МАТЕМАТИЧКИ ВЕСНИК
68, 3 (2016), 175–181
September 2016

оригинални научни рад
research paper

PARITY RESULTS FOR 13-CORE PARTITIONS

Kuwali Das

Abstract. We find some interesting congruences modulo 2 for 13-core partitions.

1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a natural number n is a finite sequence of non-increasing positive integer parts λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. The Ferrers-Young diagram of the partition λ of n is formed by arranging n nodes in k rows so that the i^{th} row has λ_i nodes. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. Let λ'_j denote the number of nodes in column j. The hook number $H(i,j)$ of the (i,j) node is defined as the number of nodes directly below and to the right of the node including the node itself. That is, $H(i,j) = \lambda_i + \lambda'_j - j - i + 1$. A partition λ is said to be a t-core if and only if it has no hook numbers that are multiples of t. If $a_t(n)$ denotes the number of partitions of n that are t-cores, then the generating function for $a_t(n)$ satisfies the identity [9, Equation 2.1]

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}, \tag{1.1}$$

where as customary, for any complex numbers a and q with $|q| < 1$,

$$(a;q)_\infty := \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

A number of results on $a_t(n)$ have been proven by various mathematicians. Garvan, Kim and Stanton [9] gave analytic and bijective proofs of the identity $a_5(5n+4) = 5a_5(n)$. Granville and Ono [10] proved that for $t \geq 4$, every natural number n has a t-core, thereby settling a conjecture of Brauer regarding the existence of defect zero characters for finite simple groups. E. X. W. Xia [15] established some new

2010 Mathematics Subject Classification: 11P83, 05A17
Keywords and phrases: t-core partition; theta function; dissection; congruence.

Ramanujan-type congruences modulo 2 and 4 for t-core partitions, (see [5,6,10,13–15] for further results). In this paper, we prove the following parity results on 13-cores.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} a_{13}\,(104n+6)\,q^n \equiv (q;q)_\infty^3 \pmod 2$$

and

$$\sum_{n=0}^{\infty} a_{13}\,(4(26n+i)+2)\,q^n \equiv 0 \pmod 2,$$

where $i=0$ *or* $2 \le i \le 25$.

Theorem 1.2. *Let* $n \ge 0$. *Then for any positive integer* k *we have*

$$a_{13}\left(104 \cdot 3^{2k} n + 13 \cdot 3^{2k} - 7\right) \equiv a_{13}(104n+6) \pmod 2,$$

$$a_{13}\left(104 \cdot 5^{2k} n + 5 \cdot \frac{13 \cdot 5^{2k-1}+1}{3}\right) \equiv a_{13}(104n+6) \pmod 2,$$

and

$$a_{13}\left(104 \cdot 7^{2k} n + 7 \cdot (13 \cdot 7^{2k-1} - 1)\right) \equiv a_{13}(104n+6) \pmod 2.$$

Theorem 1.3. *If* $p \ge 5$ *is a prime with* $\left(\frac{-13}{p}\right) = -1$, *then for all nonnegative integers* n *and* k *we have*

$$a_{13}\left(4 \cdot p^{2k+1}(pn+j) + 7 \cdot (p^{2k+2}-1)\right) \equiv 0 \pmod 2,$$

where $1 \le j \le p-1$.

Theorem 1.4. *If* $p \ge 5$ *is a prime with* $\left(\frac{-2}{p}\right) = -1$, *then for all nonnegative integers* n *and* k *we have*

$$a_{13}\left(4 \cdot p^{2k+1}(pn+j) + 13 \cdot p^{2k+2} - 7\right) \equiv 0 \pmod 2,$$

where $1 \le j \le p-1$.

2. Background

For $|ab|<1$, Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In this notation, Jacobi's famous triple product identity [4, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Two important special cases of the above are

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2.1}$$

where the last equality in (2.1) is Euler's famous pentagonal number theorem. We will also need the following results.

