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LIGHTLIKE SUBMANIFOLDS OF INDEFINITE PARA-SASAKIAN MANIFOLDS

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Abstract. In this paper, we study invariant, slant and screen slant lightlike submanifolds of indefinite para-Sasakian manifolds. We obtain necessary and sufficient conditions for existence of slant and screen slant lightlike submanifolds of indefinite para-Sasakian manifolds and also provide non-trivial examples of such submanifolds. We obtain integrability conditions of distributions D and $RadTM$ on screen slant lightlike submanifolds of indefinite para-Sasakian manifold. Further we obtain sufficient condition for induced connection on screen slant lightlike submanifolds of indefinite para-Sasakian manifold to be metric connection.

1. Introduction

A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric on it is degenerate. In [3], Duggal and Bejancu introduced the geometry of arbitrary lightlike submanifolds of semi-Riemannian manifolds. Lightlike geometry has its applications in general relativity, particularly in black hole theory, which gave impetus to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures. Lightlike submanifolds of an indefinite Sasakian manifold have been studied by Duggal and Sahin in [5]. In 2009, Sahin [9] study screen slant lightlike submanifolds of indefinite Kaehler manifold. In [11], authors introduced the notion of an ϵ-para-Sasakian structure and gave some examples.

In this article, we study lightlike submanifolds of an ϵ-para-Sasakian manifold, which is called an indefinite para-Sasakian manifold. The paper is arranged as follows. Section 2 contains some basic results and definitions. In Section 3, we study invariant lightlike submanifolds of an indefinite para-Sasakian manifold giving some examples. Section 4 deals with slant lightlike submanifolds of an indefinite para-Sasakian manifold. In Section 5, we study screen slant lightlike submanifolds of an indefinite para-Sasakian manifold and obtain integrability conditions of distributions D and $RadTM$.

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2. Preliminaries

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called an ϵ-almost paracontact metric manifold [11] if there exists a $(1,1)$ tensor field ϕ, a vector field V called the characteristic vector field and a 1-form η, satisfying

$$\phi^2 X = X - \eta(X)V, \quad \eta(V) = \epsilon, \quad \eta \circ \phi = 0, \quad \phi V = 0, \tag{2.1}$$

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\overline{M}), \tag{2.2}$$

where $\epsilon = 1$ or -1. It follows that

$$\overline{g}(V, V) = \epsilon, \qquad \overline{g}(X, V) = \eta(X),$$
$$\overline{g}(X, \phi Y) = \overline{g}(\phi X, Y), \quad \forall X, Y \in \Gamma(T\overline{M}). \tag{2.3}$$

Then $(\phi, V, \eta, \overline{g})$ is called an ϵ-almost paracontact metric structure on $\overline{M}$.

An ϵ-almost paracontact metric structure $(\phi, V, \eta, \overline{g})$ is called an indefinite para-Sasakian structure [11] if

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, \phi Y)V - \epsilon\eta(Y)\phi^2 X, \quad \forall X, Y \in \Gamma(T\overline{M}), \tag{2.4}$$

where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

A semi-Riemannian manifold endowed with an indefinite para-Sasakian structure is called an indefinite para-Sasakian manifold. From (2.4), we get

$$(\overline{\nabla}_X V) = \phi X, \quad \forall X \in \Gamma(T\overline{M}). \tag{2.5}$$

Let $(\overline{M}, \overline{g}, \phi, V, \eta)$ be an ϵ-almost paracontact metric manifold. If $\epsilon = 1$, then $\overline{M}$ is said to be a spacelike ϵ-almost paracontact metric manifold and if $\epsilon = -1$, then $\overline{M}$ is called a timelike ϵ-almost paracontact metric manifold. In this paper we consider indefinite para-Sasakian manifold with spacelike characteristic vector field V.

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [3] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM).$$

Now consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi- Riemannian complementary vector bundle of $RadTM$ in $TM^\perp$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $[S(TM)]^\perp$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^\perp),$$
$$T\overline{M}|_M = TM \oplus tr(TM),$$
$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^\perp).$$

The following are four cases of a lightlike submanifold $\left(M, g, S(TM), S(TM^{\perp})\right)$:

Case 1. r-lightlike if $r < min\,(m, n)$,

Case 2. co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case 3. isotropic if $r = m < n$, $S\,(TM) = \{0\}$,

Case 4. totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.6}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall V \in \Gamma(tr(TM)), \tag{2.7}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.6) and (2.7), we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l\,(X, Y) + h^s\,(X, Y)\,, \quad \forall X, Y \in \Gamma(TM), \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X\,(N) + D^s\,(X, N)\,, \quad \forall N \in \Gamma(ltr(TM)), \tag{2.9}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X\,(W) + D^l\,(X, W)\,, \quad \forall W \in \Gamma(S(TM^{\perp})), \tag{2.10}$$

where $h^l(X, Y) = L\,(h(X, Y))$, $h^s(X, Y) = S\,(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. $\nabla^l$and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively. For any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{2.11}$$

where PX and FX are tangential and transversal parts of ϕX respectively.

Now by using (2.6), (2.8)–(2.10) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y),$$
$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X).$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \quad \forall X, Y \in \Gamma(TM),$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \quad \xi \in \Gamma(RadTM).$$

By using above equations, we obtain

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_\xi X, \overline{P}Y),$$
$$\overline{g}(h^*(X, \overline{P}Y), N) = g(A_N X, \overline{P}Y),$$
$$\overline{g}(h^l(X, \xi), \xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.12}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.8), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y).$$

Definition 2.1. [3] A submanifold M of semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be totally geodesic lightlike submanifold of $\overline{M}$ if any geodesic of M, with respect to Levi-Civita connection $\overline{\nabla}$, is a geodesic of $\overline{M}$, i.e., $h^l = h^s = 0$ on M.

