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DECOMPOSITIONS OF NORMALITY AND INTERRELATION AMONG ITS VARIANTS

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Abstract. Interrelation among some existing variants of normality is discussed and characterizations of these variants are obtained. It is verified that Urysohn's type lemma and Tietze's type extension hold for some of these variants of normality. Decomposition of normality in terms of near normality and some factorizations of normality in presence of some lower separation axioms are given.

1. Introduction and preliminaries

To properly study normality in general topology several forms of decomposition exist in the literature, e.g., Singal and Arya [20] provided decomposition in terms of almost normality and seminormality. Kohli and Das [8] gave a decomposition in terms of θ-normality and θ-regularity whereas Kohli and Singh [11] decomposed normality in terms of weak δ-normality by using Σ-normality. Arhangel'skii and Ludwig [1] introduced the notion of β-normality and studied the space in detail in which the authors provided one more decomposition of normality by using β-normality and κ-normality. Das [3] introduced (weak) (functional) Δ-normality and decomposed normality in terms of Δ-regularity which is a simultaneous generalization of regularity and normality. In this paper, interrelation among some existing variants of normality is established and some characterizations of these variants are given. Further, normality is decomposed by using a newly defined notion of near regularity and in terms of β-normality in presence of other lower separation axioms.

Let X be a topological space and $A \subset X$. Throughout the present paper closure of A is denoted by $\overline{A}$ and interior is denoted by $\operatorname{int} A$. A set A is said to be regularly closed [12] if $U = \operatorname{int} \overline{U}$. Complement of a regularly closed set is known as regularly open. A finite union of regular open sets is called π-open set and a finite intersection of regular closed sets is called π-closed set. A topological space X is said to be almost regular [19] if every regularly closed set A and a point

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outside it can be separated by disjoint open sets. A space X is said to be almost compact [2] if every open cover of X has a finite subcover whose closures covers X. A space X is said to be almost normal [20] if every pair of disjoint closed sets, one of which is regularly closed, are contained in disjoint open sets. A space is κ-normal [22] (mildly normal [21]) if any two disjoint regularly closed sets are contained in disjoint open sets. A topological space is a Lindelöf space if every open cover has a countable subcover. A point x is said to be θ-limit point [24] of A if closure of every neighborhood containing x intersects A. A set A_θ is the θ-closure of A which contains all θ-limit points of A. A set A is θ-closed if $A = A_\theta$. Complement of a θ-closed set is known as θ-open set. Similarly a point x is called δ-limit point of A if every regularly open neighborhood of x intersects A. A set A is said to be δ-closed if A_δ containing all δ-limit points of A is same as A. Complement of a δ-closed set is known as δ-open set.

2. Interrelation among variants of normality

Definition 2.1. A topological space X is said to be:

(i) θ-normal [8] if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets.

(ii) functionally θ-normal(fθ-normal) [8] if for every pair of disjoint closed sets A and B one of which is θ-closed there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

(iii) weakly θ-normal (wθ-normal) [8] if every pair of disjoint θ-closed sets are contained in disjoint open sets.

(iv) weakly functionally θ-normal (wfθ-normal) [8] if for every pair of disjoint θ-closed sets A and B there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

(v) Δ-normal [3] if every pair of disjoint closed sets one of which is δ-closed are contained in disjoint open sets.

(vi) weakly Δ-normal (wΔ-normal) [3] if every pair of disjoint δ-closed sets are contained in disjoint open sets.

(vii) weakly functionally Δ-normal (wfΔ-normal) [3] if for every pair of disjoint δ-closed sets A and B there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

(viii) α-normal [1] if for any two disjoint closed subsets A and B of X there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B.

(ix) β-normal [1] if any two disjoint closed sets A and B of X there exist disjoint open subsets U and V of X such that $\overline{(A \cap U)} = A$, $\overline{(B \cap V)} = B$ and $\overline{U} \cap \overline{V} = \phi$.

