. [2, 13] if for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x+h\|^2 \le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \tag{6}$$

for all x, h in E, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X$, $\langle\cdot,\cdot\rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} \frac{b(t)}{t} = 0$;

(iii) *Opial condition* [8] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \ne x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \ne 2$ fail to satisfy Opial condition.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

First we state the following lemmas to be used later on.

Lemma 1. [11] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1. [5] *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

2. Rate of convergence

In this section, we show that our process (5) converges faster than (4). Analytic proof is given on the lines similar to those of Sahu [10]. A numerical example in support of our claim is given afterwards.

Theorem 3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a contraction with a contraction factor $k \in (0,1)$ and fixed point q. Let $\{u_n\}$ be defined by the iteration process*

$$\begin{cases} u_1 = x \in C, \\ u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n T v_n, \\ v_n = (1-\beta_n)\, u_n + \beta_n T u_n, \quad n \in \mathbb{N} \end{cases}$$

and $\{x_n\}$ by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n T z_n, \\ y_n = (1-\beta_n)\, Tx_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)\, x_n + \gamma_n T x_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$. Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our process (5) converges faster than (4).

Proof. As proved in Theorem 3.6 of Sahu [10], $\|u_{n+1} - q\| \leq k^n[1-(1-k)\alpha\beta]^n \|u_1 - q\|$ for all $n \in \mathbb{N}$. Let

$$a_n = k^n[1-(1-k)\alpha\beta]^n \|u_1 - q\|.$$

Now

$$\begin{aligned} \|z_n - q\| &= \|(1-\gamma_n)\, x_n + \gamma_n T x_n - q\| \leq (1-\gamma_n) \|x_n - q\| + k\gamma_n \|x_n - q\| \\ &= (1-(1-k)\,\gamma_n) \|x_n - q\|, \end{aligned}$$

so that

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)\, Tx_n + \beta_n T z_n - q\| \leq k\,(1-\beta_n) \|x_n - q\| + \beta_n k \|z_n - q\| \\ &\leq k\,[(1-\beta_n) + \beta_n (1-(1-k)\,\gamma_n)] \|x_n - q\| \\ &= k\,[1-(1-k)\,\beta_n\gamma_n] \|x_n - q\|. \end{aligned}$$

Thus

$$
\begin{aligned}
\|x_{n+1}-q\| &= \|(1-\alpha_n)Ty_n+\alpha_n Tz_n-q\| \\
&\le \left[(1-\alpha_n)k^2(1-(1-k)\,\beta_n\gamma_n)+\alpha_n k\,(1-(1-k)\,\gamma_n)\right]\|x_n-q\| \\
&< k\,[1-\alpha_n-(1-k)\,(1-\alpha_n)\beta_n\gamma_n+\alpha_n-(1-k)\,\alpha_n\gamma_n]\,\|x_n-q\| \\
&\le k\,[1-(1-k)\,\alpha_n\beta_n\gamma_n+(1-k)\,\alpha_n\beta_n\gamma_n-(1-k)\,\alpha_n\beta_n\gamma_n]\,\|x_n-q\| \\
&= k\,[1-(1-k)\,\alpha_n\beta_n\gamma_n]\,\|x_n-q\|\,.
\end{aligned}
$$

Let

$$b_n = k^n\,[1-(1-k)\,\alpha\beta\gamma]^n\,\|x_1-q\|\,.$$

Then

$$
\begin{aligned}
\frac{b_n}{a_n} &= \frac{k^n\,[1-(1-k)\,\alpha\beta\gamma]^n\,\|x_1-q\|}{k^n[1-(1-k)\alpha\beta]^n\,\|u_1-q\|} = \left[\frac{1-(1-k)\,\alpha\beta\gamma}{1-(1-k)\alpha\beta}\right]^n \frac{\|x_1-q\|}{\|u_1-q\|} \\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. ■

Now, we present an example which shows that our iteration process (5) converges at a rate faster than both Agarwal et al. iteration process (4) and Picard iteration process (1). Note that Sahu [10] has already given an example that Agarwal et al. process (4) converges at a rate faster than both Picard (1) and Mann iteration processes (2).