Definition 2.2. [1] A lightlike submanifold $\big(M, g, S(TM), S(TM^\perp)\big)$ of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is minimal if $h^s = 0$ on $Rad(TM)$ and $tr(h) = 0$, where trace is written with respect to g restricted to $S(TM)$.

Definition 2.3. [4] A lightlike submanifold $\big(M, g, S(TM), S(TM^\perp)\big)$ of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X,Y) = H\overline{g}(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{2.13}$$

From (2.8) and (2.13), it is easy to see that M is totally umbilical if and only if on each coordinate neighbourhood U, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$, such that

$$h^l(X,Y) = H^l\overline{g}(X,Y) \quad \text{and} \quad h^s(X,Y) = H^s\overline{g}(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{2.14}$$

3. Invariant lightlike submanifolds

Definition 3.1. A lightlike submanifold M, tangent to the structure vector field V, of an indefinite para-Sasakian manifold $\overline{M}$ is said to be invariant lightlike submanifold if the following condition is satisfied:

$$\phi(RadTM) = RadTM \quad \text{and} \quad \phi(D) = D, \tag{3.1}$$

where $S(TM) = D \perp \{V\}$ and D is complementary nondegenerate distribution to $\{V\}$ in $S(TM)$.

From (2.4), (2.5), (2.8) and (3.1), we get

$$h^l(X,V) = 0, \quad h^s(X,V) = 0, \quad \nabla_X V = PX, \tag{3.2}$$

$$h(X,\phi Y) = \phi h(X,Y) = h(\phi X, Y), \quad \forall X,Y \in \Gamma(TM). \tag{3.3}$$

Let $(\mathbb{R}^{2m+1}_q, \overline{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_q$ with its usual para-Sasakian structure given by

$$\eta = \tfrac{1}{2}(dz - \textstyle\sum_{i=1}^m y^i dx^i), \quad V = 2\partial z,$$

$$\overline{g} = \eta \otimes \eta + \tfrac{1}{4}(-\textstyle\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=\frac{q}{2}+1}^m dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi(\textstyle\sum_{i=1}^m (X_i \partial x_i + Y_i \partial y_i) + Z\partial z) = \sum_{i=1}^m (Y_i \partial x_i + X_i \partial y_i) + \sum_{i=1}^m Y_i y^i \partial z,$$

where $(x^i; y^i; z)$ are the cartesian coordinates on $\mathbb{R}^{2m+1}_q$. Now we construct some examples of invariant lightlike submanifolds of an indefinite para-Sasakian manifold.

Example 1. Let $(\mathbb{R}^7_2, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,-,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial y_1, \partial y_2, \partial y_3, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^7_2$ given by $x^1 = y^2 = u_1$, $x^2 = y^1 = u_2$, $x^3 = u_3$, $y^3 = u_4$, $z = u_5$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),$$

$$Z_3 = 2(\partial x_3 + y^3 \partial z), \quad Z_4 = 2\partial y_3 \quad \text{and} \quad Z_5 = V = 2\partial z.$$

Hence $RadTM = span\{Z_1, Z_2\}$, $S(TM) = span\{Z_3, Z_4, V\}$ and $ltr(TM)$ is spanned by $N_1 = \partial x_1 - \partial y_2 + y^1 \partial z$, $N_2 = -\partial x_2 + \partial y_1 - y^2 \partial z$.

It follows that $\phi Z_1 = Z_2$, $\phi Z_2 = Z_1$, $\phi Z_3 = Z_4$, $\phi Z_4 = Z_3$, $\phi N_1 = N_2$ and $\phi N_2 = N_1$. Thus $\phi RadTM = RadTM$, $\phi D = D$ and $\phi ltr(TM) = ltr(TM)$. Hence M is an invariant 2-lightlike submanifold of $\mathbb{R}^7_2$.

Example 2. Let $(\mathbb{R}^9_2, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^9_2$ given by $x^1 = y^2 = u_1$, $x^2 = y^1 = u_2$, $-x^3 = y^4 = u_3$, $-x^4 = y^3 = u_4$, $z = u_5$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z), \quad Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z),$$

$$Z_3 = 2(-\partial x_3 + \partial y_4 - y^3 \partial z), \quad Z_4 = 2(-\partial x_4 + \partial y_3 - y^4 \partial z),\ Z_5 = V = 2\partial z.$$

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z$, $N_2 = \partial x_2 - \partial y_1 + y^2 \partial z$ and $S(TM^\perp)$ is spanned by $W_1 = 2(\partial x_3 + \partial y_4 + y^3 \partial z), W_2 = 2(\partial x_4 + \partial y_3 + y^4 \partial z)$.

It follows that $\phi Z_1 = Z_2$, $\phi Z_2 = Z_1$, $\phi Z_3 = -Z_4$, $\phi Z_4 = -Z_3$, $\phi N_1 = N_2$, $\phi N_2 = N_1$, $\phi W_1 = W_2$ and $\phi W_2 = W_1$. Thus $\phi RadTM = RadTM$, $\phi D = D$, $\phi ltr(TM) = ltr(TM)$ and $\phi S(TM^\perp) = S(TM^\perp)$. Hence M is an invariant 2-lightlike submanifold of $\mathbb{R}^9_2$.