(x) nearly normal [14] if for any pair of nonempty disjoint sets A and B in X, of which one is δ-closed and the other regularly closed, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

(xi) quasi-normal [22] if for any two disjoint π-closed subsets A and B of X there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

(xii) π-normal [7] if for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

THEOREM 2.2. *Every almost normal space is nearly normal.*

Proof. Since every δ-closed set is closed the proof is obvious. ∎

THEOREM 2.3 *Every nearly normal space is κ-normal.*

Proof. The proof directly follows from the fact that, every regularly closed set is δ-closed and every δ-closed set is closed. ∎

REMARK 2.4. In [16], Mukherjee and Mandal independently introduced the notion of semi nearly normal spaces during the study of countably nearly paracompact space whereas the same notion was earlier introduced and studied by Das in [3] with a different name as "Δ-normality". For the sake of continuity of the work, the notion semi nearly normal or Δ-normal spaces will be referred as Δ-normal spaces throughout the present paper. The following implications are obvious from the definitions and above observations. However, none of these implications is reversible (see [3, 4, 8–10, 17] and Example 2.5 and Example 2.8 below).

$$\begin{tikzcd}
 & \text{normal} \arrow[dl] \arrow[d] & \pi\text{-normal} \arrow[r] & \text{almost normal} \arrow[d] & \\
\beta\text{-normal} \arrow[d] & \Delta\text{-normal} \arrow[ur] \arrow[r] \arrow[d] & \text{wf}\Delta\text{-normal} \arrow[d] \arrow[dr] & \text{nearly normal} \arrow[dr] & \\
\alpha\text{-normal} & \text{f}\theta\text{-normal} \arrow[r] \arrow[d] & \text{wf}\theta\text{-normal} \arrow[d] & \text{w}\Delta\text{-normal} \arrow[dl] \arrow[d] & \kappa\text{-normal} \\
 & \theta\text{-normal} \arrow[r] & \text{w}\theta\text{-normal} & \text{quasi-normal} \arrow[ur] &
\end{tikzcd}$$

EXAMPLE 2.5. Example of a space which is nearly normal but not almost normal. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{b\}, \{c\}, \{a, b, c\}, \{c, d\}, \{b, c\}, \{b, c, d\}\}$. This space is vacuously nearly normal but not almost normal because there does not exist disjoint open sets that separate disjoint regularly closed set $\{a, b\}$ and closed set $\{d\}$ respectively.

Urysohn's type lemma holds for most of the generalized notions of normality. Whereas Urysohn's type lemma does not hold for the class of θ-normal and weak θ-normal spaces defined by Kohli and Das in [8]. Thus the authors in [8] defined two new notions (weakly)functionally θ-normal spaces. Similarly, Δ-normality defined in [3] satisfies Urysohn's type lemma in contrary to weak Δ-normality. Near normality was defined by Mukherjee and Debray in [14] to study nearly paracompact spaces and observed that in a Hausdorff space near paracompactness implies near normality.

Theorem 2.6. *For a space X, the following are equivalent:*

(1) *X is nearly normal.*

(2) *For every δ-closed set A and for every regularly open set B containing A, there is an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq B$.*

(3) *For every regularly open set B containing a δ-closed set A, there exists a regularly open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B$.*

(4) *For every regularly closed set B disjoint from the δ-closed set A, there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $\overline{U} \cap \overline{V} = \phi$.*

Proof. The implications (1) ⇒ (2), (3) ⇒ (4) and (4) ⇒ (1) are obvious.

To prove (2) ⇒ (3), let B be a regularly open set containing a δ-closed set A, then there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq B$. If $\mathrm{int}(\overline{V}) = U$, then $A \subseteq U \subseteq \overline{U} \subseteq B$ where U is regularly open. ■

The following result of [15] shows that Urysohn's type lemma holds for nearly normal spaces which is useful in the sequel to obtain an example of a κ-normal space which is not nearly normal.

Theorem 2.7. [15] *A space X is nearly normal if and only if for every disjoint δ-closed set A and regularly closed set B, there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

Example 2.8. A space which is κ-normal but not nearly normal.