EXAMPLE 1. Let $X = R$ and $C = [1, 50]$. Let $T : C \to C$ be an operator defined by $T(x) = \sqrt{x^2-8x+40}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, with the initial value $x_1 = 30$. The corresponding our iteration process, Agarwal et al. iteration process (4) and Picard iteration process (1) are respectively given below.

No. of iterations	Our Scheme	Agarwal et al.	Picard
1	30	30	30
2	24.7190745016294	25.5882938061377	26.4575131106459
3	19.6259918011617	21.2975671390087	22.9856454143631
4	14.8333116816574	17.1825447168428	19.6075172268171
5	10.5657980903686	13.3383202216318	16.3583188007017
6	7.27549523377657	9.93830773498313	13.2939100184926
7	5.53883084294713	7.28483266161142	10.5060346197715
8	5.07812107001585	5.7202701653486	8.14423025667036
9	5.00964607610839	5.15153820231574	6.4167471837608
10	5.00116164194966	5.02585594531354	5.46266116011379
11	5.00013945663076	5.00418380568606	5.11266835119445
12	5.00001673565453	5.00067063562347	5.02374669542064
13	5.00000200829091	5.00010733321846	5.00480342235184
14	5.00000024099509	5.0000171741223	5.00096289903954
15	5.00000002891941	5.00000274788024	5.00019266881324

All sequences converges to $x^* = 5$. Comparison shows that our iteration process (5) converges fastest than Agarwal et al. iteration process (4) and Picard iteration process (1).

3. Convergence theorems

In this section, we give some convergence theorems using our iteration process (5). We start with proving a key theorem for later use.

THEOREM 4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mappings of C. Let $\{x_n\}$ be defined by the iteration process* (5) *where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$. If $F(T) \neq \varnothing$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Let $q \in F(T)$. Then

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1-\alpha_n)Ty_n + \alpha_n Tz_n - q\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n \|z_n - q\| \\
&\leq (1-\alpha_n)(1-\beta_n)\|x_n - q\| + (1-\alpha_n)\beta_n \|z_n - q\| + \alpha_n \|z_n - q\| \\
&\leq (1-\alpha_n)(1-\beta_n)\|x_n - q\| \\
&\qquad + [(1-\alpha_n)\beta_n + \alpha_n]\,[(1-\gamma_n)\|Tx_n - q\| + \gamma_n \|x_n - q\|] \\
&\leq [(1-\alpha_n)(1-\beta_n) + (1-\alpha_n)\beta_n + \alpha_n]\,\|x_n - q\| \\
&= \|x_n - q\|.
\end{aligned}$$

Thus $\lim_{n\to\infty} \|x_n - q\|$ exists. Call it c.

Now $\|z_n - q\| \leq (1-\gamma_n)\|x_n - q\| + \gamma_n \|x_n - q\| = \|x_n - q\|$, implies that

$$\limsup_{n\to\infty} \|z_n - q\| \leq c. \tag{7}$$

Similarly,

$$\begin{aligned}
\|y_n - q\| &\leq (1-\beta_n)\|x_n - q\| + \beta_n \|z_n - q\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n \|x_n - q\| = \|x_n - q\|,
\end{aligned}$$

implies that

$$\limsup_{n\to\infty} \|y_n - q\| \leq c. \tag{8}$$

Next, $\|Ty_n - q\| \leq \|y_n - q\|$, gives by (8) that

$$\limsup_{n\to\infty} \|Ty_n - q\| \leq c. \tag{9}$$

Similarly,

$$\limsup_{n\to\infty} \|Tz_n - q\| \leq c.$$

Moreover, $c = \lim_{n\to\infty} \|x_{n+1} - q\| = \lim_{n\to\infty} \|(1-\alpha_n)(Ty_n - q) + \alpha_n (Tz_n - q)\|$ gives by Lemma 1, $\lim_{n\to\infty} \|Ty_n - Tz_n\| = 0$. Now

$$\|x_{n+1} - q\| = \|(Ty_n - q) + \alpha_n (Ty_n - Tz_n)\| \leq \|Ty_n - q\| + \alpha_n \|Ty_n - Tz_n\|,$$

yields that $c \leq \liminf_{n\to\infty} \|Ty_n - q\|$, so that (9) gives $\lim_{n\to\infty} \|Ty_n - q\| = c$.