Theorem 3.1. *Let $(M, g, S(TM), S(TM^\perp))$ be an invariant lightlike submanifold, tangent to the structure vector field V of an indefinite para-Sasakian manifold $\overline{M}$. If the second fundamental forms h^l and h^s of M are parallel then M is totally geodesic.*

Proof. Suppose h^l is parallel. Then $(\nabla_X h^l)(Y, V) = 0$, $\forall X, Y \in \Gamma(TM)$, which implies

$$\nabla_X h^l(Y, V) - h^l(\nabla_X Y, V) - h^l(Y, \nabla_X V) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{3.4}$$

From (3.2) and (3.4), we get $h^l(Y, \nabla_X V) = 0$, $\forall X, Y \in \Gamma(TM)$. Thus from above, we have $h^l(Y, PX) = 0$, $\forall X, Y \in \Gamma(TM)$. Hence $h^l = 0$. Similarly $h^s = 0$. Thus M is totally geodesic. ∎

Theorem 3.2. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to the structure vector field V of an indefinite para-Sasakian manifold $\overline{M}$. If M is totally umbilical then it is totally geodesic.*

Proof. Let M be a totally umbilical lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then, from (2.8), we have

$$\overline{\nabla}_X V = \nabla_X V + h^l(X,V) + h^s(X,V), \quad \forall X \in \Gamma(TM). \tag{3.5}$$

From (2.5), (2.11) and (3.5), we get

$$PX + FX = \nabla_X V + h^l(X,V) + h^s(X,V), \quad \forall X \in \Gamma(TM). \tag{3.6}$$

Equating transversal parts in (3.6), we get

$$h^l(X,V) + h^s(X,V) = FX. \tag{3.7}$$

Replacing X by V in (3.7), we get

$$h^l(V,V) + h^s(V,V) = FV. \tag{3.8}$$

Now from (2.1), (2.11) and (3.8), we get

$$h^l(V,V) = 0 \quad \text{and} \quad h^s(V,V) = 0. \tag{3.9}$$

From (2.14) and (3.9), we have $H^l\overline{g}(V,V) = 0$ and $H^s\overline{g}(V,V) = 0$.

Since V is non-null vector, we have $H^l = H^s = 0$. Thus from (2.14), we obtain $h^l(X,Y) = 0$ and $h^s(X,Y) = 0$. Hence, M is totally geodesic. ∎

Theorem 3.3. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of nullity degree two of an indefinite para-Sasakian manifold $\overline{M}$. Then, $RadTM$ defines a totally geodesic foliation on M.*

Proof. Let M be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. By definition of lightlike submanifold, $RadTM$ defines a totally geodesic foliation if and only if $\overline{g}(\nabla_X Y, Z) = 0$, $\forall X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Since $rank(RadTM) = 2$, we can write $X, Y \in \Gamma(RadTM)$ as a linear combination of ξ and $\phi\xi$, that is $X = A_1\xi + B_1\phi\xi$ and $Y = A_2\xi + B_2\phi\xi$. Now since $\overline{\nabla}$ is a metric connection, using (2.8), we get

$$\begin{aligned}\overline{g}(\nabla_X Y, Z) &= X\overline{g}(Y,Z) - \overline{g}(Y, \overline{\nabla}_X Z)\\ &= -\overline{g}(Y, \overline{\nabla}_X Z) = -\overline{g}(Y, h^l(X,Z))\\ &= -\overline{g}(A_2\xi + B_2\phi\xi, h^l(A_1\xi + B_1\phi\xi, Z))\\ &= -A_1A_2\overline{g}(\xi, h^l(\xi,Z)) - B_1A_2\overline{g}(\xi, h^l(\phi\xi,Z)) - B_2A_1\overline{g}(\phi\xi, h^l(\xi,Z))\\ &\quad - B_2A_2\overline{g}(\phi\xi, h^l(\phi\xi,Z)), \text{ for all } X, Y \in RadTM \text{ and } Z \in \Gamma(S(TM)).\end{aligned} \tag{3.10}$$

From (2.12), (3.3) and (3.10), we get $\overline{g}(\nabla_X Y, Z) = 0$, which completes the proof. ∎

4. Slant lightlike submanifolds

At first, we state the following lemmas for later use:

Lemma 4.1. *Let M be an r-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ of index $2q$ with structure vector field tangent to M. Suppose that $\phi RadTM$ is a distibution on M such that $RadTM \cap \phi RadTM = \{0\}$.*

Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi RadTM \cap \phi ltr(TM) = \{0\}$.

LEMMA 4.2. *Let M be a q-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$, of index $2q$ with structure vector field tangent to M. Suppose $RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi ltr(TM) \oplus \phi RadTM$ in $S(TM)$ is Riemannian.*

The proofs of Lemma 4.1 and Lemma 4.2 follow as in Lemma 3.1 and Lemma 3.2 respectively of [10], so we omit them.

DEFINITION 4.1. Let M be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then we say that M is slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $RadTM$ is a distribution on M such that $\phi RadTM \cap RadTM = \{0\}$,

(ii) For each non-zero vector field X tangent to D at $x \in U \subset M$, the angle $\theta(X)$ between ϕX and the vector space D_x is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where D is complementary distribution to $(\phi RadTM \oplus \phi ltr(TM)) \perp \{V\}$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called slant angle of distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp D \perp \{V\}. \tag{4.1}$$

From Definition 4.1, we conclude that the class of slant lightlike submanifolds does not include invariant lightlike submanifolds of an indefinite para-Sasakian manifold.

