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SOME CALCULUS OF THE COMPOSITION OF FUNCTIONS IN BESOV-TYPE SPACES

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Abstract. In the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, we will prove that the composition operator $T_f : g \to f \circ g$ takes both $B^s_{\infty,q}(\mathbb{R}^n) \cap B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $W^1_\infty(\mathbb{R}^n) \cap B^{s,\tau}_{p,q}(\mathbb{R}^n)$ to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, under some restrictions on s, τ, p, q, and if the real function f vanishes at the origin and belongs locally to $B^{s+1}_{\infty,q}(\mathbb{R})$.

1. Introduction and the main result

To a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we will associate the composition operator $T_f : g \to f \circ g$ and we will study its boundedness on Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ under some restrictions on the parameters s, τ, p and q.

The problem of composition in a real-valued function space E consists of the conditions satisfied by f such that $T_f(E) \subseteq E$ holds. The properties of the operator T_f strongly depend on the space E, see, e.g. [1, Section 4] and [3, Section 4]. The operator T_f is *nonlinear* unless f is a linear function. For instance, it has been proved that the inclusion $T_f(E) \subseteq E$ implies that $f(t) = ct$ for some constant c, in the following cases:

– $E = W^m_p(\mathbb{R}^n)$ the Sobolev space, for $1 \le p < \infty$ and $1 + 1/p < m < n/p$, see [5],

– $E = B^s_{p,q}(\mathbb{R}^n)$ the Besov space, for $1 \le p < \infty$ and $1 + 1/p < s < n/p$, see e.g. [1, Theorem 3.3],

– $E = F^s_{p,q}(\mathbb{R}^n)$ the Triebel-Lizorkin space, for $1 \le p < \infty$ and $1 + 1/p < s < n/p$, see e.g. [1, Theorem 3.3],

– $E = B^s_{p,q}(\mathbb{R}^n)$, for $1 \le p < \infty$, $q > 1$ (or $E = F^s_{p,q}(\mathbb{R}^n)$, for $1 < p < \infty$, $q \ge 1$) and $1 + 1/p = s < n/p$, see e.g. [1, Theorem 3.3] or [13, Lemma 5.3.1/2, p. 308].

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The acting of T_f on Besov spaces $B^s_{p,q}(\mathbb{R})$ in the *one*-dimensional case has been studied in several works, e.g. [4, 11]. However in the n-case (i.e. $B^s_{p,q}(\mathbb{R}^n)$) the composition problem is not trivial and we have some results which can be found in [9,10,13], where some of them are on the intersection spaces.

In the context of intersections, we want to extend the result given in [9] for $B^s_{p,q}(\mathbb{R}^n)$, to the case of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. Then we will prove the following result.

THEOREM 1.1. *Let $0 < p, q \leq \infty$, $(n/p - n)_+ < s \neq 1$ and $0 \leq \tau \leq 1/p$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function such that $f(0) = 0$ and $f \in B^{s+1}_{\infty,q}(\mathbb{R})_{loc}$.*

(i) If $s < 1$, then T_f takes $W^1_\infty(\mathbb{R}^n) \cap B^{s,\tau}_{p,q}(\mathbb{R}^n)$ to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$.

(ii) If $s > 1$, then T_f takes $B^s_{\infty,q}(\mathbb{R}^n) \cap B^{s,\tau}_{p,q}(\mathbb{R}^n)$ to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$.

REMARK 1.2. From the embedding $B^{s,\beta}_{\infty,q}(\mathbb{R}) \hookrightarrow B^s_{\infty,q}(\mathbb{R})$ if $\beta \geq 0$ (see [16, p. 40]), Theorem 1.1 also holds if one replaces $B^{s+1}_{\infty,q}(\mathbb{R})_{loc}$ by $B^{s+1,\beta}_{\infty,q}(\mathbb{R})_{loc}$.

Besov-type spaces coincide with Besov spaces for some values of τ, s, p and q, e.g., we have $B^{s,0}_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$ (see [16, Lemma 2.1, p. 22]), then Theorem 1.1 covers the case of $B^s_{p,q}(\mathbb{R}^n)$, in particular the Hölder space $B^s_{\infty,\infty}(\mathbb{R}^n)$, and yields the result in [9] which was given only in the case $p, q \geq 1$ and $0 < s \neq 1$. This presents our principal contribution, and we will also extend it to the case $s = 1$ (see Section 4 below).

The proof of Theorem 1.1 is based essentially on three aspects:

– the "paralinearization" method (see e.g. [2, p. 95] or [8]) which concerns the possibility to linearize T_f,

– an almost orthogonality estimate (see Proposition 3.3 below),

– the boundedness of T_f on $B^s_{\infty,q}(\mathbb{R}^n)$, see [3, Theorem 4] and [9, Proposition 3.1], also, Proposition 3.1 below.

However in the case $0 < q < 1$, the Fatou lemma and the precise estimate resulting from the acting of T_f on $B^s_{\infty,q}(\mathbb{R}^n)$ (cf. (17)) are also main tools for the proof.

Notation

As usual, $\mathbb{N}$ denotes the set of natural numbers including 0, $\mathbb{Z}$ the integers, and $\mathbb{R}$ the real numbers. All functions are assumed to be real valued, except in Subsections 2.1–2.2. For $a \in \mathbb{R}$ we put $a_+ := \max(0, a)$. The symbol $\hookrightarrow$ indicates a continuous embedding. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual. For $0 < p \leq \infty$ we denote by $\|\cdot\|_p$ the quasi-norm (norm if $1 \leq p \leq \infty$) of $L_p(\mathbb{R}^n)$. For $f \in L_1(\mathbb{R}^n)$, we denoted by $\mathcal{F}f$ (or $\widehat{f}$) the Fourier transform and by $\mathcal{F}^{-1}f$ the inverse Fourier transform. They are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. $W^1_\infty(\mathbb{R}^n)$ is the usual Sobolev space of bounded and Lipschitz functions on $\mathbb{R}^n$. For a tempered function space E, the local associated space is denoted by E_{loc} and is the set of

$f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\varphi f \in E$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. For $\nu := (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$ we denote by

$$P_{k,\nu} := \{x \in \mathbb{R}^n : \nu_j \leq 2^k x_j < \nu_j + 1, j = 1, 2, \ldots, n\} \tag{1}$$

the dyadic cube. Finally, the constants $c, c_1, \ldots$ are positive and depend only on the fixed parameters $s, p, q, \ldots$, and their values may change from line to line.