Lemma 2.1. [8, Theorem 2.2] *For any prime $p \geq 5$,*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}), \tag{2.2}$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6; \\ \dfrac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, if $\dfrac{-(p-1)}{2} \leq k \leq \dfrac{(p-1)}{2}$ and $k \neq \dfrac{(\pm p-1)}{6}$, then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.2. [1] *For any prime $p \geq 5$, we have*

$$f^3(-q) = \sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\frac{pn+2k+1}{2}} + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} f^3(-q^{p^2}). \tag{2.3}$$

Furthermore, if $k \neq \dfrac{p-1}{2}$ and $0 \leq k \leq p-1$, then

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p.$$

3. Congruences modulo 2 for 13-core partitions

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} a_{13}(4n)\, q^n \equiv (q;q)_\infty^3 (q^{13};q^{13})_\infty^3 \pmod 2$$

and

$$\sum_{n=0}^{\infty} a_{13}\,(4n+2)\,q^n \equiv q(q^{26};q^{26})_\infty^3 \pmod 2. \tag{3.1}$$

Proof. For $t > 1$ a partition is called t-regular if none of its parts is divisible by t, and we denote by $b_t(n)$ the number of t-regular partitions of n. Then the generating function for $b_t(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty}.$$

Putting $t = 13$ in (1.1), we have

$$\sum_{n=0}^{\infty} a_{13}(n)q^n = \frac{(q^{13};q^{13})_\infty^{13}}{(q;q)_\infty}. \tag{3.2}$$

Using binomial expansion and then taking congruence modulo 2, we have

$$(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2. \tag{3.3}$$

Employing (3.3) in (3.2), we find that

$$\sum_{n=0}^{\infty} a_{13}(n)q^n \equiv \frac{(q^{26};q^{26})_\infty^6(q^{13};q^{13})_\infty}{(q;q)_\infty} \equiv (q^{26};q^{26})_\infty^6 \sum_{n=0}^{\infty} b_{13}(n)q^n \pmod 2. \tag{3.4}$$

Extracting the terms involving even powers of q from both sides of (3.4) yields

$$\sum_{n=0}^{\infty} a_{13}(2n)q^n \equiv (q^{13};q^{13})_\infty^6 \sum_{n=0}^{\infty} b_{13}(2n)q^n \pmod 2. \tag{3.5}$$

From [7, Theorem 2] we recall that

$$\sum_{n=0}^{\infty} b_{13}(2n)q^n \equiv (q^2;q^2)_\infty^3 + q^3(q^{26};q^{26})_\infty^3 \pmod 2. \tag{3.6}$$

Applying (3.6) in (3.5), we obtain

$$\sum_{n=0}^{\infty} a_{13}(2n)q^n \equiv (q^{26};q^{26})_\infty^3(q^2;q^2)_\infty^3 + q^3(q^{26};q^{26})_\infty^6 \pmod 2.$$

Extracting the even and odd parts respectively, we obtain

$$\sum_{n=0}^{\infty} a_{13}(4n)q^{2n} \equiv (q^{26};q^{26})_\infty^3(q^2;q^2)_\infty^3 \pmod 2$$

and

$$\sum_{n=0}^{\infty} a_{13}(4n+2)q^{2n+1} \equiv q^3(q^{26};q^{26})_\infty^6 \equiv q^3(q^{52};q^{52})_\infty^3 \pmod 2.$$

Replacing q^2 by q in the above two congruences, we can easily obtain the required result. ■

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} a_{13}\,(104n+6)\,q^n \equiv (q;q)_\infty^3 \pmod 2 \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} a_{13}\left(4(26n+i)+2\right)q^n \equiv 0 \pmod 2,$$

where $i=0$ *or* $2 \le i \le 25$.

Proof. This follows directly from the fact that the series on the right hand side of (3.1) only involves powers of q that are congruent to 1 modulo 26. ∎

Theorem 3.3. *Let* $n \ge 0$. *Then for any positive integer* k *we have*

$$a_{13}\left(104\cdot 3^{2k}n + 13\cdot 3^{2k} - 7\right) \equiv a_{13}(104n+6) \pmod 2, \tag{3.8}$$

$$a_{13}\left(104\cdot 5^{2k}n + 5\cdot\frac{13\cdot 5^{2k-1}+1}{3}\right) \equiv a_{13}(104n+6) \pmod 2 \tag{3.9}$$

and

$$a_{13}\left(104\cdot 7^{2k}n + 7\cdot(13\cdot 7^{2k-1} - 1)\right) \equiv a_{13}(104n+6) \pmod 2. \tag{3.10}$$