Let X be the simplified Arens square in which S be the unit square and $X = \mathrm{int}\, S \cup \{(0,0),(1,0)\}$. Define the topology on X as defined in [23] by taking usual Euclidean neighbourhood for every point in *intS*. The points $(0,0)$ and $(1,0)$ have neighbourhoods of the form U_n and V_n respectively, where

$$U_n = \{(0,0)\} \cup \big\{(x,y) : 0 < x < \frac{1}{2}, 0 < y < \frac{1}{n}\big\}$$

and

$$V_n = \{(1,0)\} \cup \big\{(x,y) : \frac{1}{2} < x < 1, 0 < y < \frac{1}{n}\big\}.$$

Since every pair of disjoint regularly closed sets in X are separated by disjoint open sets, the space is κ-normal. But the δ-closed set $A = \{(0,0)\}$ disjoint from regularly closed set $B = \{(x,y) : 0 < x \le \frac{1}{2}, 0 < y \le \frac{1}{2}\}$ can't be functionally separated. Thus by Theorem 2.7, the space is not nearly normal.

The following theorem gives characterization of nearly normal spaces in terms of θ-open sets. Since θ-closure of a set defined by Veličko may not be θ-closed [6], θ-closure operator is not Kuratowski closure operator. In [13], M. Mršević and D. Andrijević observed that θ-closure operator is Čech closure operator. Kohli and Das in [10], obtained $u\theta$-closure operator which is a Kuratowski closure operator.

Definition 2.9. [9] Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a $u\theta$-limit point of A if every θ-open set U containing x intersects A. Let $A_{u\theta}$ denote the set of all $u\theta$-limit points of A.

Lemma 2.10. [10] *The correspondence* $A \mapsto A_{u\theta}$ *is a Kuratowski closure operator.*

It turns out that the set $A_{u\theta}$ is the smallest θ-closed set containing A. Above discussed $u\theta$-closure operator is the closure operator in the θ-topology [τ_θ for the space (X, τ)].

Theorem 2.11. *For a topological space* X*, the following statements are equivalent*

(1) X *is nearly normal.*

(2) *For every pair of disjoint closed sets* A *and* B *out of which* A *is* δ*-closed and* B *is regularly closed there exist disjoint* θ*-open sets* U *and* V *such that* $A \subset U$ *and* $B \subset V$*.*

(3) *For every* δ*-closed set* A *and for regularly open set* U *containing* A *there exists a* θ*-open set* V *such that* $A \subset V \subset V_{u\theta} \subset U$*.*

(4) *For every regularly closed set* A *and every* δ*-open set* U *containing* A *there exists a* θ*-open set* V *such that* $A \subset V \subset V_{u\theta} \subset U$*.*

(5) *For every pair of disjoint closed sets* A *and* B*, one of which is* δ*-closed and the other is regularly closed, there exist* θ*-open sets* U *and* V *such that* $A \subset U$*,* $B \subset V$ *and* $U_{u\theta} \cap V_{u\theta} = \phi$*.*

Proof. To prove (1)⇒(2), let X be a nearly normal space and A be a δ-closed set disjoint from the regularly closed set B. Thus by Theorem 2.7, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. Since the inverse image of a θ-open set is θ-open under a continuous mapping, $f^{-1}[0, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, 1]$ are two disjoint θ-open sets containing A and B respectively.

To prove (2)⇒(3), let A be a δ-closed set contained in a regularly open set U. Here A is a δ-closed set disjoint from the regularly closed set $B = X - U$. So by (2), there exist disjoint θ-open sets V and W such that $A \subset V$ and $B \subset W$. Thus $A \subset V \subset X - W \subset U$. As $X - W$ is θ-closed and $V_{u\theta}$ is the smallest θ-closed set containing V, $A \subset V \subset V_{u\theta} \subset U$.