2. Preliminaries

We start with the Littlewood-Paley decomposition. Let ρ be a C^∞ positive and radial function, such that $\rho(\xi) = 0$ if $|\xi| \geq 3/2$ and $\rho(\xi) = 1$ if $|\xi| \leq 1$, which is the so-called *cut-off* function. We put $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$; then γ is supported by the compact annulus $1/2 \leq |\xi| \leq 3/2$. We assume that ρ and γ are fixed thorughout the paper. We obtain $\sum_{k\in\mathbb{Z}} \gamma(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$ and $\rho(\xi) + \sum_{k\geq 1} \gamma(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^n$. We define pseudodifferential operators $S_j := \rho(2^{-j}D)$ $(j = 0, 1, \ldots)$ and $Q_k := \gamma(2^{-k}D)$ $(k = 1, 2, \ldots)$. We put $Q_0 := S_0$. Using the Young inequality in $L_p(\mathbb{R}^n)$, the families of operators $(S_j)_{j\in\mathbb{N}}$ and $(Q_j)_{j\in\mathbb{N}}$ constitute bounded subsets of the normed space $\mathcal{L}(L_p(\mathbb{R}^n))$ for any $p \in [1, \infty]$. Also, it is not difficult to prove that for every $N \in \mathbb{N}$, there exist $c > 0$ and $M \in \mathbb{N}$, such that

$$\|Q_j f\|_p \leq c 2^{-jN} \sup_{|\alpha|\leq M} \sup_{x\in\mathbb{R}^n} (1 + |x|)^M |f^{(\alpha)}(x)| \tag{2}$$

holds, for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $j \in \mathbb{N}$. These estimates easily yield that the series $f = S_j f + \sum_{k>j} Q_k f$ for all $j \in \mathbb{N}$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

2.1 The Besov spaces

We first define the "ordinary" Besov spaces.

Definition 2.1. Let $s \in \mathbb{R}$ and $p, q \in]0, \infty]$. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|S_0 f\|_p + \Big(\sum_{j\geq 1} (2^{sj} \|Q_j f\|_p)^q \Big)^{1/q} < \infty$.

The spaces $B^s_{p,q}(\mathbb{R}^n)$ are quasi-Banach in this quasi-norm. For their properties we recall that, e.g.,

- $B^{s_0}_{p,q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p,q_1}(\mathbb{R}^n)$ if $s_0 > s_1$, and $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$ if $s > 0$,
- if $f \in B^s_{p,q}(\mathbb{R}^n)$ then $\partial_j f \in B^{s-1}_{p,q}(\mathbb{R}^n)$ $(j = 1, \ldots, n)$.

We also recall that $B^s_{p,q}(\mathbb{R}^n)$ have the Fatou property, see [6]. We do not go into details about Besov spaces but refer instead to e.g. [13, 14].

2.2 The Besov-type spaces

Here we also begin by the definition of the Besov-type spaces.

Definition 2.2. Let $s, \tau \in \mathbb{R}$ and $p, q \in]0, \infty]$. The Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k} \Big(\sum_{j\geq k_+} (2^{sj}\|Q_j f\|_{L_p(P_{k,\nu})})^q \Big)^{1/q} < \infty,$$

where the dyadic cube $P_{k,\nu}$ is defined in (1).

$B^{s,\tau}_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces in the above quasi-norm, where $B^{s,\tau}_{p,q}(\mathbb{R}^n) = \{0\}$ if $\tau < 0$. We refer to [16] for some properties of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and recall the following remark.

Remark 2.3. The space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ is independent of the choices of ρ, i.e. if we choose another cut-off function ρ_1 with the same properties as ρ, the space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ remains unchanged and the resulting quasi-norm is equivalent to the one defined by ρ.

The following assertion is useful, which is an estimate of Nikol'skij-type and will play a major role in this paper.

Proposition 2.4. *Let $p, q \in]0, \infty]$, $s > (n/p - n)_+$ and $\tau \geq 0$. Let $b > 0$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{u_j}$ is supported by the ball $|\xi| \leq b2^j$ and $A := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k} \Big(\sum_{j\geq k_+} (2^{sj}\|u_j\|_{L_p(P_{k,\nu})})^q \Big)^{1/q} < \infty$. Then the series $\sum_{j\geq 0} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a limit u satisfying $\|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq cA$, where the constant c depends only on n, s, τ, p, q and b.*

For the proof, we need to use the following three lemmas, where the proof of the first one is completely similar to [15, Lemma 3.8, p. 155], and the second one is a Marschall pointwise estimate proved in, e.g. [16, Lemma 6.1, p. 150]; however the third lemma is essentially given in [7, p. 782, (2.11)].

Lemma 2.5. *Let $a > 1$ and $0 < q \leq \infty$. Then, there exists a constant $c > 0$, such that for all $l \in \mathbb{Z}$ and all sequences $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive real numbers satisfying $A := \big(\sum_{k\geq l_+} \varepsilon_k^q \big)^{1/q} < \infty$, it holds $\big(\sum_{j\geq l_+} \big(\sum_{k\geq j} a^{j-k}\varepsilon_k \big)^q \big)^{1/q} \leq cA$.*

Lemma 2.6. *Let $C > 0$, $R \geq 1$ and $t \in]0, 1]$. Let $h \in \mathcal{D}(\mathbb{R}^n)$ and $\theta \in C^\infty(\mathbb{R}^n)$ be such that h and $\widehat{\theta}$ are supported by the balls $|\xi| \leq C$ and $|\xi| \leq CR$, respectively. Then the inequality $|(\theta * \mathcal{F}^{-1}h)(x)| \leq c(CR)^{n/t-n}\|h\|_{\dot{B}^{n/t}_{1,t}(\mathbb{R}^n)} (M|\theta|^t(x))^{1/t}$ holds, where M and $\dot{B}^{n/t}_{1,t}(\mathbb{R}^n)$ denote the Hardy-Littlewood maximal function on $\mathbb{R}^n$ and the homogeneous Besov space, respectively. The constant c is independent of θ, h, C, R and x.*

Lemma 2.7. *Let $0 < p \leq \infty$. Then there exists a constant $c > 0$ such that the inequality $\sup_{x\in P_{j,\nu}} |\psi(x)| \leq c2^{jn/p} \sup_{\mu\in\mathbb{Z}^n} \|\psi\|_{L_p(P_{j,\mu})}$ holds, for all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{\psi}$ is supported by the ball $|\xi| \leq 2^{j+1}$ ($j \in \mathbb{Z}$), all $\nu \in \mathbb{Z}^n$ and all $x \in \mathbb{R}^n$.*

Proof (Proof of Proposition 2.4). Let $\widetilde{\gamma}$ be a radial function in $\mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $\gamma\widetilde{\gamma} = \gamma$. We put $\widetilde{Q}_j := \widetilde{\gamma}(2^{-j}D)$. Also, for the time being and for brevity, we denote by "g" the series $\sum_{k\geq 0} u_k$. Since $\widehat{u_k}$ is supported by the ball $|\xi| < b2^k$, there exists an integer m_0 (which will be used along this proof), which depends only on b, such that $Q_j u_k = 0$ if $k \leq j + m_0$ (m_0 is the nearest integer to the real number $-\log_2(2b)$), but if $k \geq 0$, then $S_0 g = \sum_{k\geq 0} S_0 u_k$ and $Q_j g = \sum_{k\geq (j+m_0)_+} Q_j u_k$ $(j = 1, 2, \ldots)$.

