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A REMARK ON THE PROPERTY $\mathcal{P}$ AND PERIODIC POINTS OF ORDER ∞

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Abstract. In this paper, we considered the relationship between periodic points, fixed points, and the property $\mathcal{P}$. We also presented an extended version of periodic points together with their behaviors in topological spaces and cone metric spaces.

1. Introduction

It is obvious that if x_* is a fixed point of a mapping f, then it is also a fixed point of iterates f^n for all $n \in \mathbb{N}$. The converse is trivially not true in general. Whenever the converse is true for the mapping f, we say that it satisfies the *property* $\mathcal{P}$. The point x_* such that $x_* = f^h x_*$ for some $h \in \mathbb{N}$ is called a *periodic point* of f. In this case, h is called the *order* of x_* and if it is the least positive integer such that x_* is a fixed point of f^h, then we call it the *prime order* of x_*.

The studies about periodic and fixed points are not limited only in metric spaces, but also in various generalizations of a metric space, for example, in G-metric spaces, probabilistic metric spaces and cone metric spaces (see, e.g., [1–5, 8, 11, 12]).

In this paper, we study the property $\mathcal{P}$ under the framework of cone metric spaces and also its special case in metric spaces. For the last section, we extend the notion of a periodic point to the one of order ∞ and, also, a new class of mappings is as well considered for the existence and behavior of its periodic points of both finite and infinite order.

2. Cone metric spaces

This section is devoted to a recollection of basic definitions appeared in cone metric space theory (for details see, e.g., [7, 9]). For instance, we consider a Banach space E with the zero element θ. A nonempty closed solid subset $P \subseteq E$ is called a cone in E if

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(a) $aP + bP \subseteq P$ for each $a, b \geq 0$;

(b) $P \cap -P = \{\theta\}$.

This cone defines a partial ordering $\preceq$ on E, $x \preceq y$ if and only if $y - x \in P$. Naturally, if $x \neq y$ and $x \preceq y$, we write $x \prec y$. Moreover, we write $x \ll y$ if $y - x \in \mathrm{Int}(P)$, where $\mathrm{Int}(P)$ denotes the interior of P. When $\mathrm{Int}(P) \neq \emptyset$, such cone is called *solid*.

A cone in E is said to be *normal* if there exists some constant $K > 0$ such that $x \preceq y$ implies $\|x\| \leq K\|y\|$. The least constant K as such is called the *normality constant*. It is proved that there does not exist a cone with normality constant $K < 1$ and there always exist a cone with normality constant $K > h$ for each given $h > 1$.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to P$ is called a *cone metric* if the following conditions are satisfied: for all $x, y, z \in X$,

(a) $d(x, y) = \theta$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \preceq d(x, z) + d(z, y)$.

In this case, the pair (X, d) is called a *cone metric space*.

In what follows, unless otherwise specified, we assume that every cone metric d is induced by the Banach space E with a solid cone P, and assume that $\preceq$ is the partial ordering on E constructed by the cone P.

Definition 2.2. [7] Let (X, d) be a cone metric space. The sequence $\{x_n\}$ in X is called:

1. a *Cauchy sequence* if, for any $e \gg \theta$, there exists $N \in \mathbb{N}$ such that, for all $m, n > N \in \mathbb{N}$, $d(x_m, x_n) \ll e$;
2. *convergent* if, for any $e \gg \theta$, there exists $N \in \mathbb{N}$ such that, for all $n > N \in \mathbb{N}$, $d(x_n, x) \ll e$ for some fixed $x \in X$. We write $\lim_{n\to\infty} x_n = x$.

If every Cauchy sequence is convergent, then X is said to be *complete*.

3. The property $\mathcal{P}$

In this section, we give some general conditions for a mapping to satisfy the property $\mathcal{P}$ and also review some results from the past years to which our theorems can be applied. Our results may also improve the quality of future researches in this field.

Theorem 3.1. *Let X be a Hausdorff topological space and f be a self-mapping on X. Suppose that the orbit $\{f^n x\}$ converges for any $x \in X$. Then f satisfies the property $\mathcal{P}$.*

Proof. Assume that there exists a point $x_* \in \mathrm{Fix}(f^m) \setminus \mathrm{Fix}(f)$, where $m \in \mathbb{N}$ is the least number in the sense that $x_* = f^m x_*$ and $x_* \neq f^n x_*$ for all $n \in \mathbb{N}$ with

$n < m$. Thus the sequence $\{f^n x_*\}$ may be written as follows:

$$f^n x_* = f^{n \pmod m} x_*$$

for each $n \in \mathbb{N}$. It is obvious that $\{f^n x_*\}$ does not converge. This contradicts the hypothesis that $\{f^n x\}$ converges for any $x \in X$. Therefore, we conclude that f satisfies the property $\mathcal{P}$. ∎