EXAMPLE 1. Let $(\mathbb{R}^9_2, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^9_2$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $x^3 = 0$, $x^4 = u_3$, $y^1 = u_4$, $y^3 = u_5 \sin\theta$, $y^4 = u_5 \cos\theta$, $z = u_6$, where $\theta \in (0, \frac{\pi}{2})$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = 2(-\partial x_1 + \partial y_2 - y^1 \partial z)$, $Z_2 = 2(\partial x_2 + y^2 \partial z)$, $Z_3 = 2(\partial x_4 + y^4 \partial z)$,

$Z_4 = 2\partial y_1$, $Z_5 = 2(\sin\theta \partial y_3 + \cos\theta \partial y_4)$, $Z_6 = V = 2\partial z$.

Hence $RadTM = span\,\{Z_1\}$ and $S(TM) = span\,\{Z_2, Z_3, Z_4, Z_5, V\}$.

Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^\perp)$ is spanned by $W_1 = 2(\partial x_3 + y^3 \partial z), W_2 = 2(\cos\theta \partial y_3 - \sin\theta \partial y_4)$. It follows that $\phi Z_1 = 2(\partial x_2 - \partial y_1 + y^2 \partial z) = Z_2 - Z_4$, $\phi N = \partial x_2 + \partial y_1 + y^2 \partial z = \frac{1}{2}(Z_2 + Z_4)$ and $g(\phi Z_1, \phi N) = 1$. Thus $\phi RadTM$ and $\phi ltr(TM)$ are distributions on M and $D = span\,\{Z_3, Z_5\}$ is a slant distribution with slant angle θ. Thus $TM = RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp D \perp \{V\}$. Hence M is a slant lightlike submanifold of $\mathbb{R}^9_2$.

THEOREM 4.3. *Let M be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M such that $\phi RadTM \cap$*

$RadTM = \{0\}$. *Then* M *is slant lightlike submanifold if and only if there exists a constant* $\lambda \in [0,1]$ *such that* $P^2X = \lambda(X - \eta(X)V)$, $\forall X \in \Gamma(D)$.

Proof. Let M be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Suppose there exists a constant λ, such that $P^2X = \lambda(X - \eta(X)V) = \lambda\phi^2X$, $\forall X \in \Gamma(D)$. Now

$$\cos\theta(X) = \frac{g(\phi X, PX)}{|\phi X||PX|} = \frac{g(X, \phi PX)}{|\phi X||PX|} = \frac{g(X, P^2X)}{|\phi X||PX|} = \lambda\frac{g(X, \phi^2 X)}{|\phi X||PX|} = \lambda\frac{g(\phi X, \phi X)}{|\phi X||PX|}.$$

From above equation, we get

$$\cos\theta(X) = \lambda\frac{|\phi X|}{|PX|}. \tag{4.2}$$

Also $|PX| = |\phi X|\cos\theta(X)$, which implies

$$\cos\theta(X) = \frac{|PX|}{|\phi X|}. \tag{4.3}$$

From (4.2) and (4.3), we get $\cos^2\theta(X) = \lambda(\text{constant})$. Hence, M is a slant lightlike submanifold.

Conversely, suppose that M is a slant lightlike submanifold. Then $\cos^2\theta(X) = \lambda$, where λ is a constant. From (4.3), we have $\frac{|PX|^2}{|\phi X|^2} = \lambda$. Now $g(PX, PX) = \lambda g(\phi X, \phi X)$, which gives $g(X, P^2X) = \lambda g(X, \phi^2X)$. Thus $g(X, (P^2 - \lambda\phi^2)X) = 0$. Since X is non-null vector, we have $(P^2 - \lambda\phi^2)X = 0$. Hence, $P^2X = \lambda\phi^2X = \lambda(X - \eta(X)V)$, $\forall X \in \Gamma(D)$. ∎

Corollary 4.4. *Let* M *be a slant lightlike submanifold of an indefinite para-Sasakian manifold* $\overline{M}$ *with slant angle* θ. *Then*

$$g(PX, PY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D),$$
$$g(FX, FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D).$$

Proof. Since $g(PX, PY) = g(X, P^2Y) = g(X, \lambda\phi^2Y) = \lambda g(X, \phi^2Y) = \lambda g(\phi X, \phi Y)$, $\forall X, Y \in \Gamma(D)$, we have

$$g(PX, PY) = \cos^2\theta g(\phi X, \phi Y), \quad \forall X, Y \in \Gamma(D). \tag{4.4}$$

Thus $\quad g(PX, PY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D)$.

From (4.4), we obtain $g(PX, PY) = (1 - \sin^2\theta)g(\phi X, \phi Y)$, $\forall X, Y \in \Gamma(D)$, which implies $g(\phi X, \phi Y) - g(PX, PY) = \sin^2\theta g(\phi X, \phi Y)$, $\forall X, Y \in \Gamma(D)$, which gives $g(FX, FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y))$, $\forall X, Y \in \Gamma(D)$. This completes the proof. ∎

Now, we denote the projections on $RadTM, \phi RadTM, \phi ltr(TM)$ and D in TM by P_1, P_2, P_3 and P_4, respectively. Similarly, we denote the projections on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2, respectively. Then, we get

$$X = P_1X + P_2X + P_3X + P_4X + \eta(X)V, \quad \forall X \in \Gamma(TM). \tag{4.5}$$
$$W = Q_1W + Q_2W, \quad \forall W \in \Gamma(tr(TM)). \tag{4.6}$$

Now applying ϕ to (4.5), we have

$$\phi X = \phi P_1 X + \phi P_2 X + \phi P_3 X + fP_4 X + FP_4 X, \quad \forall X \in \Gamma(TM),$$

where fP_4X (resp. FP_4X) denotes the tangential (resp. screen transversal) component of $\phi P_4 X$. Thus we get

$$\phi P_1 X \in \phi RadTM, \ \ \phi P_2 X \in \Gamma(RadTM), \ \ \phi P_3 X \in \Gamma(ltr(TM)),$$
$$fP_4 X \in \Gamma(D), \ \ FP_4 X \in \Gamma(S(TM^{\perp})).$$

Applying ϕ to (4.6), we obtain $\phi W = \phi Q_1 W + BQ_2 W + CQ_2 W$, where BQ_2W (resp. CQ_2W) denote the tangential (resp. transversal) component of $\phi Q_2 W$.