Step 1: convergence in $\mathcal{S}'(\mathbb{R}^n)$. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We put $g_1 := \sum_{j\geq 1} Q_j g$ and $g_2 := S_0 g$. We will estimate $|\langle g_1, f\rangle|$ and $|\langle g_2, f\rangle|$ separately.

Substep 1.1: estimate of $|\langle g_1, f\rangle|$. Let $0 < d < 1$. By the assumption on $\widetilde{\gamma}$, we have $\langle Q_j g, f\rangle = \langle Q_j g, \widetilde{Q}_j f\rangle$, and then by Bernstein inequality we get

$$|\langle g_1, f\rangle| \leq c \sum_{j\geq 1} 2^{-jn(1-1/d)} \Big(\int_{\mathbb{R}^n} |Q_j g(x) \widetilde{Q}_j f(x)|^d \mathrm{d}x \Big)^{1/d}.$$

Now, we decompose "$\int_{\mathbb{R}^n} \ldots$" with respect to $\bigcup_{\nu\in\mathbb{Z}^n} P_{j,\nu}$ for $j \in \mathbb{N}$, and thus we find

$$\begin{aligned}|\langle g_1, f\rangle| &\leq c \sum_{j\geq 1} 2^{-jn(1-1/d)} \Big(\sum_{\nu\in\mathbb{Z}^n} \int_{P_{j,\nu}} |Q_j g(x) \widetilde{Q}_j f(x)|^d \mathrm{d}x \Big)^{1/d} \\ &\leq c \sum_{j\geq 1} 2^{-jn(1-1/d)} \|\widetilde{Q}_j f\|_d \sup_{\nu\in\mathbb{Z}^n} \sup_{x\in P_{j,\nu}} |Q_j g(x)|.\end{aligned}$$

By using (2), let $N \in \mathbb{N}$ (which will be chosen later on) be such that

$$|\langle g_1, f\rangle| \leq c \sum_{j\geq 1} 2^{-j(N+n-n/d)} \sup_{\nu\in\mathbb{Z}^n} \sup_{x\in P_{j,\nu}} |Q_j g(x)|. \tag{3}$$

So, the problem remains to estimate $\sup_{\nu\in\mathbb{Z}^n} \sup_{x\in P_{j,\nu}} |Q_j g(x)|$. We apply Lemma 2.7 with $\psi := Q_j g$. It holds

$$\sup_{x\in P_{j,\nu}} |Q_j g(x)| \leq c 2^{jn/p} \sup_{\mu\in\mathbb{Z}^n} \|Q_j g\|_{L_p(P_{j,\mu})}. \tag{4}$$

Applying now Lemma 2.6 with

$$\theta := u_k, \quad h := \gamma(2^{-j}(\cdot)), \quad C := 3 \cdot 2^{j-1} \quad \text{and} \quad R := b2^{k-j+1}, \tag{5}$$

we have $b2^k \leq CR$ ($\operatorname{supp} \widehat{\theta} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq CR\}$), also the condition $R \geq 1$ is guaranteed by the fact that $k \geq (j + m_0)_+$. Then we obtain, for some $t \in]0, 1]$,

$$|Q_j u_k(x)| \leq c\, 2^{k(n/t-n)} \|\gamma(2^{-j}(\cdot))\|_{\dot{B}^{n/t}_{1,t}(\mathbb{R}^n)} (M|u_k|^t(x))^{1/t}. \tag{6}$$

Using the $\dot{B}^{n/t}_{1,t}(\mathbb{R}^n)$'s property, i.e. $\|\gamma(2^{-j}(\cdot))\|_{\dot{B}^{n/t}_{1,t}(\mathbb{R}^n)} \leq c\, 2^{j(n-n/t)} \|\gamma\|_{\dot{B}^{n/t}_{1,t}(\mathbb{R}^n)}$ for all $j \in \mathbb{N}$, we get

$$|Q_j g(x)| \leq c \sum_{k\geq (j+m_0)_+} 2^{(k-j)(n/t-n)} \big(M|u_k|^t(x)\big)^{1/t}, \quad \forall\, x \in \mathbb{R}^n.$$

For any $l \in \mathbb{Z}$ we take the $L_p(P_{l,\mu})$ of the last inequality and use the following elementary inequality

$$\Big(\sum_{j\geq 0} \varepsilon_j\Big)^\alpha \leq \sum_{j\geq 0} \varepsilon_j^\alpha \qquad (0 < \alpha \leq 1,\, \varepsilon_j \geq 0,\, j = 0, 1, \ldots), \tag{7}$$

with $\alpha := t$, to obtain $\|Q_j g\|_{L_p(P_{l,\mu})} \leq c \Big\| \sum_{k\geq (j+m_0)_+} 2^{(k-j)(n-nt)} M|u_k|^t(\cdot) \Big\|^{1/t}_{L_{p/t}(P_{l,\mu})}$.

We choose first $t < \min(1,p)$ (i.e. $p/t > 1$). Then the maximal function satisfies $\|Mf\|_{L_{p/t}(P_{l,\mu})} \le c\|f\|_{L_{p/t}(P_{l,\mu})}$ for all j and all μ; indeed, let $1_{P_{l,\mu}}$ be the indicatrix function of $P_{l,\mu}$; then for any cube Q satisfying $Q \subset P_{l,\mu}$ it holds

$$\Big(\int_Q 1_{P_{l,\mu}}(x)\mathrm{d}x\Big)\Big(\int_Q 1_{P_{l,\mu}}(x)^{1/(1-p)}\mathrm{d}x\Big)^{p-1} \le c|Q|^p,$$

and we have a weighted norm inequalities for M in $L_{p/t}(1_{P_{l,\mu}};\mathrm{d}x)$, [12, Theorem 9], but $L_{p/t}(1_{P_{l,\mu}};\mathrm{d}x) = L_{p/t}(P_{l,\mu})$; see also [2, Theorem 1.14, p. 13]. We apply the Minkowski inequality (i.e. $\ell_1(\mathbb{N};L_{p/t}(P_{l,\mu})) \hookrightarrow L_{p/t}(P_{l,\mu};\ell_1(\mathbb{N}))$), and we obtain

$$\begin{aligned}\|Q_j g\|_{L_p(P_{l,\mu})} &\le c\Big(\sum_{k\ge(j+m_0)_+} 2^{(k-j)(n-nt)}\|M|u_k|^t\|_{L_{p/t}(P_{l,\mu})}\Big)^{1/t} \\ &\le c\,2^{j(n-n/t)}\Big(\sum_{k\ge(j+m_0)_+} 2^{k(n-st-nt)}(2^{ks}\|u_k\|_{L_p(P_{l,\mu})})^t\Big)^{1/t} \qquad (\forall l\in\mathbb{Z}). \qquad (8)\end{aligned}$$