Proof. Note that for a non-zero integer r and a nonnegative integer n, the general partition function $p_r(n)$ is defined as the coefficient of q^n in the expansion of $(q;q)_\infty^r$. From (3.7), we have

$$\sum_{n=0}^{\infty} a_{13}\left(104n+6\right)q^n \equiv \sum_{n=0}^{\infty} p_3(n)q^n \pmod 2. \tag{3.11}$$

From [3], we have

$$p_3\left(3^{2k}n + \frac{3^{2k}-1}{8}\right) = (-3)^k p_3(n),$$

$$p_3\left(5^{2k}n + \frac{5^{2k}-1}{24}\right) = 5^k p_3(n)$$

and

$$p_3\left(7^{2k}n + \frac{7^{2k}-1}{8}\right) = (-7)^k p_3(n).$$

Employing the above three identities in (3.11), we can easily obtain (3.8), (3.9) and (3.10). ∎

Theorem 3.4. *If* $p \ge 5$ *is a prime with* $\left(\frac{-13}{p}\right) = -1$, *then for all nonnegative integers* k *we have*

$$\sum_{n=0}^{\infty} a_{13}\left(4\cdot p^{2k}n + 7\cdot(p^{2k}-1)\right)q^n \equiv (q;q)_\infty^3 (q^{13};q^{13})_\infty^3 \pmod 2. \tag{3.12}$$

Proof. Note first that (3.1) is the $k=0$ case of (3.12). Now suppose (3.12) holds for some $k \ge 0$, and consider the congruence

$$\frac{(\ell^2+\ell)}{2} + 13\cdot\frac{(m^2+m)}{2} \equiv 14\cdot\frac{(p^2-1)}{8} \pmod p, \tag{3.13}$$

for $0 \leq \ell, m \leq p-1$. Since the above congruence is equivalent to

$$(2\ell+1)^2 + 13\cdot(2m+1)^2 \equiv 0 \pmod{p},$$

and $\left(\frac{-13}{p}\right) = -1$, it follows that (3.13) has only one solution, namely $k = m = (p-1)/2$. Therefore, extracting the terms involving $q^{pn+7(\frac{p^2-1}{4})}$ from both sides of (3.12), by (2.3) we deduce that

$$\sum_{n=0}^{\infty} a_{13}\Big(4\cdot p^{2k}(pn+7(\frac{p^2-1}{4}))+7\cdot(p^{2k}-1)\Big)q^n \equiv (q^p;q^p)_\infty^3(q^{13p};q^{13p})_\infty^3 \pmod{2}. \tag{3.14}$$

Again, extracting terms involving q^{pn} from both sides of the above congruence and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_{13}\left(4\cdot p^{2k+2}n + 7\cdot(p^{2k+2}-1)\right)q^n \equiv (q;q)_\infty^3(q^{13};q^{13})_\infty^3 \pmod{2},$$

which is the $k+1$ case of (3.12). ∎

We observe that in (3.14), there are no terms involving q^{pn+j} with $1 \leq j \leq p-1$. This implies the following result.

Theorem 3.5. *If $p \geq 5$ is a prime with $\left(\frac{-13}{p}\right) = -1$, then for all nonnegative integers k we have*

$$a_{13}\left(4\cdot p^{2k+1}(pn+j) + 7\cdot(p^{2k+2}-1)\right) \equiv 0 \pmod{2},$$

where $1 \leq j \leq p-1$.

Theorem 3.6. *If $p \geq 5$ is a prime with $\left(\frac{-2}{p}\right) = -1$, then for all nonnegative integers k we have*

$$\sum_{n=0}^{\infty} a_{13}\left(104\cdot p^{2k}n + 13\cdot p^{2k} - 7\right)q^n \equiv (q;q)_\infty(q^2;q^2)_\infty \pmod{2}. \tag{3.15}$$