Similarly, the implications (3)⇒ (4) and (4)⇒(5) can be proved and the prove of (5)⇒(1) is obvious as every θ-open set is open. ■

Recall that a covering $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of a topological space X is said to be a shrinking of X if there exists a cover $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of X such that $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in \Lambda$.

Theorem 2.12. [14] *If* X *is a nearly normal space then every point finite regularly open cover of* X *is shrinkable.*

Corollary 2.13. *In an almost normal space every point finite regularly open cover of* X *is shrinkable.*

Theorem 2.14. [3] *A topological space* X *is weakly* Δ*-normal iff for every* δ*-closed set* A *and a* δ*-open set* U *containing* A *there is an open set* V *such that* $A \subset V \subset \overline{V} \subset U$*.*

Theorem 2.15. *A topological space X is weakly Δ-normal if and only if every point finite δ-open cover of X is shrinkable.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a point finite open cover of a weakly Δ-normal space X. Well order the set by assuming $\Lambda = \{1, 2, 3, \ldots, \alpha, \ldots\}$. Since arbitrary union of δ-open sets is δ-open [24], $\bigcup_{\alpha>1} U_\alpha$ is δ-open. Thus $A_1 = X - \bigcup_{\alpha>1} U_\alpha$ is a δ-closed set contained in the δ-open set U_1. So by Theorem 2.14, there exists an open set V_1 such that $A_1 \subset V_1 \subset \overline{V_1} \subset U_1$. As closure of an open set is regularly closed and every regularly closed set is δ-closed, $\overline{V_1}$ is δ-closed. Let $A_2 = X - \left[V_1 \cup \bigcup_{\alpha>2} U_\alpha\right]$. Here A_2 is a δ-closed set contained in the δ-open set U_2. Again by Theorem 2.14, there exists a open set V_2 such that $A_2 \subset V_2 \subset \overline{V_2} \subset U_2$. Continuing in this manner we obtain a collection $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of open sets which is a shrinking of $\mathcal{U}$ provided $\mathcal{V}$ covers X. Let $x \in X$, since $\mathcal{V}$ is a point finite open cover of X, x belongs to finitely many members of $\mathcal{U}$, say $U_{\alpha 1}, U_{\alpha 2}, \ldots, U_{\alpha k}$. Suppose $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Now $x \notin U_\lambda$ for $\lambda > \alpha$ and if $x \notin V_\beta$ for $\beta < \alpha$, then $x \in A_\alpha \in B_\alpha$. Thus in any case $x \in V_\beta$ for $\beta \leq \alpha$. Hence $\mathcal{V}$ is a shrinking of $\mathcal{U}$.

Conversely, let A and B be two disjoint δ-closed sets in X. Let $U_1 = X - A$ and $U_2 = X - B$. Here $\mathcal{U} = \{U_1, U_2\}$ is a point finite δ-open cover of X and thus by the hypothesis, there exists a shrinking $\mathcal{V} = \{V_1, V_2\}$ of $\mathcal{U} = \{U_1, U_2\}$. Since $\overline{V_1} \subset U_1$ and $\overline{V_2} \subset U_2$, $X - \overline{V_1}$ and $X - \overline{V_2}$ are two disjoint open sets containing A and B respectively. Hence X is weakly Δ-normal. ■

Corollary 2.16. *In a weakly functionally Δ-normal space every point finite δ-open cover of X is shrinkable.*

Corollary 2.17. *In a Δ-normal space every point finite δ-open cover of X is shrinkable.*

The following formulation of δ-embedded set is useful to establish that Tietze's type theorem holds for Δ-normal spaces.

Definition 2.18. A subset A of a topological space X is said to be δ-embedded in X if every δ-closed set in the subspace topology of A is the intersection of A with a δ-closed set in X.