Secondly, we choose t such that $n - st - nt < 0$, which implies that

$$\sum_{k\ge(j+m_0)_+} 2^{k(n-st-nt)} \le \sum_{k\ge 0} 2^{k(n-st-nt)} \le c. \qquad (9)$$

Then t will be chosen such that

$$\frac{n}{s+n} < t \le \min(1,p), \qquad (10)$$

which is possible since $s > (n/p - n)_+$. On the other hand, we have

$$\sup_{k\ge(j+m_0)_+}\ \sup_{\mu\in\mathbb{Z}^n} 2^{ks}\|u_k\|_{L_p(P_{j,\mu})} \le c2^{-n\tau j}A. \qquad (11)$$

Indeed, if $m_0 \ge 0$, which implies $(j+m_0)_+ = j+m_0 \ge j$, then we use the fact that $\sup_{k\ge(j+m_0)_+}\cdots \le \sup_{k\ge j}\ldots$; if $m_0 < 0$, we have $P_{j,\mu} \subset P_{j+m_0,2^{m_0}\mu}$ with $2^{m_0}\mu \in \mathbb{Z}^n$ and use the inequality $\|u_k\|_{L_p(P_{j,\mu})} \le \|u_k\|_{L_p(P_{j+m_0,2^{m_0}\mu})} \le \sup_{\nu\in\mathbb{Z}^n}\|u_k\|_{L_p(P_{j+m_0,\nu})}$. Then choosing $l = j$ in (8), and inserting, both (9) and (11) in (8), we get

$$\|Q_j g\|_{L_p(P_{j,\mu})} \le c2^{j(n-n\tau-n/t)}A \qquad (\forall j\in\mathbb{N},\ \forall\mu\in\mathbb{Z}^n). \qquad (12)$$

Now we turn to (3). By inserting, both (4) and (12) in (3), and by choosing the natural number N such that $N + n\tau - n/p - n/d + n/t > 0$, we derive that $|\langle g_1, f\rangle|$ is bounded by $c_1 A\sum_{j\ge1} 2^{-j(N+n\tau-n/p-n/d+n/t)}$ which gives the bound c_2A.

Substep 1.2: estimate of $|\langle g_2, f\rangle|$. This estimate is similar to that of the above substep, but only a few changes are needed. Indeed, we begin with

$$|\langle g_2, f\rangle| \le \sum_{\nu\in\mathbb{Z}^n}\int_{P_{0,\nu}} |S_0 g(x)||f(x)|\,\mathrm{d}x \le \|f\|_1 \sup_{\nu\in\mathbb{Z}^n}\ \sup_{x\in P_{0,\nu}} |S_0 g(x)|. \qquad (13)$$

To estimate the last term of (13) we consider the following two cases:

- *The case 1:* $b \ge 3/2$. We will apply Lemma 2.6 as in (6), and we find, for some $t \in]0,1]$, $|S_0 u_k(x)| \le c\,2^{k(n/t-n)}(M|u_k|^t(x))^{1/t}$, where we have used

  $$\theta := u_k, \quad h := \rho, \quad C := 3/2 \quad \text{and} \quad R := b2^{k+1}/3, \qquad (14)$$

  with $R \ge 1$ for all $k \ge 0$ by the assumption on b (recall that $\widehat{\theta}$ is supported by the ball $|\xi| \le CR = b2^k$). Then we continue by choosing t such that $t < \min(1,p)$

and obtain, as in (8) (with $\mu = \nu$ and $l = 0$),

$$\|S_0 g\|_{L_p(P_{0,\nu})} \le c\Big(\sum_{k\ge 0} 2^{k(n-st-nt)} (2^{ks}\|u_k\|_{L_p(P_{0,\nu})})^t \Big)^{1/t}. \tag{15}$$

Now we write $2^{ks}\|u_k\|_{L_p(P_{0,\nu})} \le 2^{n\tau 0}\Big(\sum_{l\ge 0}(2^{ls}\|u_l\|_{L_p(P_{0,\nu})})^q\Big)^{1/q}$. Since $2^{n\tau 0} = 1$, then $2^{ks}\|u_k\|_{L_p(P_{0,\nu})} \le \sup_{j\in\mathbb{N}} 2^{n\tau j}\Big(\sum_{l\ge j}(2^{ls}\|u_l\|_{L_p(P_{j,\nu})})^q\Big)^{1/q} \le cA$ holds. From (15), by choosing also t such that $n - st - nt < 0$ (cf. (10)), we get that

$$\|S_0 g\|_{L_p(P_{0,\nu})} \le cA \qquad (\forall \nu \in \mathbb{Z}^n). \tag{16}$$

Now, by applying again Lemma 2.7 with $\psi := S_0 g$, ($\widehat{\psi}$ is supported in $|\xi| \le 3/2$), we get $\sup_{x\in P_{0,\nu}} |S_0 g(x)| \le c \sup_{\mu\in\mathbb{Z}^n} \|S_0 g\|_{L_p(P_{0,\mu})}$ $(\forall \nu \in \mathbb{Z}^n)$. Finally, by inserting this last inequality in (13) and taking (16) into account, we obtain $|\langle g_2, f\rangle| \le c\|f\|_1 A$ which yields the desired result.

Now the function g exists and belongs to $\mathcal{S}'(\mathbb{R}^n)$. We put $u := g$.

- *The case 2:* $b < 3/2$. We first replace ρ by another function with the same properties. Let $r > 0$. Let ρ_r be a cut-off function such that $\rho_r(\xi) = 0$ if $|\xi| \ge r$ and $\rho_r(\xi) = 1$ if $|\xi| \le 3r/2$. We put $\gamma_r(\xi) := \rho_r(\xi) - \rho_r(2\xi)$ which is supported by the compact annulus $r/2 \le |\xi| \le 3r/2$, and associate the operators $S_{r,k} := \rho_r(2^{-k}D)$ $(k = 0, 1, \ldots)$ and $Q_{r,j} := \gamma_r(2^{-j}D)$ $(j = 1, 2, \ldots)$. Again, we write $g := g_1 + g_2$ where $g_1 := \sum_{j\ge 1} Q_{r,j}g$ and $g_2 := S_{r,0}g$, and we estimate $|\langle g_1, f\rangle|$ and $|\langle g_2, f\rangle|$ similarly as in Substeps 1.1 and 1.2/Case 1, respectively. Indeed, we only note the following three situations:

  – m_0 is the nearest integer to the real number $\log_2(r/(2b))$, where $Q_{r,j}u_k = 0$ if $k \le j + m_0$,

  – as in (5), the constants C and R become $C := 3r2^{j-1}$ and $R := b2^{k-j+1}/r$ with $R \ge 1$, the estimate of $|\langle g_1, f\rangle|$ follows,

  – by choosing r such that $0 < r < 2b/3$ we obtain as in (14), $C := 3r/2$ and $R := b2^{k+1}/(3r)$ with $R \ge 1$ for all $k \ge 0$ and the estimate of $|\langle g_2, f\rangle|$ follows too.