Now, by using (2.4), (4.5) and (2.8)–(2.10) and equating tangential, lightlike transversal and screen transversal components, we obtain

$$\begin{aligned} -\overline{g}(\phi X, \phi Y)V - \eta(Y)\phi^2 X = \nabla_X \phi P_1 X + \nabla_X \phi P_2 X - A_{\phi P_3 Y} X + \nabla_X fP_4 Y \\ - A_{FP_4 Y} X - \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y \\ - fP_4 \nabla_X Y - \phi h^l(X,Y) - Bh^s(X,Y), \end{aligned} \tag{4.7}$$

$$\begin{aligned} h^l(X, \phi P_1 Y) + h^l(X, \phi P_2 Y) + h^l(X, fP_4 Y) = -\nabla^l_X \phi P_3 Y - D^l(X, FP_4 Y) \\ + \phi P_3 \nabla_X Y, \end{aligned}$$

$$\begin{aligned} h^s(X, \phi P_1 Y) + h^s(X, \phi P_2 Y) + h^s(X, fP_4 Y) = -D^s(X, \phi P_3 Y) - \nabla^s_X FP_4 Y \\ + FP_4 \nabla_X Y - Ch^s(X,Y). \ \blacksquare \end{aligned}$$

THEOREM 4.5. *Let M be a proper slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field V tangent to M. Then induced connection ∇ is never a metric connection.*

Proof. Suppose that the induced connection is a metric connection. Then $\nabla_X \phi P_2 Y \in \Gamma(RadTM)$ and $h^l(X,Y) = 0$. Thus for $Y \in \phi RadTM$ and $X \in \phi ltr(TM)$, (4.7) becomes

$$-\overline{g}(X,Y)V = \nabla_X \phi P_2 X - \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y - fP_4 \nabla_X Y - Bh^s(X,Y).$$

Since $TM = RadTM \oplus \phi RadTM \oplus \phi ltr(TM) \oplus D \oplus V$, from (4.8), we get

$$\begin{aligned} \phi P_1 \nabla_X Y = 0, \quad \nabla_X \phi P_2 X + \phi P_2 \nabla_X Y = 0, \\ \overline{g}(X,Y)V = 0, \quad fP_4 \nabla_X Y + Bh^s(X,Y) = 0. \end{aligned} \tag{4.9}$$

Now, taking $X = \phi N$ and $Y = \phi\xi$ in (4.9), we get $\overline{g}(N,\xi)V = 0$. Thus $V = 0$, which is a contradiction. Hence M does not have a metric connection. $\blacksquare$

5. Screen slant lightlike submanifolds

At first, we state the following lemma for later use:

LEMMA 5.1. *Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$, of index $2q$ such that $2q < dim(M)$ with structure vector field tangent* *to M. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

The proof of above lemma follows as in Lemma 4.1 of [10], so we omit it.

Definition 5.1. Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then we say that M is screen slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:

(i) $RadTM$ is invariant with respect to ϕ, i.e. $\phi(RadTM) = RadTM$,

(ii) For each non-zero vector field X tangent to D at $x \in U \subset M$, the angle $\theta(X)$ between ϕX and the vector space D_x is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where D is complementary nondegenerate distribution to $\{V\}$ in $S(TM)$ such that $S(TM) = D\perp\{V\}$.

This constant angle $\theta(X)$ is called the slant angle of distribution D. A screen slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = RadTM\perp D\perp\{V\}. \tag{5.1}$$

From Definitions 4.1 and 5.1, we conclude that the class of screen slant lightlike submanifolds does not include slant lightlike submanifolds of an indefinite para-Sasakian manifold and vice-versa.

Theorem 5.2. *Let M be a screen slant lightlike submanifold of $\overline{M}$. Then M is invariant (resp. screen real) if and only if $\theta = 0$(resp. $\theta = \frac{\pi}{2}$).*

Proof of the above theorem follows from Proposition 4.1 of [10].

Example 1. Let $(\mathbb{R}^9_2, \overline{g}, \phi, \eta, V)$ be an indefinite para-Sasakian manifold, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^9_2$ given by $x^1 = y^2 = u_1$, $x^2 = y^1 = u_2$, $x^3 = u_3 \cos\theta$, $x^4 = u_3 \sin\theta$, $y^3 = u_4 \sin\theta$, $y^4 = u_4 \cos\theta$, $z = u_5$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$, where

$Z_1 = 2(\partial x_1 + \partial y_2 + y^1 \partial z)$, $Z_2 = 2(\partial x_2 + \partial y_1 + y^2 \partial z)$,

$Z_3 = 2(\cos\theta \partial x_3 + \sin\theta \partial x_4 + y^3 \cos\theta \partial z + y^4 \sin\theta \partial z)$,

$Z_4 = 2(\sin\theta \partial y_3 + \cos\theta \partial y_4)$, $Z_5 = V = 2\partial z$.

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, V\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2 - y^1 \partial z$, $N_2 = \partial x_2 - \partial y_1 + y^2 \partial z$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(-\sin\theta \partial x_3 + \cos\theta \partial x_4 - y^3 \sin\theta \partial z + y^4 \cos\theta \partial z)$,

$W_2 = 2(\cos\theta \partial y_3 - \sin\theta \partial y_4)$.