In turn, $\|Ty_n - q\| \le \|Ty_n - Tz_n\| + \|Tz_n - q\| \le \|Ty_n - Tz_n\| + \|z_n - q\|$ implies

$$c \le \liminf_{n\to\infty} \|z_n - q\|. \tag{10}$$

By (7) and (10), we obtain

$$\lim_{n\to\infty} \|z_n - q\| = c.$$

Thus $c = \lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|\,(1-\gamma_n)\,(x_n - q) + \gamma_n\,(Tx_n - q)\,\|$ gives by Lemma 1 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. ■

LEMMA 5. *Assume that the conditions of Theorem 4 are satisfied. Then, for any $p_1, p_2 \in F(T)$, $\lim_{n\to\infty} \langle x_n, J(p_1 - p_2)\rangle$ exists; in particular, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.*

Proof. Take $x = p_1 - p_2$ with $p_1 \ne p_2$ and $h = t(x_n - p_1)$ in the inequality (6) to get

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\,\langle x_n - p_1, J(p_1 - p_2)\rangle \\
&\quad \le \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\quad \le \frac{1}{2}\|p_1 - p_2\|^2 + t\,\langle x_n - p_1, J(p_1 - p_2)\rangle + b(t\,\|x_n - p_1\|).
\end{aligned}$$

As $\sup_{n\ge 1} \|x_n - p_1\| \le M'$ for some $M' > 0$, it follows that

$$\begin{aligned}
&\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle \\
&\quad \le \frac{1}{2}\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|^2 \\
&\quad \le \frac{1}{2}\|p_1 - p_2\|^2 + b(tM') + t\liminf_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle\,.
\end{aligned}$$

That is,

$$\limsup_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle \le \liminf_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'}M'.$$

If $t \to 0$, then $\lim_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F(T)$; in particular, we have $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. ■

We now give our weak convergence theorem.

THEOREM 6. *Let E be a uniformly convex Banach space and let C, T and $\{x_n\}$ be taken as in Theorem 4. Assume that (a) E satisfies Opial's condition or (b) E has a Fréchet differentiable norm. If $F(T) \ne \phi$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Theorem 4. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For, let u and

v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Theorem 4, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 2, therefore we obtain $Tu = u$. Again in the same manner, we can prove that $v \in F(T)$. Next, we prove the uniqueness. To this end, first assume (a) is true. If u and v are distinct, then by Opial condition,

$$\begin{aligned}\lim_{n\to\infty}\|x_n - u\| &= \lim_{n_i\to\infty}\|x_{n_i} - u\| < \lim_{n_i\to\infty}\|x_{n_i} - v\| = \lim_{n\to\infty}\|x_n - v\| \\ &= \lim_{n_j\to\infty}\|x_{n_j} - v\| < \lim_{n_j\to\infty}\|x_{n_j} - u\| = \lim_{n\to\infty}\|x_n - u\|.\end{aligned}$$

This is a contradiction, so $u = v$. Next assume (b). By Lemma 5, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. Therefore $\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0$ implies $u = v$. Consequently, $\{x_n\}$ converges weakly to a point of $F(T)$ and this completes the proof. ■

A mapping $T : C \to C$, where C is a subset of a normed space E, is said to satisfy Condition (A) [12] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$. It is to be noted that Condition (A) is weaker than compactness of the domain C.

THEOREM 7. *Let E be a real Banach space and let $C, T, F(T), \{x_n\}$ be taken as in Theorem 4. Then $\{x_n\}$ converges to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.*

Proof. Necessity is obvious. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Theorem 4, $\lim_{n\to\infty}\|x_n - w\|$ exists for all $w \in F(T)$, therefore $\lim_{n\to\infty} d(x_n, F(T))$ exists. But by hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We will show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\varepsilon > 0$, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, F(T)) < \frac{\varepsilon}{2}.$$

Particularly, $\inf\{\|x_{n_0} - p\| : p \in F(T)\} < \frac{\varepsilon}{2}$. Hence, there exist $p^* \in F(T)$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$. Now, for $m, n \geq n_0$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\,\|x_{n_0} - p^*\| < \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed in the Banach space E, so that there exists a point q in C such that $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$. Since F is closed, $q \in F(T)$. ■

Applying Theorem 7, we obtain a strong convergence of the process (5) under Condition (A) as follows.