Again, the function g now exists and belongs to $\mathcal{S}'(\mathbb{R}^n)$, and we also put $u := g$.

Step 2: proof of $\|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \le cA$. Consider a number r such that $r > 2b$. Based on Remark 2.3, we will use the sequences $(S_{r,k})_{k\ge 0}$ and $(Q_{r,j})_{j\ge 1}$ defined above in Substep 1.2/Case 2. The condition $r > 2b$ implies $m_0 \ge 0$. Now, we first write the inequality (8) and recall that it holds for any $l \in \mathbb{Z}$; we put $\sigma := n - st - nt$ for brevity, and get

$$\|Q_{r,j}u\|_{L_p(P_{l,\nu})} \le 2^{j(n-n/t)}\Big(\sum_{k\ge (j+m_0)_+} 2^{k\sigma}(2^{ks}\|u_k\|_{L_p(P_{l,\nu})})^t\Big)^{1/t}$$

which is bounded by $2^{j(n-n/t)}\big(\sum_{k\geq j+m_0} 2^{k\sigma}(2^{ks}\|u_k\|_{L_p(P_{l,\nu})})^t\big)^{1/t}$ where t is given in (10). We continue by summation with respect to j and take into account that in the right-hand side it holds that $\sum_{k\geq j+m_0}\cdots \leq \sum_{k\geq j}\cdots$. Then

$$\Big(\sum_{j\geq l_+}(2^{js}\|Q_{r,j}u\|_{L_p(P_{l,\nu})})^q\Big)^{1/q} \leq c_1\Big(\sum_{j\geq l_+}\Big(\sum_{k\geq j} 2^{(k-j)\sigma}(2^{ks}\|u_k\|_{L_p(P_{l,\nu})})^t\Big)^{q/t}\Big)^{1/q}.$$

Applying Lemma 2.5, since $\sigma < 0$, the right-hand side of the last inequality is bounded by $c2^{-n\tau l}A$, and the desired result follows. □

3. Proof of Theorem 1.1

As mentioned in the Introduction, the main tools of the proof are the following statements, where we need a cut-off function: we fix φ a C^∞-function on $\mathbb{R}$, such that $\varphi(x) = 1$ if $x \in [-1, 1]$ and $\varphi(x) = 0$ if $x \notin [-2, 2]$. We put $\varphi_t := \varphi(t^{-1}(\cdot))$ for all $t > 0$, which will be used in what follows as in the following equality $f \circ g = f\varphi_t \circ g$ if $g \in L_\infty(\mathbb{R}^n)$ and $t \geq \max(1, \|g\|_\infty)$.

Proposition 3.1. *[3, 9] Let $0 < s \neq 1$ and $0 < q \leq \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function in $B^s_{\infty,q}(\mathbb{R})_{loc}$.*

(i) If $s > 1$, then T_f takes $B^s_{\infty,q}(\mathbb{R}^n)$ to itself.

(ii) If $s < 1$, then T_f takes $W^1_\infty(\mathbb{R}^n)$ to $B^s_{\infty,q}(\mathbb{R}^n)$.

Moreover, there exists a continuous increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on n, q and s, such that, for all such functions f, and all g in various function spaces in (i) and (ii), it holds

$$\|T_f(g)\|_{B^s_{\infty,q}(\mathbb{R}^n)} \leq \|f\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(\mathcal{N}(g)), \tag{17}$$

where $t \geq \max(1, \|g\|_\infty)$, $\mathcal{N}(g) := \|g\|_{B^s_{\infty,q}(\mathbb{R}^n)}$ in the case (i) and $\mathcal{N}(g) := \|g\|_{W^1_\infty(\mathbb{R}^n)}$ in the case (ii).

Remark 3.2. Concerning Proposition 3.1, the cases $s > 1$ and $0 < s < 1$ are proved in [3, Theorem 4] and [9, Proposition 3.1], respectively. These two references provide the proofs for $q \geq 1$, however the extension to $0 < q < 1$ is easy. Also, the precise estimate (17) occurs in both [3] and the proof given in [9].

Proposition 3.3. *Let $0 < p, q \leq \infty$, $s > (n/p - n)_+$ and $0 \leq \tau \leq 1/p$. Let $b > 0$. Let $(\chi_j)_{j\in\mathbb{N}}$ be a sequence of functions in $B^s_{\infty,q}(\mathbb{R}^n)$. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{f_j}$ is supported by the ball $|\xi| \leq b2^j$ and*

$$A := \sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\Big(\sum_{j\geq k_+}(2^{sj}\|f_j\|_{L_p(P_{k,\nu})})^q\Big)^{1/q} < \infty.$$

Then it holds $\big\|\sum_{j\geq 0}\chi_j f_j\big\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq cA\sup_{j\geq 0}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}$, where the constant c depends only on n, p, q, s, τ and b.

Proof. For all $j \in \mathbb{N}$, we have $\chi_j = S_j\chi_j + \sum_{m\geq j+1} Q_m\chi_j$, then we put $\sum_{j\geq 0}\chi_j f_j = g_1 + g_2$, where $g_1 := \sum_{j\geq 0} f_j S_j\chi_j$ and $g_2 := \sum_{m\geq 1}\sum_{j=0}^{m-1} f_j Q_m\chi_j$.

Step 1: estimate of g_1. The function $\widehat{f_j S_j\chi_j}$ is supported by the ball $|\xi| \leq (b + 3/2)2^j$, hence from Proposition 2.4, we have

$$\|g_1\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\Big(\sum_{j\geq k_+} 2^{sjq}\|f_j S_j\chi_j\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}.$$

Using the inequality $\|S_j\chi_j\|_\infty \leq c\|\chi_j\|_\infty$ $(\forall j\geq 0)$, and the embedding $B^s_{\infty,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ $(s>0)$, we get $\|f_j S_j\chi_j\|_{L_p(P_{k,\nu})} \leq c\|f_j\|_{L_p(P_{k,\nu})}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}$ and $\|g_1\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}$ is bounded by $cA\sup_{j\geq 0}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}$.

Step 2: estimate of g_2. The function $\mathcal{F}(\sum_{j=0}^{m-1} f_j Q_m\chi_j)$ is supported by the ball $|\xi| \leq (b/2 + 3/2)2^m$ where $m\geq 1$. Then Proposition 2.4 gives us

$$\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\Big(\sum_{m\geq 1+k_+} 2^{smq}\Big\|\sum_{j=0}^{m-1} f_j Q_m\chi_j\Big\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}. \tag{18}$$

We continue the proof by the following substeps with respect to p and q.