It follows that $\phi Z_1 = Z_2$, $\phi Z_2 = Z_1$, which implies that $RadTM$ is invariant, i.e., $\phi RadTM = RadTM$. On other hand, we can see that $D = span\{Z_3, Z_4\}$ is a slant distribution with slant angle 2θ. Hence M is screen slant 2-lightlike submanifold of $\mathbb{R}^9_2$.

Now, we denote the projections on $RadTM$ and D in TM by P_1 and P_2 respectively. Similarly, we denote the projections on $ltr(TM)$ and $S(TM^{\perp})$ by Q_1 and Q_2 respectively. Then, we get

$$X = P_1X + P_2X + \eta(X)V, \quad \forall X \in \Gamma(TM). \tag{5.2}$$

Now applying ϕ to (5.2), we have $\phi X = \phi P_1X + \phi P_2X$, which gives

$$\phi X = \phi P_1X + fP_2X + FP_2X, \quad \forall X \in \Gamma(TM), \tag{5.3}$$

where fP_2X (resp. FP_2X) denotes the tangential (resp. transversal) component of ϕP_2X. Thus we get $\phi P_1X \in RadTM, fP_2X \in \Gamma(D), FP_2X \in \Gamma(S(TM^{\perp}))$. Also, we have

$$W = Q_1W + Q_2W, \quad \forall W \in \Gamma(tr(TM)). \tag{5.4}$$

Applying ϕ to (5.4), we obtain

$$\phi W = \phi Q_1W + \phi Q_2W, \tag{5.5}$$

which gives

$$\phi W = \phi Q_1W + BQ_2W + CQ_2W, \tag{5.6}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of ϕQ_2W.

Now, by using (2.4), (5.3), (5.6) and (2.8)–(2.10) and equating tangential, lightlike transversal and screen transversal components, we obtain

$$\begin{aligned} -\overline{g}(\phi X, \phi Y)V - \eta(Y)\phi^2X = \nabla_X\phi P_1Y + \nabla_X fP_2Y - A_{FP_2Y}X \\ - \phi P_1\nabla_XY - fP_2\nabla_XY + Bh^s(X,Y), \end{aligned} \tag{5.7}$$

$$\begin{aligned} h^l(X, \phi P_1Y) + h^l(X, fP_2Y) &= \phi h^l(X,Y) - D^l(X, FP_2Y), \\ h^s(X, \phi P_1Y) + h^s(X, fP_2Y) &= Ch^s(X,Y) - \nabla^s_X FP_2Y - FP_2\nabla_XY. \end{aligned} \tag{5.8}$$

THEOREM 5.3. *Let M be a $2q$-lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then M is screen slant lightlike submanifold if and only if*

(i) the lightlike transversal vector bundle $ltr(TM)$ is invariant with respect to ϕ,

(ii) there exists a constant $\lambda \in [0,1]$ such that $P^2X = \lambda(X - \eta(X)V)$, $\forall X \in \Gamma(D)$.

Proof. Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then its radical distribution $RadTM$ is invariant with respect to ϕ, i.e., $\phi X = X$ $\forall X \in \Gamma RadTM$.

Now, for $N \in \Gamma ltr(TM)$ and $X \in \Gamma D$, using (2.3) and (5.3), we obtain

$$\overline{g}(\phi N, X) = \overline{g}(N, \phi X) = \overline{g}(N, fX + FX) = \overline{g}(N, fX) + \overline{g}(N, FX) = 0.$$

Thus ϕN does not belong to $\Gamma(D)$.

For $N \in \Gamma ltr(TM)$ and $W \in \Gamma S(TM^{\perp})$, from (2.3) and (5.6), we have

$\overline{g}(\phi N, W) = \overline{g}(N, \phi W) = \overline{g}(N, BW + CW) = \overline{g}(N, BW) + \overline{g}(N, CW) = 0.$

Hence we conclude that ϕN does not belong to $\Gamma S(TM^{\perp})$.

Now, suppose that $\phi N \in \Gamma(RadTM)$. Then $\phi(\phi N) = \phi^2 N = -N + \eta(N)V \in \Gamma(ltrTM) \oplus span\{V\}$, which contradicts that $RadTM$ is invariant. Hence $ltr(TM)$ is invariant with respect to ϕ.

Since $|PX| = |\phi X| \cos\theta(X)$, $\forall X \in \Gamma(D)$, we have

$$\cos\theta(X) = \frac{|PX|}{|\phi X|}. \tag{5.9}$$

In view of (5.9), we get $\cos^2\theta(X) = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX,PX)}{g(\phi X,\phi X)} = \frac{g(X,P^2X)}{g(X,\phi^2 X)}$, which gives

$$g(X, P^2X) = \cos^2\theta g(X, \phi^2 X). \tag{5.10}$$

Since M is screen slant lightlike submanifold, $\cos^2\theta(X) = \lambda(constant) \in [0, 1]$.

Therefore from (5.10), we get $g(X, P^2X) = \lambda g(X, \phi^2 X) = g(X, \lambda\phi^2 X)$, which implies $g(X, (P^2 - \lambda\phi^2)X) = 0$. Since X is non-null vector, we have $(P^2 - \lambda\phi^2)X = 0$, which implies

$$P^2X = \lambda\phi^2 X = \lambda(X - \eta(X)V), \quad \forall X \in \Gamma(D).$$

This proves (ii).