Theorem 2.19. *Let X be a Δ-normal space and A be a δ-closed, δ-embedded subset of X then*

(1) *Every continuous function $f : A \to [-1, 1]$ defined on the set A can be extended to a continuous function $g : X \to [-1, 1]$.*
(2) *Every continuous function $f : A \to R$ can be extended to a continuous function $g : X \to R$.*

Proof. To prove (1), let X be a Δ-normal space and A be a δ-closed, δ-embedded subset of X. Suppose $f : A \to [-1, 1]$ is a continuous function. Let $A_1 = \{x \in A : f(x) \geq \frac{1}{3}\}$ and $B_1 = \{x \in A : f(x) \leq -\frac{1}{3}\}$. Since the inverse image of a δ-closed set is δ-closed under a continuous mapping, A_1 and B_1 are two disjoint

δ-closed sets in A. Since A is δ-embedded in X, there exist two δ-closed sets P_1 and P_2 in X such that $A_1 = P_1 \cap A$ and $B_1 = P_2 \cap A$. As intersection of two δ-closed sets is δ-closed [24], A_1 and B_1 are δ-closed in X. Since X is Δ-normal by [3, Theorem 2.2], there exists a continuous function $f_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $f_1(A_1) = \frac{1}{3}$ and $f_1(B_1) = -\frac{1}{3}$. It is obvious that $|f(x) - f_1(x)| \leq \frac{2}{3}$ for every $x \in A$. So $g_1 = f - f_1$ is a mapping of A into $[-\frac{2}{3}, \frac{2}{3}]$. Clearly $A_2 = \{x \in A : g_1(x) \geq \frac{2}{9}\}$ and $B_2 = \{x \in A : g_1(x) \leq -\frac{2}{9}\}$ are two disjoint δ-closed sets in A. As A is δ-embedded and as per the argument mentioned above, A_2 and B_2 are δ-closed in X. Again applying Δ-normality of X, there exists a continuous function $f_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ such that $f_2(A_2) = \frac{2}{9}$ and $f_2(B_2) = -\frac{2}{9}$. Here $|g - f_2| \leq (\frac{2}{3})^2$ on A. Thus $|f - (f_1 + f_2)| \leq (\frac{2}{3})^2$ on A. Continuing this process we obtained a sequence of continuous functions $\{f_n\}$ defined on A such that $|f - \sum_{i=1}^{n} f_i| \leq (\frac{2}{3})^n$ on A. It can be easily verified that the function $g : X \to R$ defined by $g(x) = \sum_{i=1}^{\infty} f_i(x)$ for every $x \in X$ is a continuous extension of the function f.

The proof of (2) is similar to the proof of (1). ∎

Remark 2.20. In a similar fashion the Tietze's type theorem also holds for weakly functionally Δ-normal spaces.

3. Decompositions of normality

Definition 3.1. A topological space X is said to be nearly regular if for every regularly closed set A and each open set U containing A there exists a δ-open set V such that $A \subset V \subset U$.

Definition 3.2. [25] A topological space X is said to be seminormal if for every closed set A and each open set U containing A there exists a regularly open set V such that $A \subset V \subset U$.

Definition 3.3. [3] A topological space X is said to be Δ-regular if for every closed set F and each open set U containing F, there exists a δ-open set V such that $F \subset V \subset U$.

Theorem 3.4. *Every Δ-regular space is nearly regular.*

From the definitions and above result, the following implications hold. However, none of these implications is reversible (see [3] and Example 3.5 below).

$$\begin{tikzcd}
\text{normal} \arrow[r] & \text{seminormal} \arrow[r] & \Delta\text{-regular} \arrow[r] & \text{nearly regular} \\
\text{regular} \arrow[r] & \text{semiregular} \arrow[ur] & &
\end{tikzcd}$$

Example 3.5. Example of a nearly regular space which is not Δ-regular. Let X be an infinite set with co-finite topology in which the only regularly closed set is the whole space X. Thus the space is vacuously nearly regular but not Δ-regular as for a closed set F containing finite points and contained in an open set U there does not exists an δ-open set satisfying $F \subset V \subset U$.

The following decomposition of almost normality directly follows from definitions.