Substep 2.1: the case $p\geq 1$ *and* $q\geq 1$. By Minkowski inequality with respect to $\ell_q(\mathbb{N})$, we get

$$\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \leq c_1\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\sum_{j\geq 0}\Big(\sum_{m\geq j+1} 2^{qsm}\|f_j Q_m\chi_j\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}.$$

Now we have easily

$$\Big(\sum_{m\geq j+1} 2^{qsm}\|f_j Q_m\chi_j\|^q_{L_p(P_{k,\nu})}\Big)^{1/q} \leq \Big(\sum_{m\geq 0} 2^{qsm}\|Q_m\chi_j\|^q_\infty\Big)^{1/q}\|f_j\|_{L_p(P_{k,\nu})},$$

and then we obtain that $\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}$ is bounded by $c_2\sup_{j\geq 0}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}(A_1 + A_2)$ where $A_1 := \sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\sum_{j\geq 1+k_+}\|f_j\|_{L_p(P_{k,\nu})}$ and $A_2 := \sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\sum_{j=0}^{k_+}\|f_j\|_{L_p(P_{k,\nu})}$.

Then, by Hölder inequality it holds

$$A_1 \leq \sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k}\sum_{j\geq k_+} 2^{-sj}(2^{sj}\|f_j\|_{L_p(P_{k,\nu})}) \leq cA. \tag{19}$$

Now we prove that $A_2 \leq cA$. By the inequality $\|f_j\|_{L_p(P_{k,\nu})} \leq 2^{-kn/p}\|f_j\|_\infty$ we get $A_2 \leq \sup_{k\in\mathbb{Z}} 2^{kn(\tau-1/p)}\sum_{j=0}^{k_+}\|f_j\|_\infty$. For the estimate of $\|f_j\|_\infty$, we use the same calculus given in the proof of [16, Proposition 2.6, p. 46]. We obtain $\|f_j\|_\infty \leq c2^{j(n/p-n\tau-s)}A$ for all $j\geq 0$, and by assumptions $0\leq\tau\leq 1/p$ and $s>0$ we get that

$$\begin{aligned} A_2 &\leq c_1 A\sup_{k\in\mathbb{Z}} 2^{kn(\tau-1/p)}\sum_{j=0}^{k_+} 2^{-jn(\tau-1/p)}2^{-sj} \\ &\leq c_1 A\Big(1 + \sup_{k\geq 1} 2^{kn(\tau-1/p)}\sum_{j=0}^{k} 2^{-jn(\tau-1/p)}2^{-sj}\Big) \leq c_2 A. \end{aligned} \tag{20}$$

Now, by (19) and (20) it follows that $\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}$ is bounded by $cA\sup_{j\geq 0}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}$.

Substep 2.2: the case $p\geq 1$ *and* $0<q<1$. In (18) using (7) with $\alpha:=q$, we have

$$\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}\leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{m\geq 1+k_+}\sum_{j=0}^{m-1}2^{smq}\|f_jQ_m\chi_j\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}.$$

Using the following estimate

$$\|f_jQ_m\chi_j\|_{L_p(P_{k,\nu})}\leq\|Q_m\chi_j\|_\infty\|f_j\|_{L_p(P_{k,\nu})},\tag{21}$$

we obtain

$$\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}\leq c\sup_{j\geq 0}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}(A_3+A_4),\tag{22}$$

where $A_3:=\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{j\geq 1+k_+}\|f_j\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}$,
$A_4:=\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{j=0}^{k_+}\|f_j\|^q_{L_p(P_{k,\nu})}\Big)^{1/q}$. The estimates of A_3 and A_4 are completely similar to that of A_1 and A_2, respectively.

Substep 2.3: the case $0<q\leq p<1$. From (18), and using twice (7) with respect to $\ell_p(\{0,\dots,m-1\})$ and with respect to $\ell_{q/p}(\{k_++1,k_++2,\dots\})$, we have

$$\begin{aligned}\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}&\leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{m\geq 1+k_+}\Big(\sum_{j=0}^{m-1}\int_{P_{k,\nu}}2^{psm}|f_jQ_m\chi_j(x)|^p\mathrm{d}x\Big)^{q/p}\Big)^{1/q}\\&\leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{m\geq 1+k_+}\sum_{j=0}^{m-1}2^{smq}\|f_jQ_m\chi_j\|^q_{L^p(P_{k,\nu})}\Big)^{1/q}.\end{aligned}$$

Then, again we proceed as in (21) and (22).

Substep 2.4: the case $0<p<1$*,* $p<q$ *and* $0<q\leq\infty$. Here also from (18) and using (7) with respect to $\ell_p(\{0,\dots,m-1\})$, we obtain

$$\begin{aligned}\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}&\leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{m\geq 1+k_+}\Big(\sum_{j=0}^{m-1}\int_{P_{k,\nu}}2^{smp}|f_jQ_m\chi_j(x)|^p\mathrm{d}x\Big)^{q/p}\Big)^{1/q}\\&\leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\Big\{\sum_{m\geq k_+}\Big(\sum_{j\geq 0}2^{smp}\|f_jQ_m\chi_j\|^p_{L^p(P_{k,\nu})}\Big)^{q/p}\Big\}^{p/q}\Big)^{1/p}.\end{aligned}$$

Now by Minkowski inequality with respect to $\ell_{q/p}(\mathbb{N})$, it holds

$$\|g_2\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}\leq c\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big\{\sum_{j\geq 0}\Big(\sum_{m\geq k_+}2^{smq}\|f_jQ_m\chi_j\|^q_{L^p(P_{k,\nu})}\Big)^{p/q}\Big\}^{1/p},$$

and by (21) we obtain the bound $c\sup_{j\geq 0}\|\chi_j\|_{B^s_{\infty,q}(\mathbb{R}^n)}(A_5+A_6)$ where

$$A_5:=\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{j\geq 1+k_+}\|f_j\|^p_{L_p(P_{k,\nu})}\Big)^{1/p},\ A_6:=\sup_{k\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}2^{n\tau k}\Big(\sum_{j=0}^{k_+}\|f_j\|^p_{L_p(P_{k,\nu})}\Big)^{1/p},$$

and the estimates of A_5 and A_6 are similar to that of A_1 and A_2, respectively, however some technical changes are needed. Indeed, by Hölder inequality with exponents q/p

and $q/(q-p)$, it holds

$$\sum_{j\geq 1+k_+}\|f_j\|^p_{L_p(P_{k,\nu})}=\sum_{j\geq 1+k_+}2^{-sjp}(2^{sj}\|f_j\|_{L_p(P_{k,\nu})})^p\leq c\Big(\sum_{j\geq k_+}(2^{sj}\|f_j\|_{L_p(P_{k,\nu})})^q\Big)^{p/q}$$

for all $k\in\mathbb{Z}$, which yields $A_5\leq cA$. For A_6, by using the estimate $\|f_j\|_{L_p(P_{j,\nu})}\leq c2^{-j(n\tau+s)}A$ ($\forall j\geq 0$, $\forall\nu\in\mathbb{Z}^n$), then as in (20) we get

$$A_6\leq c_1A\Big(1+\sup_{k\geq 1}2^{kn(\tau p-1)}\sum_{j=0}^{k}2^{-jn(\tau p-1)}2^{-sjp}\Big)^{1/p}\leq c_2A.$$