THEOREM 8. *Let E be a real uniformly convex Banach space and let $C, T, F(T)$ and $\{x_n\}$ be taken as in Theorem 4. Let T satisfy Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We proved in Theorem 4 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{11}$$

From Condition (A) and (11), we get

$$\lim_{n\to\infty} f(d(x_n, F(T)) \le \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

i.e., $\lim_{n\to\infty} f(d(x_n, F(T)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Now all the conditions of Theorem 7 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of $F(T)$. ∎

4. Application to constrained optimization problems and split feasibility problems

This section is devoted to some applications. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $T : C \to H$ a nonlinear operator. T is said to be:

(1) monotone if $\langle Tx - Ty, x - y\rangle \ge 0$ for all $x, y \in C$,

(2) λ-strongly monotone if there exists a constant $\lambda > 0$ such that $\langle Tx - Ty, x - y\rangle \ge \lambda \|x - y\|^2$ for all $x, y \in C$,

(3) v-inverse strongly monotone (v-ism) if there exists a constant $v > 0$ such that $\langle Tx - Ty, x - y\rangle \ge v \|Tx - Ty\|^2$ for all $x, y \in C$.

The variational inequality problem defined by C and T will be denoted by $VI(C,T)$. These were initially studied by Kinderlehrer and Stampachhia [7]. The variational inequality problem $VI(C,T)$ is the problem of finding a vector z in C such that $\langle Tz, z - k\rangle \ge 0$ for all $k \in C$. The set of all such vectors which solve variational inequality $VI(C,T)$ problem is denoted by $\Omega(C,T)$. The variational inequality problem is connected with various kinds of problems such as the convex minimization problem, the complementarity problem, the problem of finding a point $u \in H$ satisfying $0 = Tu$ and so on. The existence and approximation of solutions are important aspects in the study of variational inequalities. The variational inequality problem $VI(C,T)$ is equivalent to the fixed point problem, that is

$$\text{to find } x^* \in C \text{ such that } x^* = F_\mu x^* = P_C(I - \mu T)x^*,$$

where $\mu > 0$ is a constant and P_C is the metric projection from H onto C and $F_\mu := P_C(I - \mu T)$. If T is L-Lipschitzian and λ-strongly monotone, then the operator F_μ is a contraction on C provided that $0 < \mu < 2\lambda/L^2$. In this case, an application of Banach contraction principle implies that $\Omega(C,T) = \{x^*\}$ and the sequence of the Picard iteration process, given by

$$x_{n+1} = F_\mu x_n,\ n \in \mathbb{N}$$

converges strongly to x^*.

Construction of fixed points of nonexpansive operators is an important subject in the theory of nonexpansive operators and has applications in a number of applied areas such as image recovery and signal processing (see, [6, 9, 14]. For instance, split feasibility problem of C and T (denoted by $SFP(C,T)$) is

$$\text{to find a point } x \text{ in } C \text{ such that } Tx \in Q, \tag{12}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of another Hilbert space H_2 and $T : H_1 \to H_2$ is a bounded linear operator. The $SFP(C,T)$ is said to be consistent if (12) has a solution. It is easy to see that $SFP(C,T)$ is consistent if and only if the following fixed point problem has a solution:

$$\text{find } x \in C \text{ such that } x = P_C(I - \gamma T^*(I - P_Q)T)x, \tag{13}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively; $\gamma > 0$, and T^* is the adjoint of T. Note that for sufficient small $\gamma > 0$, the operator $P_C(I - \gamma T^*(I - P_Q)T)$ in (13) is nonexpansive.

In the view of Theorem 4, we have the following sharper results which contain iterative process (5) faster than the one defined by (4). These results deal with variational inequality problems.