THEOREM 3.6. *A space is almost normal iff it is nearly normal and nearly regular.*

THEOREM 3.7. *In a nearly regular seminormal space the following are equivalent:*

(1) *X is normal.*

(2) *X is Δ-normal.*

(3) *X is π-normal.*

(4) *X is almost normal.*

(5) *X is nearly normal.*

Proof. The proof of (1)$\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) is obvious from definitions.

Now we have to show (5) $\Longrightarrow$ (1). Let X be a nearly normal, nearly regular, seminormal space and let A, B be disjoint closed sets in X. By Theorem 3.6, X is an almost normal, seminormal space. Since $A \subseteq X - B$, by seminormality , there exists a regular open set U such that $A \subset U \subset X - B$. Then $X - U$ is regularly closed set disjoint from closed set A. Thus by almost normality there exist disjoint open sets W and Z such that $X - U \subseteq W$ and $A \subseteq Z$ which implies $A \subseteq Z$ and $B \subseteq W$. Hence X is normal. ∎

THEOREM 3.8. *In a β-normal space the following are equivalent:*

(1) *X is normal.*

(2) *X is Δ-normal.*

(3) *X is π-normal.*

(4) *X is almost normal.*

(5) *X is nearly normal.*

(6) *X is $wf\Delta$-normal.*

(7) *X is $w\Delta$-normal.*

(8) *X is quasi-normal.*

(9) *X is κ-normal.*

Proof. The proof of (1)$\Longrightarrow$ (2)$\Longrightarrow$ (3)$\Longrightarrow$ (4)$\Longrightarrow$ (5)$\Longrightarrow$ (9) and (1)$\Longrightarrow$(2)$\Longrightarrow$ (6)$\Longrightarrow$ (7) $\Longrightarrow$ (8) $\Longrightarrow$ (9) are obvious from definitions. The prove of (9) $\Longrightarrow$ (1) directly follows from the fact that a space is normal iff it is κ-normal and β-normal [1]. ∎

LEMMA 3.9. [10] *A space X is almost regular iff for every open set U in X,* $\mathrm{int}(\overline{U})$ *is θ-open.*

Theorem 3.10. *An almost regular space X is normal iff it is β-normal and weakly θ-normal.*

Proof. Normality implies β-normality and wθ-normality is evident from [1, 8]. Conversely, let X be an almost regular β-normal, weakly θ-normal space. Let A and B be two disjoint closed sets in X by β-normality of X, there exist open sets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $\overline{U} \cap \overline{V} = \phi$. This implies $A \subset \overline{U}$ and $B \subset \overline{V}$. Since U and V are open sets, by almost regularity of X and by Lemma 3.9, $\overline{U}$ and $\overline{V}$ are disjoint θ-closed sets in X. Thus by weak θ-normality of X, there exist open sets O and P such that $\overline{U} \subset O$, $\overline{V} \subset P$. Hence O and P are two disjoint open sets separating two disjoint closed sets A and B. Hence the space is normal. ■

Corollary 3.11. *Every almost regular, almost compact β-normal space is normal.*

Proof. Since every almost compact space is weakly θ-normal [8], the proof is immediate from the above theorem. ■

Corollary 3.12. *Every almost regular, Lindelöf β-normal space is normal.*

Proof. As every Lindelöf space is weakly θ-normal [8], the result is obvious. ■

Corollary 3.13. *An almost regular space X is normal iff it is β-normal and wΔ-normal.*

Proof. The result immediately follows from the implication that every wΔ-normal space is wθ-normal [3]. ■

Theorem 3.14. *A Hausdorff space X is normal iff it is β-normal and wΔ-normal.*

Proof. Every normal space is β-normal as well as wΔ-normal. Conversely, let X be a Hausdorff, β-normal, wΔ-normal space. Since a Hausdorff wΔ-normal space is almost regular [3], by Theorem 3.10, X is normal. ■

Theorem 3.15. *A T_1 space is normal iff it is β-normal and wθ-normal.*

Proof. Let X be a T_1 wθ-normal and β-normal space. Since every T_1 β-normal space is regular [1], X is a regular wθ-normal space. Again by [5, 2.4] a space X is regular if and only if every closed set in X is θ-closed. Thus X is normal. ■

Theorem 3.16. *A T_1 space is normal iff it is α-normal and θ-normal.*

Proof. Let X be a T_1 α-normal, θ-normal space. Since every T_1 α-normal space is Hausdorff, by [8, Theorem 3.5], X is normal. ■

REFERENCES

[**1**] A. V. Arhangel'skii and L. Ludwig, *On α-normal and β-normal spaces*, Comment. Math. Univ. Carolin., **42** (3) (2001), 507–519.