The proof is complete. □

Proof (Proof of Theorem 1.1). Let g be a function in $W^1_\infty(\mathbb{R}^n)\cap B^{s,\tau}_{p,q}(\mathbb{R}^n)$, $s<1$, (in the case $s>1$ the function g is taken in $B^s_{\infty,q}(\mathbb{R}^n)\cap B^{s,\tau}_{p,q}(\mathbb{R}^n)$). We easily get, both $\lim_{j\to\infty}f\circ S_jg=f\circ g$ in $L_\infty(\mathbb{R}^n)$ and the following linearization:

$$f\circ g=f\circ S_0g+\sum_{j\geq 0}(f\circ S_{j+1}g-f\circ S_jg), \tag{23}$$

(for more details, see [8, 9]). Now, we introduce a sequence of operators $(R_j)_{j\in\mathbb{N}}$ defined by $R_0(f,g):=\int_0^1 f\circ(zS_0g)\mathrm{d}z$, $R_j(f,g):=\int_0^1 f\circ(S_{j-1}g+zQ_jg)\mathrm{d}z$ ($j=1,2,\ldots$). From (23) we have

$$f\circ g=\sum_{j\geq 0}R_j(f',g)Q_jg. \tag{24}$$

On the other hand, there exist two positive constants c_1 and c_2 such that

$$\|S_0g\|_\infty\leq c_1\|g\|_\infty \text{ and } \|S_{j-1}g+zQ_jg\|_\infty\leq c_2\|g\|_\infty \ (\forall z\in[0,1],\ j=1,2,\ldots).$$

By taking $t\geq\max(1,c_1\|g\|_\infty,c_2\|g\|_\infty)$ we arrive at

$$R_j(f',g)=R_j(\varphi_tf',g), \tag{25}$$

where the cut-off function φ is defined in the beginning of this section. The function $f'\varphi_t$ belongs to $B^s_{\infty,q}(\mathbb{R})$. Indeed, we may write $f'\varphi_t=(f\varphi_t)'-f\varphi_t'$, then both $(f\varphi_t)'\in B^s_{\infty,q}(\mathbb{R})$ and $f\varphi_t'\in B^{s+1}_{\infty,q}(\mathbb{R})\hookrightarrow B^s_{\infty,q}(\mathbb{R})$ yield the desired assertion. Now we establish the following *claim*: the sequence $(R_j(f',g))_{j\in\mathbb{N}}$ is bounded in $B^s_{\infty,q}(\mathbb{R}^n)$.

In the case $q\geq 1$, the equality (25) and Proposition 3.1 give the claim. However, this argument does not work in the case $0<q<1$ since it is not possible to apply the Minkowski inequality. Then the integral (in R_j) can be interpreted as the limit of Riemann sums, i.e. we first prove

$$R_0(f',g)=\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}f'\Big(\frac{k}{m}S_0g\Big)\quad\text{in}\quad\mathcal{S}'(\mathbb{R}^n). \tag{26}$$

We set $U_{m,(0)}:=\frac{1}{m}\sum_{k=0}^{m-1}f'\big(\frac{k}{m}S_0g\big)$. Indeed, using Proposition 3.1 (see also (17)), there exits a continuous increasing function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ depending only on n,q and s, such that

$$\Big\|f'\Big(\frac{k}{m}S_0g\Big)\Big\|_{B^s_{\infty,q}(\mathbb{R}^n)}\leq\|f'\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(\mathcal{N}(S_0g))\qquad(k=0,\ldots,m-1). \tag{27}$$

Here we have used $\phi\big(\mathcal{N}\big(\frac{k}{m}S_0 g\big)\big) \le \phi(\mathcal{N}(S_0 g))$, where

$$t \ge \max(1, c\|g\|_\infty) \ge \max(1, \|S_0 g\|_\infty) \ge \max\Big(1, \Big\|\frac{k}{m}S_0 g\Big\|_\infty\Big),$$

and $\mathcal{N}(\cdot)$ is defined in Proposition 3.1, i.e., $\mathcal{N}(S_0 g) \le c\|g\|_{W^1_\infty(\mathbb{R}^n)}$ if $s < 1$, or $\mathcal{N}(S_0 g) \le c\|g\|_{B^s_{\infty,q}(\mathbb{R}^n)}$ if $s > 1$, (they follow from $\|S_0 g\|_\infty \le c\|g\|_\infty$), so we conclude that $\mathcal{N}(S_0 g) \le c\mathcal{N}(g)$ in each case, and consequently

$$\|U_{m,(0)}\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le \|f'\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(c\mathcal{N}(g)) \qquad (\forall m \ge 1). \tag{28}$$

Now, by the embedding $B^s_{\infty,q}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ the estimate (27) yields

$$\|U_{m,(0)}\|_\infty \le \|f'\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(c\mathcal{N}(g)) \qquad (\forall m \ge 1), \tag{29}$$

where $t \ge \max(1, c\|g\|_\infty)$ and the right-hand side of (29) is independent of m. Let now $\psi \in \mathcal{S}(\mathbb{R}^n)$. We apply Dominated Convergence Theorem, and we deduce that

$$\lim_{m\to\infty}\langle U_{m,(0)}, \psi\rangle = \int_{\mathbb{R}^n}\lim_{m\to\infty} U_{m,(0)}(x)\psi(x)\,\mathrm{d}x = \langle R_0(f', g), \psi\rangle,$$

and (26) is proved. Now we put $U_{m,(j)} := \frac{1}{m}\sum_{k=0}^{m-1} f'\big(S_{j-1}g + \frac{k}{m}Q_j g\big)$, $(j = 1, 2, \ldots)$, and the same proof yields the following:

$$\|U_{m,(j)}\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le \|f'\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(c\mathcal{N}(g)) \qquad (\forall j \ge 1, \forall m \ge 1), \tag{30}$$

$$\|U_{m,(j)}\|_\infty \le \|f\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(c\mathcal{N}(g)) \quad (\forall j, m \ge 1), \tag{31}$$

$$\lim_{m\to\infty} U_{m,(j)} = R_j(f', g) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n). \tag{32}$$