Conversely suppose that conditions (i) and (ii) are satisfied. We can show that $RadTM$ is invariant in similar way that $ltr(TM)$ is invariant. From (ii) we have $P^2X = \lambda\phi^2X$, $\forall X \in \Gamma(D)$, where λ(constant) $\in [0, 1]$.

Now, $\cos\theta(X) = \frac{g(\phi X,PX)}{|\phi X||PX|} = \frac{g(X,\phi PX)}{|\phi X||PX|} = \frac{g(X,P^2X)}{|\phi X||PX|} = \lambda\frac{g(X,\phi^2X)}{|\phi X||PX|} = \lambda\frac{g(\phi X,\phi X)}{|\phi X||PX|}$. From the above equation, we get

$$\cos\theta(X) = \lambda\frac{|\phi X|}{|PX|}. \tag{5.11}$$

Therefore (5.9) and (5.11) give $\cos^2\theta(X) = \lambda$(constant). Hence M is a screen slant lightlike submanifold. ■

Corollary 5.4 *Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with slant angle θ, then*

$$\begin{aligned} g(PX, PY) &= \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D), \\ g(FX, FY) &= \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)), \quad \forall X, Y \in \Gamma(D). \end{aligned} \tag{5.12}$$

The proof of above corollary follows using the steps as in proof of Corollary 3.2 of [9].

Lemma 5.5. *Let M be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then we have* *(i)* $g(\nabla_X Y, V) = -\overline{g}(Y, \phi X), \quad \forall X, Y \in \Gamma(TM) - \{V\}$,

(ii) $g([X, Y], V) = 0, \quad \forall X, Y \in \Gamma(TM) - \{V\}$.

Proof. Let M be a lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then from (2.8), we have

$$g(\nabla_X Y, V) = \overline{g}(\overline{\nabla}_X Y, V), \quad \forall X, Y \in \Gamma(TM) - \{V\}. \tag{5.13}$$

Since $\overline{\nabla}$ is a metric connection, from (5.13) we get

$$g(\nabla_X Y, V) = -\overline{g}(Y, \overline{\nabla}_X V), \quad \forall X, Y \in \Gamma(TM) - \{V\}. \tag{5.14}$$

From (2.5) and (5.14), we obtain

$$g(\nabla_X Y, V) = -\overline{g}(Y, \phi X), \quad \forall X, Y \in \Gamma(TM) - \{V\}. \tag{5.15}$$

On interchanging X and Y in (5.15), we get

$$g(\nabla_Y X, V) = -\overline{g}(X, \phi Y), \quad \forall X, Y \in \Gamma(TM) - \{V\}. \tag{5.16}$$

From (2.3), (5.15) and (5.16), we have

$$g([X, Y], V) = 0, \quad \forall X, Y \in \Gamma(TM) - \{V\}. \quad \blacksquare \tag{5.17}$$

Theorem 5.6. *Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then*

(i) the radical distribution $RadTM$ is integrable if and only if $h^s(Y, \phi X) = h^s(X, \phi Y)$ and $(\nabla_X \phi P_1)Y = (\nabla_Y \phi P_1)X$, $\forall X, Y \in \Gamma(RadTM)$,

(ii) the distribution D is integrable if and only if $P_1(\nabla_X fY - \nabla_Y fX) = P_1(A_{FY}X - A_{FX}Y)$, $\forall X, Y \in \Gamma(D)$.

Proof. Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. From (5.8), we get

$$h^s(X, \phi Y) = Ch^s(X, Y) - FP_2\nabla_X Y, \quad \forall X, Y \in \Gamma(RadTM). \tag{5.18}$$

Interchanging X and Y in (5.18), we get

$$h^s(Y, \phi X) = Ch^s(Y, X) - FP_2\nabla_Y X, \quad \forall X, Y \in \Gamma(RadTM). \tag{5.19}$$

From (5.18) and (5.19), we get

$$h^s(Y, \phi X) - h^s(X, \phi Y) = FP_2(\nabla_X Y - \nabla_Y X) = FP_2[X, Y]. \tag{5.20}$$

From (5.7), we have

$$\nabla_X \phi P_1 Y - \phi P_1 \nabla_X Y - fP_2\nabla_X Y + Bh^s(X, Y) = 0, \quad \forall X, Y \in \Gamma(RadTM). \tag{5.21}$$

On interchanging X and Y in (5.21), we get

$$\nabla_Y \phi P_1 X - \phi P_1 \nabla_Y X - fP_2\nabla_Y X + Bh^s(Y, X) = 0, \quad \forall X, Y \in \Gamma(RadTM). \tag{5.22}$$

From (5.21) and (5.22), we have

$$(\nabla_X \phi P_1)Y - (\nabla_Y \phi P_1)X = fP_2([X, Y]), \quad \forall X, Y \in \Gamma(RadTM). \tag{5.23}$$

Proof of (i) follows from (5.17), (5.20) and (5.23).