Proof. Note that (3.7) is the $k=0$ case of (3.15). Now suppose (3.15) holds for some $k \geq 0$, and consider the congruence

$$\frac{(3\ell^2+\ell)}{2} + 2\cdot\frac{(3m^2+m)}{2} \equiv 3\cdot\frac{(p^2-1)}{24} \pmod{p}, \tag{3.16}$$

for $0 \leq \ell,\ m \leq p-1$. The above congruence is equivalent to

$$(6\ell+1)^2 + 2\cdot(6m+1)^2 \equiv 0 \pmod{p},$$

and $\left(\frac{-2}{p}\right) = -1$, it follows that (3.16) has only one solution, namely $\ell = m = (\pm p-1)/6$. Therefore, extracting the terms involving $q^{pn+(\frac{p^2-1}{8})}$ from both sides of (3.15), by (2.2) we deduce that

$$\sum_{n=0}^{\infty} a_{13}\Big(104\cdot p^{2k}(pn+\frac{p^2-1}{8})+13\cdot p^{2k}-7\Big)q^n \equiv (q^p;q^p)_\infty(q^{13p};q^{13p})_\infty \pmod{2}. \tag{3.17}$$

Extracting the terms involving q^{pn} from both sides of (3.17) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} a_{13}\left(104 \cdot p^{2k+2}n + 13 \cdot p^{2k+2} - 7\right) q^n \equiv (q;q)_\infty (q^2;q^2)_\infty \pmod 2,$$

which is the $k+1$ case of (3.15). ■

From (3.17), we can easily obtain the following result.

THEOREM 3.7. *If $p \geq 5$ is a prime with $\left(\frac{-2}{p}\right) = -1$, then for all nonnegative integers k we have*

$$a_{13}\left(4 \cdot p^{2k+1}(pn+j) + 13 \cdot p^{2k+2} - 7\right) \equiv 0 \pmod 2,$$

where $1 \leq j \leq p-1$.

REFERENCES

[**1**] Z. Ahmed and N. D Baruah, *New congruences for ℓ-regular partitions for $\ell \in \{5,6,7,49\}$*, Ramanujan J., 20pp., DOI 10.1007/s11139-015-9752-2, 2016.

[**2**] N.D. Baruah and K. K. Ojah, *Analogues of Ramanujan's partition identities and congruences arising from his theta functions and modular equations*, Ramanujan J. **28** (2012), 385–407.

[**3**] N.D. Baruah and B.K. Sarmah, *Identities and congruences for the general partition and Ramanujans tau function*, Indian J. Pure Appl. Math. **44** (2013), 643-671.

[**4**] B.C. Berndt, *Ramanujan's Notebooks, Part III* (Springer, New York, 1991).

[**5**] A. Berkovich and H. Yesilyurt, *New identities for 7-cores with prescribed BG-rank*, Discrete Math. **308** (2008), 5246–5259.

[**6**] A. Berkovich and H. Yesilyurt, *On the representations of integers by the sextenary quadratic forms $x^2+y^2+z^2+7s^2+7t^2+7u^2$*, J. Number Theory. **79** (2009), 507–512.

[**7**] N. Calkin, N. Drake, K. James, S. Law, P. Lee, D. Penniston and J. Radder, *Divisibility properties of the* 5*-regular and* 13*-regular partition functions*, Integers **8** (2008), A60.

[**8**] S.P. Cui and N.S.S. Gu, *Arithmetic properties of ℓ-regular partitions*, Adv. Appl. Math. **51** (2013), 507-523.

[**9**] F.G. Garvan, D. Kim, D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), 1-17.

[**10**] A. Granville and K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), 331–347.

[**11**] M.D. Hirschhorn and J. A. Sellers, *Two congruences involving 4-cores*, Electron. J. Combin. **3** (2) (1996), R10.

[**12**] M.D. Hirschhorn and J.A. Sellers, *Elementary proofs of various facts about 3-cores*, Bull. Aust. Math. Soc. **129** (2009), 1366–1378.

[**13**] B. Kim, *On inequalities and linear relations for 7-core partitions*, Discrete Math. **310** (2010), 861–868.

[**14**] K. Ono and L. Sze, *4-core partitions and class numbers*, Acta Arith. **80** (1997), 249–272.

[**15**] E.X.W. Xia, *New congruences modulo powers of* 2 *for broken* 3*-diamond partitions and* 7*-core partitions*, J. Number Theory, **141** (2014), 119–135.

(received 09.07.2015; in revised form 11.05.2016; available online 20.05.2016)

Department of Mathematical Sciences, Bodoland University, Kokrajhar-783370, Assam, India

E-mail: kwldas90@gmail.com