Applying the Fatou property to the sequence $(U_{m,(j)})_{m\in\mathbb{N}}$, by (26)–(32), we get

$$\|R_j(f', g)\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le c_1\|f'\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})}\phi(c_2\mathcal{N}(g)) \qquad (\forall j \ge 0),$$

where $t \ge \max(1, \|g\|_\infty)$, and the claim is proved. Finally, by applying Proposition 3.3 to the series (24) (with $\chi_j := R_j(f', g)$ and $f_j := Q_j g$), we obtain

$$\|T_f(g)\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \le c_1\|f\varphi_t\|_{B^{s+1}_{\infty,q}(\mathbb{R})}\phi(c_2\mathcal{N}(g))\|g\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}, \tag{33}$$

where t and $\mathcal{N}(g)$ are defined above. Here we have also used $\|f'\varphi_t\|_{B^s_{\infty,q}(\mathbb{R})} \le c\|f\varphi_t\|_{B^{s+1}_{\infty,q}(\mathbb{R})}$ for all $t > 0$. Now concerning the assumption $f(0) = 0$, by testing the zero function in (33), we obtain this condition, and the proof of Theorem 1.1 is complete. □

4. Some extensions and remarks

Now, we deal with the case $s = 1$, where we need the following notation: we denote by $\dot{W}^m_\infty(\mathbb{R}^n)$ $(m = 1, 2, \ldots)$ the homogeneous Sobolev space of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^{(\alpha)} \in L_\infty(\mathbb{R}^n)$ for $|\alpha| = m$, and endowed with the semi-norm $\|f\|_{\dot{W}^m_\infty(\mathbb{R}^n)} := \sum_{|\alpha|=m}\|f^{(\alpha)}\|_\infty$. We have $\|f + \mathcal{P}\|_{\dot{W}^m_\infty(\mathbb{R}^n)} = \|f\|_{\dot{W}^m_\infty(\mathbb{R}^n)}$ for all polynomials $\mathcal{P}$ of degree less than m. So, we formulate the following statement.

Proposition 4.1. *Let $0 < q \le \infty$. If a function $f : \mathbb{R} \to \mathbb{R}$ belongs to $(\dot{W}^1_\infty(\mathbb{R}) \cap$*

$B^1_{\infty,q}(\mathbb{R}))_{loc}$, then T_f takes $\dot{W}^1_\infty(\mathbb{R}^n) \cap B^1_{\infty,q}(\mathbb{R}^n)$ to $B^1_{\infty,q}(\mathbb{R}^n)$. Moreover, there exists a constant $c = c(n,q) > 0$ such that

$$\|T_f(g)\|_{B^1_{\infty,q}(\mathbb{R}^n)} \leq c\|f\varphi_t\|_{\dot{W}^1_\infty(\mathbb{R})\cap B^1_{\infty,q}(\mathbb{R})} \left(1 + \|g\|_{\dot{W}^1_\infty(\mathbb{R}^n)\cap B^1_{\infty,q}(\mathbb{R}^n)}\right)$$

holds, for all such functions f and all $g \in \dot{W}^1_\infty(\mathbb{R}^n) \cap B^1_{\infty,q}(\mathbb{R}^n)$, where $t \geq \max(1, \|g\|_\infty)$. The function φ_t was defined in the beginning of Section 3.

Proof. In the case $0 < q \leq 1$, we have $B^1_{\infty,q}(\mathbb{R}) \cap \dot{W}^1_\infty(\mathbb{R}) = B^1_{\infty,q}(\mathbb{R})$ and $B^1_{\infty,q}(\mathbb{R}^n) \cap \dot{W}^1_\infty(\mathbb{R}^n) = B^1_{\infty,q}(\mathbb{R}^n)$, and the assertion is proved in [3, Theorem 5] with $q = 1$. The proof given in [3, Theorem 5] can be easily extended to any $q > 0$ under assumptions on f and g, since we only replace, in this proof, the $L_1(]0,\infty[; \mathrm{d}t/t)$ by $L_q(]0,\infty[; \mathrm{d}t/t)$. □

Based on this proposition, we obtain a result for the composition operator T_f on the space $B^{1,\tau}_{p,q}(\mathbb{R}^n)$ which has a proof completely similar to that of Theorem 1.1.

Theorem 4.2. *Let $0 < p, q \leq \infty$, $(n/p - n)_+ < s = 1$ and $0 \leq \tau \leq 1/p$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function such that $f(0) = 0$ and $f \in (\dot{W}^2_\infty(\mathbb{R}) \cap B^2_{\infty,q}(\mathbb{R}))_{loc}$. Then T_f takes $\dot{W}^1_\infty(\mathbb{R}^n) \cap B^1_{\infty,q}(\mathbb{R}^n) \cap B^{1,\tau}_{p,q}(\mathbb{R}^n)$ to $B^{1,\tau}_{p,q}(\mathbb{R}^n)$. Moreover, there exists a constant $c = c(n,p,q,\tau) > 0$ such that*

$$\|T_f(g)\|_{B^{1,\tau}_{p,q}(\mathbb{R}^n)} \leq c\|f\varphi_t\|_{\dot{W}^2_\infty(\mathbb{R})\cap B^2_{\infty,q}(\mathbb{R})} \left(1 + \|g\|_{\dot{W}^1_\infty(\mathbb{R}^n)\cap B^1_{\infty,q}(\mathbb{R}^n)}\right)\|g\|_{B^{1,\tau}_{p,q}(\mathbb{R}^n)}$$

holds, for all such functions f and all $g \in \dot{W}^1_\infty(\mathbb{R}^n) \cap B^1_{\infty,q}(\mathbb{R}^n) \cap B^{1,\tau}_{p,q}(\mathbb{R}^n)$, and where $t \geq \max(1, \|g\|_\infty)$.

Remark 4.3. It would be interesting to extend the result in Theorem 1.1 to:

(i) The Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$, $(p \in]0,\infty[,\ q \in]0,\infty],\ s, \tau \in \mathbb{R})$, the set of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{kn\tau} \Big\| \Big(\sum_{j \geq k_+} (2^{sj}|Q_j f|)^q \Big)^{1/q} \Big\|_{L_p(P_{k,\nu})} < \infty.$$

(ii) The homogeneous Besov-type spaces $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, $(p, q \in]0,\infty],\ s, \tau \in \mathbb{R})$, the set of the tempered distributions modulo polynomials f such that

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} 2^{n\tau k} \Big(\sum_{j\geq k} (2^{sj}\|Q_j f\|_{L_p(P_{k,\nu})})^q \Big)^{1/q} < \infty. \tag{34}$$

Here $Q_j := \gamma(2^{-j}D)$ for all $j \in \mathbb{Z}$. Recall that $\|f\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} = \|f + \mathcal{P}\|_{\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)}$ for all polynomials $\mathcal{P}$ on $\mathbb{R}^n$.

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