DEFINITION 3.2. A self-mapping f on X is called a *weak Picard's mapping* if every orbit converges and their limits are fixed points of f. In addition, if f has exactly one fixed point, then f is called a *Picard's mapping.*

COROLLARY 3.3. *Every weak Picard's mapping satisfies the property $\mathcal{P}$.*

COROLLARY 3.4. *Every Picard's mapping satisfies the property $\mathcal{P}$.*

THEOREM 3.5. *Let (X, d) be a cone metric space and f be a self-mapping on X. Suppose that, for each $x \in X$ with $x \neq fx$, $d(f^n x, f^{n+1}x) \prec d(f^{n-1}x, f^n x)$ for all $n \in \mathbb{N}$. Then f satisfies the property $\mathcal{P}$.*

Proof. Assume that there exists a point $x_* \in \mathrm{Fix}(f^m) \setminus \mathrm{Fix}(f)$. Also, suppose that $m \in \mathbb{N}$ is the least number in the sense that $x_* = f^m x_*$ and $x_* \neq f^n x_*$ for each $n \in \mathbb{N}$ with $n < m$. From the hypothesis, we have

$$d(x, fx) = d(f^m x, f^{m+1}x) \prec d(f^{m-1}x, f^m x) \prec \cdots \prec d(x, fx),$$

which is impossible. Therefore, f satisfies the property $\mathcal{P}$. ∎

Using our results, we have the following:

THEOREM 3.6. [2] *Let (X, d) be a normal and solid cone metric space. Suppose that a mapping $f : X \to X$ satisfies*

$$d(fx, f^2x) \preceq k d(x, fx)$$

for all $x \in X$, where $k \in [0, 1)$. Then f satisfies the property $\mathcal{P}$.

THEOREM 3.7. [2, 12] *Let (X, d) be a normal and solid cone metric space. Suppose that a mapping $f : X \to X$ satisfies*

$$d(fx, f^2x) \prec d(x, fx)$$

for all $x \in X$ with $x \neq fx$. Then f satisfies the property $\mathcal{P}$.

THEOREM 3.8. [10] *Let (X, d) be an orbitally complete metric space and $f : X \to X$ be an orbitally continuous mapping satisfying*

$$d(fx, fy) \le k \max\left\{ d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}, \frac{d(x, fy)d(y, fx)}{d(x, y)}, \frac{d(x, fx)d(x, fy)}{2d(x, y)} \right\}$$

for all $x, y \in X$ with $x \neq y$, where $k \in [0, 1)$. Then f satisfies the property $\mathcal{P}$.

Theorem 3.9. [6] *Let (X,d) be a metric space and f be a self-mapping on X satisfying*

$$d(fx,fy) \leq a\frac{[1+d(x,fx)]d(y,fy)}{1+d(x,y)} + b\frac{d(x,fx)d(y,fy)}{d(x,y)} + cd(x,y)$$

for all $x,y \in X$ with $x \neq y$, where $a,b,c \geq 0$ and $a+b+c<1$. Then f satisfies the property $\mathcal{P}$.

The following theorem is due to Rhoades and Abbas [11], which is incorrectly stated. This gap is found in the original proof, and it happens because of the non-arbitrarity of the chosen elements. We give here an evident counterexample to it, and we subsequently fill this gap in. The proof for our replacement will be omitted, as it can be seen explicitly with Theorems 3.1 and 3.5.

Theorem 3.10. [11] *Let f be an orbitally lower semi-continuous self-mapping on a complete metric space (X,d). Let $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ be a Lebesgue integrable function which is summable and $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon > 0$. Suppose that either of the following holds:*

(i) $\int_0^{d(fx,f^2x)} \varphi(t)\,dt \leq k\int_0^{d(x,fx)} \varphi(t)\,dt$ *for all $x \in X$, where $k \in [0,1)$, or*

(ii) $\int_0^{d(fx,f^2x)} \varphi(t)dt < \int_0^{d(x,fx)} \varphi(t)\,dt$ *for all $x \neq fx$ in the closure of the orbit $\{f^n z\}$ for some $z \in X$ in which the orbit $\{f^n z\}$ has an accumulated point $p \in X$ and f is orbitally continuous at p and fp.*

Then f satisfies the property $\mathcal{P}$.

We claim here that the conditions given in the above theorem are in fact not sufficient. The counterexample goes as follows.