THEOREM 9. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$, and defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu T)y_n + \alpha_n P_C(I-\mu T)z_n, \\ y_n &= (1-\beta_n)P_C(I-\mu T)x_n + \beta_n P_C(I-\mu T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu T)x_n, \ n \in \mathbb{N} \end{aligned}$$

converges weakly to $x^* \in \Omega(C,T)$.

COROLLARY 10. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ an L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$, and defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu T)x_n + \alpha_n P_C(I-\mu T)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n P_C(I-\mu T)x_n, \ n \in \mathbb{N}, \end{aligned}$$

converges weakly to $x^* \in \Omega(C,T)$.

COROLLARY 11. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ an L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$ is in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0,1)$ Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$, and defined by*

$$x_{n+1} = P_C(I-\mu T)[(1-\alpha_n)x_n + \alpha_n P_C(I-\mu T)x_n], \ n \in \mathbb{N},$$

converges weakly to $x^* \in \Omega(C,T)$.

Application to constrained optimization problems. Let C be a closed convex subset of a Hilbert space H, P_C the metric projection of H onto C and $T : C \to H$ a v-ism where $v > 0$ is a constant . It is well known that $P_C(I - \mu T)$ is nonexpansive operator provided that $\mu \in (0, 2v)$.

The algorithms for signal and image processing are often iterative constrained optimization processes designed to minimize a convex differentiable function T over a closed convex set C in H. It is well known that every L-Lipschitzian operator is $2/L$-ism. Therefore, we have the following result which generates the sequence of vectors in the constrained or feasible set C which converges weakly to the optimal solution which minimizes T.

THEOREM 12. *Let C be a closed convex subset of a Hilbert space H and T a convex and differentiable function on an open set D containing the set C. Assume that $\bigtriangledown T$ is an L-Lipschitz operator on D, $\mu \in (0, 2/L)$ and minimizers of T relative to the set C exist. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu\bigtriangledown T)y_n + \alpha_n P_C(I-\mu\bigtriangledown T)z_n, \\ y_n &= (1-\beta_n)P_C(I-\mu\bigtriangledown T)x_n + \beta_n P_C(I-\mu\bigtriangledown T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu\bigtriangledown T)x_n, \ n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0, 1)$. Then $\{x_n\}$ converges weakly to a minimizer of T.

Application to split feasibility problems. Recall that a mapping T in a Hilbert space H is said to be averaged if T can be written as $(1-\alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and S is a nonexpansive map on H. Set $q(x) := \frac{1}{2}\|(T - P_Q T)x\|$, $x \in C$.

Consider the minimization problem

$$\text{find } \min_{x \in C} q(x).$$

By [3], the gradient of q is $\bigtriangledown q = T^*(I - P_Q)T$, where T^* is the adjoint of T. Since $I - P_Q$ is nonexpansive, it follows that $\bigtriangledown q$ is L-Lipschitzian with $L = \|T\|^2$. Therefore, $\bigtriangledown q$ is $1/L$ism and for any $0 < \mu < 2/L$, $I - \mu\bigtriangledown q$ is averaged. Therefore, the composition $P_C(I - \mu\bigtriangledown q)$ is also averaged. Set $T := P_C(I - \mu\bigtriangledown q)$. Note that the solution set of $SFP(C, T)$ is $F(T)$.

We now present an iterative process that can be used to find solutions of $SFP(C, T)$.

THEOREM 13. *Assume that $SFP(C, T)$ is consistent. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in $(0, 1)$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu\bigtriangledown q)y_n + \alpha_n P_C(I-\mu\bigtriangledown q)z_n, \\ y_n &= (1-\beta_n)P_C(I-\mu\bigtriangledown q)x_n + \beta_n P_C(I-\mu\bigtriangledown q)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu\bigtriangledown q)x_n, \ n \in \mathbb{N}, \end{aligned}$$

where $0 < \mu < 2/\|T\|^2$. Then $\{x_n\}$ converges weakly to a solution of $SFP(C, T)$.

Proof. Since $T := P_C(I - \lambda \bigtriangledown q)$ is nonexpansive, the result follows from Theorem 6. ■

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