[**2**] A. Császár, *General Topology*, Adam Higler Ltd, Bristol, 1978.

[**3**] A.K. Das, *Δ-normal spaces and decompositions of normality*, Applied Gen. Topol., **10** (2)(2009), 197–206.

[**4**] A.K. Das, *A note on spaces between normal and κ-normal spaces*, Filomat **27**:1 (2013), 85–88.

[**5**] R.F. Dickman, Jr. and J.R. Porter, *θ-perfect and θ-absolutely closed functions*, Illinois J. Math, **21** (1977), 42–60.

[**6**] J.E. Joseph, *θ-closure and θ-subclosed graphs*, Math. Chron., **8** (1979), 99–117.

[**7**] L.N. Kalantan, *π-normal topological spaces*, Filomat **22**:1 (2008), 173–181.

[**8**] J.K. Kohli and A.K. Das, *New normality axioms and decompositions of normality*, Glasnik Mat., **37** (57) (2002), 165–175.

[**9**] J.K. Kohli and A.K. Das, *On functionally θ-normal spaces*, Applied Gen. Topol., **6** (1) (2005), 1–14.

[**10**] J.K. Kohli and A.K Das, *A class of spaces contaning all generalized absolutely closed (almost compact) spaces*, Applied Gen. Topol., **7** (2) (2006), 233–244.

[**11**] J.K. Kohli and D. Singh, *Weak normality properties and factorizations of normality*, Acta. Math. Hungar., **110**, 1-2 (2006), 67–80.

[**12**] C. Kuratowski,*Topologie I*, Hafner, New York, 1958.

[**13**] M. Mršević and D. Andrijević, *On θ-connectedness and θ-closure spaces*, Topology Appl., **123** (2002), 157–166.

[**14**] M.N. Mukherjee and A. Debray, *On nearly paracompact spaces and nearly full normality*, Mat. Vesnik, **50** (1998), 99–104.

[**15**] M.N. Mukherjee and A. Debray, *On nearly normal spaces*, Southeast Asian Bull. Math., **29** (2005), 1087–1093.

[**16**] M.N. Mukherjee and Dh. Mandal, *On countably nearly paracompact spaces*, Mat. Vesnik, **66** (3) (2014), 265–273.

[**17**] E. Murtinova, *A β-normal Tychonoff space which is not normal*, Comment. Math. Univ. Carolinae., **43** (1) (2002), 159–164.

[**18**] M.K. Singal and A.R. Singal, *Almost-continuous mapping*, Yokohama Math. J., **16** (1968), 63–73.

[**19**] M.K Singal and S.P Arya, *On almost regular spaces*, Glasnik Mat., **4** (24) (1969), 88–99.

[**20**] M.K. Singal and S.P. Arya, *On almost normal and almost completely regular spaces*, Glasnik Mat., **5** (25) (1970), 141–152.

[**21**] M.K. Singal and A.R. Singal, *Mildly normal spaces*, Kyungpook Math J., **13** (1973), 27–31.

[**22**] E.V. Stchepin, *Real-valued functions and spaces close to normal*, Sib. J. Math., **13**:5 (1972), 1182–1196.

[**23**] L.A. Steen and J.A.Seebach, Jr., *Counter Examples in Topology*, Springer Verlag, New york, 1978.

[**24**] N.V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., **2** (78) (1968), 103–118.

[**25**] G. Vigilino, *Seminormal and C-compact spaces*, Duke J. Math., **38** (1971), 57–61.

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