Example 3.11. Let $X := \left\{0,\frac{1}{2},1\right\} \cup \left\{\frac{1}{2}-\frac{1}{n} : n \in \mathbb{N}, n \geq 3\right\}$ and $d : X \times X \to \mathbf{R}_+$ be a usual metric. Thus (X,d) is complete. Now, we define a self-mapping f on X by

$$\begin{cases} f(0) = 1, \\ f\left(\frac{1}{2}\right) = \frac{1}{2}, \\ f(1) = 0, \\ f\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{1}{2}-\frac{1}{n+1}, \end{cases}$$

for all $n \in \mathbb{N}$ and $n \geq 3$. Then f is continuous on X. Setting $\varphi(t) = t$ for all $t \geq 0$, all hypotheses for φ hold. Note that the condition (i) fails since $d(f(0),f(1)) \nleq kd(0,1)$ for all $k \in [0,1)$. Now, observe that the initial points that make their orbits possess an accumulated point are exactly the points in the set $Z := X \setminus \left\{0,\frac{1}{2},1\right\}$. Also, observe that the condition (ii) holds for any $z \in Z$. However, we have $\mathrm{Fix}(f) = \left\{\frac{1}{2}\right\}$ and $\mathrm{Fix}(f^2) = \left\{0,\frac{1}{2},1\right\}$. Therefore, the property $\mathcal{P}$ does not hold.

We note that condition (ii) should be developed according to the hypotheses of Theorem 3.7, and it should read as follows.

(ii') $\int_0^{d(fx,f^2x)} \varphi(t)\,dt < \int_0^{d(x,fx)} \varphi(t)\,dt$ for all $x \in X$ with $x \neq fx$.

Note that, in the proof of Theorem 3.10, where f satisfies (ii), the point $u \neq fu$ is not arbitrarily chosen, which results in a gap of the theorem. Replacing (ii) by (ii') is enough to fix it.

4. Periodic points of order ∞

Beyond the concept of a fixed point and a periodic point of finite order, we introduce a new concept of a periodic point of order ∞ in the following

DEFINITION 4.1. Let f be a self-mapping on a topological space X. A point $x_* \in X$ is called a *periodic point of order* ∞ for f if the orbit $\{f^n x_*\}$ has at least one subsequence converging to the point x_* itself. The set of all periodic points of order ∞ for f is denoted by $\mathrm{Fix}(f^\infty)$.

It is obvious that $\bigcup_{n\in\mathbb{N}} \mathrm{Fix}(f^n) \subseteq \mathrm{Fix}(f^\infty)$. The converse is not true in general as in the following example.

EXAMPLE 4.2. Let $X := \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $d : X \times X \to \mathbf{R}_+$ be a usual metric. Let $f : X \to X$ be a mapping given by

$$\begin{cases} f(0) = 1, \\ f\left(\frac{1}{n}\right) = \frac{1}{n+1} \end{cases}$$

for all $n \in \mathbb{N}$. Then $\mathrm{Fix}(f^n) = \emptyset$ for all $n \in \mathbb{N}$ while $\mathrm{Fix}(f^\infty) = \{0\}$.

We begin with some initial results for the existence and uniqueness of periodic points having infinite order in topological spaces.

THEOREM 4.3. *Let X be a topological space and suppose that $f : X \to X$ has a convergent spanning orbit at $x_0 \in X$ (i.e., if $z \in X$, then $z = f^k x_0$ for some $k \in \mathbb{N}$). Then $\mathrm{Fix}(f^\infty)$ is nonempty. Moreover, if X is Hausdorff, then $\mathrm{Fix}(f^\infty)$ is a singleton.*

Proof. Assume that $\{f^n x_0\}$ converges to some $x_* \in X$. Also, observe that $x_* = f^p x_0$ for some $p \in \mathbb{N}$. Thus $f^n x_* = f^{p+n} x_0$ so that $\{f^n x_*\}$ converges to x_* itself.

Now, assume that X is Hausdorff and $y, z \in \mathrm{Fix}(f^\infty)$. Then $y = f^p x_0$ and $z = f^q x_0$ for some $p, q \in \mathbb{N}$ and so $f^n y = f^{p+n} x_0$ and $f^n z = f^{q+n} x_0$. Letting $n \to \infty$, we obtain $y = z$. ■

Example 4.2 also works with the above theorem.

Now, for a cone metric space (X, d), we consider a new class $\mathcal{F}(X)$ of self-mappings f on X such that there exists $\alpha > 0$, $\beta < 0$ and $\gamma \geq 0$ such that

$$\alpha d(fx, fy) \preceq (\alpha - \beta) d(x, y) + \gamma[d(x, fy) + d(y, fx)] + (\beta - \gamma)[d(x, fx) + d(y, fy)] \tag{4.1}$$

for all $x, y \in X$ with $x \neq y$. Note first that this class is nonempty since the identity mapping on X is contained in $\mathcal{F}(X)$.