Now from (5.7) and (2.2), we obtain

$$\overline{g}(\phi X,\phi Y)V+\nabla_X fY-A_{FY}X=\phi P_1\nabla_X Y+fP_2\nabla_X Y-Bh^s(X,Y),\ \forall X,Y\in\Gamma(D). \tag{5.24}$$

Interchanging X and Y in (5.24), we have

$$\overline{g}(\phi Y,\phi X)V+\nabla_Y fX-A_{FX}Y=\phi P_1\nabla_Y X+fP_2\nabla_Y X-Bh^s(Y,X),\ \forall X,Y\in\Gamma(D). \tag{5.25}$$

From (5.24) and (5.25), we get

$$\begin{aligned}\nabla_X fY-\nabla_Y fX+A_{FX}Y-A_{FY}X&\\ &=\phi P_1\nabla_X Y-\phi P_1\nabla_Y X+fP_2\nabla_X Y-fP_2\nabla_Y X\\ &=\phi P_1[X,Y]+fP_2[X,Y],\quad \forall X,Y\in\Gamma(D).\end{aligned} \tag{5.26}$$

The equation (5.26) implies

$$P_1(\nabla_X fY-\nabla_Y fX)+P_1(A_{FX}Y-A_{FY}X)=\phi P_1[X,Y],\quad \forall X,Y\in\Gamma(D). \tag{5.27}$$

Proof of (ii) follows from (5.17) and (5.27). ∎

THEOREM 5.7. *Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. Then $S(TM)$ defines a totally geodesic foliation if and only if $\nabla_X fY-A_{FY}X$ has no component in $RadTM$, $\forall X,Y\in\Gamma(D)$.*

Proof. Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. From (2.2) and (2.8), we get

$$\overline{g}(\nabla_X Y,N)=\overline{g}(-(\overline{\nabla}_X\phi)Y+\overline{\nabla}_X\phi Y,\phi N),\quad \forall X,Y\in\Gamma(D)\quad\text{and}\quad N\in ltr(TM).$$

Using (2.4) in above equation, we get

$$\overline{g}(\nabla_X Y,N)=\overline{g}(\overline{g}(\phi X,\phi Y)V+\eta(Y)\phi^2X+\overline{\nabla}_X\phi Y,\phi N). \tag{5.28}$$

From (2.1) and (5.28), we obtain

$$\overline{g}(\nabla_X Y,N)=\overline{g}(\overline{\nabla}_X\phi Y,\phi N),\quad \forall X,Y\in\Gamma(D)\quad and\quad N\in ltr(TM). \tag{5.29}$$

From (2.8), (2.10), (5.3) and (5.29), we get

$$\overline{g}(\nabla_X Y,N)=\overline{g}(\nabla_X fY+h^l(X,fY)+h^s(X,fy)-A_{FY}X+\nabla^s_X FY+D^l(X,FY),\phi N).$$

From the above equation, we get

$$\overline{g}(\nabla_X Y,N)=\overline{g}(\nabla_X fY-A_{FY}X,\phi N),\quad \forall X,Y\in\Gamma(D)\text{ and }N\in ltr(TM).$$

which completes the proof. ∎

THEOREM 5.8. *Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$ with structure vector field tangent to M. If $Bh^s(X,Y)=0$, $\forall X\in\Gamma(TM)$ and $Y\in\Gamma(RadTM)$ then the induced connection ∇ is a metric connection.*

Proof. Let M be a screen slant lightlike submanifold of an indefinite para-Sasakian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ ([3]). Since

$Bh^s(X,Y)=0$, $\forall X\in\Gamma(TM)$ and $Y\in\Gamma(RadTM)$, we have $g(Bh^s(X,Y),Z)=0$, $\forall X,Z\in\Gamma(TM)$ and $Y\in\Gamma(RadTM)$. Thus from (5.5) and (5.6), we obtain

$$\overline{g}(\phi h^s(X,Y),Z)=0,\quad \forall X,Z\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM). \tag{5.30}$$

Using (2.3) and (5.3) in (5.30), we get

$$\overline{g}(h^s(X,Y),FP_2Z)=0,\quad \forall X,Z\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM). \tag{5.31}$$

Now from (2.8), we get

$$\overline{g}(FP_2\nabla_XY,\phi h^s(X,Y))=\overline{g}(FP_2\nabla_XY,\phi\overline{\nabla}_XY-\phi\nabla_XY-\phi h^l(X,Y)),\\ \forall X\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM). \tag{5.32}$$

Since $ltr(TM)$ is invariant, from (2.4), (5.3) and (5.32), we get

$$\overline{g}(FP_2\nabla_XY,\phi h^s(X,Y))=\overline{g}(FP_2\nabla_XY,\overline{\nabla}_X\phi Y)-\overline{g}(FP_2\nabla_XY,FP_2\nabla_XY). \tag{5.33}$$

From (2.8) and (5.33), we obtain

$$\overline{g}(FP_2\nabla_XY,\phi h^s(X,Y))=\overline{g}(FP_2\nabla_XY,h^s(X,\phi Y))-\overline{g}(FP_2\nabla_XY,FP_2\nabla_XY). \tag{5.34}$$

From (5.12), (5.31) and (5.34), we get

$$\overline{g}(FP_2\nabla_XY,\phi h^s(X,Y))=\sin^2\theta g(P_2\nabla_XY,P_2\nabla_XY),\\ \forall X\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM). \tag{5.35}$$

Now from (2.2) and (5.3), we have

$$\overline{g}(fP_2\nabla_XY,\phi h^s(X,Y))=\overline{g}(FP_2\nabla_XY,\phi h^s(X,Y)),\\ \forall X\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM). \tag{5.36}$$

The equations (5.30) and (5.36) imply

$$\overline{g}(FP_2\nabla_XY,\phi h^s(X,Y))=0,\quad \forall X\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM). \tag{5.37}$$

From (5.35) and (5.37), we get

$$\sin^2\theta g(P_2\nabla_XY,P_2\nabla_XY)=0,\quad \forall X\in\Gamma(TM)\quad\text{and}\quad Y\in\Gamma(RadTM).$$

Since M is proper screen slant lightlike submanifold and D is Riemannian, we get $P_2\nabla_XY=0$. Hence $\nabla_XY\in\Gamma(RadTM)$, i.e., radical distribution $RadTM$ is parallel, which completes the proof. ■

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