Theorem 4.4. *Let (X,d) be a complete cone metric space (whose underlying cone P is not necessarily normal) and let $f \in \mathcal{F}(X)$. Then* $\mathrm{Fix}(f^\infty)$ *is nonempty.*

Proof. Assume that $\alpha > 0$, $\beta < 0$ and $\gamma \geq 0$ are the constants satisfying the inequality (4.1). Suppose that $f^{n-1}x_0 \neq f^n x_0$ for all $n \in \mathbb{N}$ and $x_0 \in X$. Observe the following:

$$\begin{aligned}
&\alpha d(f^n x_0, f^{n+1}x_0)\\
&\qquad \preceq (\alpha-\beta)d(f^{n-1}x_0, f^n x_0) + \gamma[d(f^{n-1}x_0, f^{n+1}x_0) + d(f^n x_0, f^n x_0)]\\
&\qquad\quad + (\beta-\gamma)[d(f^{n-1}x_0, f^n x_0) + d(f^n x_0, f^{n+1}x_0)]\\
&\qquad \preceq (\alpha-\beta)d(f^{n-1}x_0, f^n x_0) + \gamma[d(f^{n-1}x_0, f^n x_0) + d(f^n x_0, f^{n+1}x_0)]\\
&\qquad\quad + (\beta-\gamma)[d(f^{n-1}x_0, f^n x_0) + d(f^n x_0, f^{n+1}x_0)]\\
&\qquad = \alpha d(f^{n-1}x_0, f^n x_0) + \beta d(f^n x_0, f^{n+1}x_0).
\end{aligned}$$

Hence we have

$$d(f^n x_0, f^{n+1}x_0) \preceq \lambda d(f^{n-1}x_0, f^n x_0) \preceq \lambda^2 d(f^{n-2}x_0, f^{n-1}x_0) \preceq \cdots \preceq \lambda^n d(x_0, fx_0),$$

where $\lambda = \frac{\alpha}{\alpha-\beta} < 1$. Obviously, the sequence $\{d(f^n x_0, f^{n+1}x_0)\}$ converges in norm to θ. Also, observe that

$$\begin{aligned}
d(f^m x_0, f^n x_0) &\preceq d(f^m x_0, f^{m+1}x_0) + f^{m+1}x_0, f^{m+2}x_0) + \cdots + d(f^{n-1}x_0, f^n x_0)\\
&\preceq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1})d(x_0, fx_0)\\
&\preceq (\lambda^m + \lambda^{m+1} + \cdots)d(x_0, fx_0)\\
&= \tfrac{\lambda^m}{1-\lambda} d(x_0, fx_0).
\end{aligned}$$

For each $e \gg \theta$, there exists $\delta > 0$ such that $e + \delta B^\circ \subseteq \mathrm{Int}(P)$, where B° is the open unit ball around θ. Observe that there exists $N \in \mathbb{N}$ such that $\|-\frac{\lambda^N}{1-\lambda} d(x_0, fx_0)\| < \delta$. Thus it follows that $\frac{\lambda^m}{1-\lambda} d(x_0, fx_0) \ll e$ for all $m \geq N$. Hence $\{f^n x_0\}$ is a Cauchy sequence and so it converges to some $x_* \in X$ by the completeness of X.

Now, we show that $x_* \in \mathrm{Fix}(f^\infty)$. If $x_* = f^h x_0$ for some $h \in \mathbb{N}$, then we have $f^n x_* = f^{h+n}x_0$. Thus $f^n x_* \to x_*$ and so, in this case, $x_* \in \mathrm{Fix}(f^\infty)$. Now, if $x_* \neq f^n x_0$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\alpha d(f^n x_0, fx_*) &\preceq (\alpha-\beta)d(f^{n-1}x_0, x_*) + \gamma[d(f^{n-1}x_0, fx_*) + d(x_*, f^n x_0)]\\
&\quad + (\beta-\gamma)[d(f^{n-1}x_0, f^n x_0) + d(x_*, fx_*)].
\end{aligned}$$

Letting $n \to \infty$, and by the continuity of vector addition and the closedness of cone, we have

$$(\beta-\alpha)d(x_*, fx_*) \in P,$$

which implies $x_* = fx_*$. Therefore, $x_* \in \mathrm{Fix}(f^\infty)$. ∎

Corollary 4.5. *In addition to Theorem 4.4, if f is continuous, then* $\mathrm{Fix}(f)$ *is nonempty.*

Proof. According to the proof of Theorem 4.4, for any $x_0 \in X$, we know that $f^n x_0 \to x_*$. Thus, by the continuity of f, we have

$$fx_* = f \lim_{n\to\infty} f^n x_0 = \lim_{n\to\infty} f f^n x_0 = x_*.$$

Thus $\mathrm{Fix}(f)$ is nonempty. ∎

COROLLARY 4.6. *Suppose that X is a cone metric space and $f \in \mathcal{F}(X)$. Then f satisfies the property $\mathcal{P}$.*

Proof. See the proof of Theorem 4.4 and apply Theorem 3.1 to complete. ∎

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