MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
70, 3 (2018), 204–210
September 2018

research paper
оригинални научни рад

# A FIXED POINT THEOREM FOR MAPPINGS WITH A CONTRACTIVE ITERATE IN RECTANGULAR $b$-METRIC SPACES

**Zoran D. Mitrović**

**Abstract**. In this paper, we give a proof for Sehgal-Guseman theorem of fixed point in rectangular $b$-metric spaces. Our result is supported with a suitable example. As a corollary of our results, we obtain fixed point results of contraction mappings in $b$-metric spaces.

## 1. Introduction and preliminaries

In 1922, Banach proved the following contraction mapping principle.

Theorem 1.1. *Let $(X, d)$ be a complete metric space. Let $T$ be a contractive mapping on $X$, that is, one for which exists $q \in [0, 1)$ satisfying*

$$d(Tx, Ty) \leq qd(x, y)$$

*for all $x, y \in X$. Then there exists a unique fixed point $x \in X$ of $T$.*

This theorem is a forceful tool in nonlinear analysis, has many applications and has been extended by a great number of authors. In 1969, Sehgal [8] proved the following generalization of the contraction mapping principle.

Theorem 1.2. *Let $(X, d)$ be a complete metric space, $q \in [0, 1)$ and $T : X \to X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $k = k(x)$ such that*

$$d(T^{k(x)}x, T^{k(x)}y) \leq qd(x, y)$$

*for all $y \in X$, then $T$ has a unique fixed point $u \in X$. Moreover, for any $x \in X$, $u = \lim_{n\to\infty} T^n x$.*

In 1970, Guseman [5] generalized the result of Sehgal to mappings which are both necessarily continuous and which have a contractive iterate at each point in a (possibly proper) subset of the space.

*2010 Mathematics Subject Classification*: 47H10
*Keywords and phrases*: Fixed points, rectangular $b$-metric space.

In the paper [4] authors introduced the concept of rectangular $b$-metric space, which is not necessarily Hausdorff and which generalizes the concept of metric space, rectangular metric space (RMS) and $b$-metric space.

Definition 1.3. [4] Let $X$ be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(RbM1) $d(x, y) = 0$ if and only if $x = y$;

(RbM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(RbM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$

for all $x, y \in X$ and all distinct points $u, v \in X \backslash \{x, y\}$.

Then $d$ is called a rectangular $b$-metric on $X$ with coefficient $s$ and $(X, d, s)$ is called a rectangular $b$-metric space (in short RbMS).

Note that every rectangular metric space is a rectangular $b$-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true.

Also in [4] the concept of convergence in such spaces is similar to that of standard metric spaces (see for example [6, 7]).

Definition 1.4. [4] Let $(X, d)$ be a $b$-rectangular metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then:

(a) The sequence $\{x_n\}$ is said to be convergent in $(X, d)$ and converges to $x$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in $(X, d)$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) $(X, d)$ is said to be a complete $b$-rectangular metric space if every Cauchy sequence in X converges to some $x \in X$.

In the papers of Bakhtin [1] and Czerwik [2], the notion of $b$-metric space has been introduced and some fixed point theorems for single-valued and multi-valued mappings in $b$-metric spaces were proved.

Definition 1.5. Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a $b$-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

A triplet $(X,d,b)$, is called a $b$-metric space.

Note that a metric space is included in the class of $b$-metric spaces with coefiicent $s \geq 1$. Note also that every $b$-metric space is a rectangular $b$-metric space (with coefficient $s^2$) but the converse is not necessarily true ( [4], Examples 2.7).

We have the following diagram where arrows stand for inclusions. The inverse inclusions do not hold.

$$\begin{array}{ccc} \text{metric space} & \longrightarrow & b\text{-metric space} \\ \downarrow & & \downarrow \\ \text{rectangular metric space} & \longrightarrow & b\text{-rectangular metric space} \end{array}$$

The aim of this paper is to obtain Theorem 1.2. in rectangular $b$-metric spaces.

## 2. Main result

LEMMA 2.1. *Let $(X,d,s)$ be a complete rectangular $b$-metric space and $T : X \to X$ a mapping satisfying the condition: for each $x \in X$ there exists $k(x) \in \mathbb{N}$ such that*

$$d\left(T^{k(x)}x, T^{k(x)}y\right) \leq \lambda d(x,y),$$

*for all $y \in X$, where $\lambda \in (0,1)$. Then for each $x \in X, r(x) = \sup\{d(T^n(x),x) : n \in \mathbb{N}\}$ is finite or $T$ has a fixed point.*

*Proof.* Let $x \in X$ and let

$$l(x) = \sup\{d(T^k(x),x) : k \in \{1,\ldots,k_1+k_2+\cdots+k_{n_0}+k_{n_0+1}\}\},$$

where $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \frac{1}{2s}$ and

$$k_1 = k(x), k_2 = k(T^{k_1}x), k_3 = k(T^{k_2+k_1}x), \ldots, k_{n_0+1} = k(T^{k_{n_0}+\ldots+k_1}x).$$

Let $S = k_1+k_2+\cdots+k_{n_0}$ and $S_1 = k_1+k_2+\cdots+k_{n_0}+k_{n_0+1}$. We have,

$$\begin{aligned} d(T^Sx, T^{S+m}x) &= d(T^{k_1+k_2+\ldots+k_{n_0}}x, T^{k_1+k_2+\ldots+k_{n_0}}(T^m)x) \\ &\leq \lambda d(T^{k_1+k_2+\ldots+k_{n_0-1}}x, T^{k_1+k_2+\ldots+k_{n_0-1}}(T^m)x) \\ &\vdots \\ &\leq \lambda^{n_0} d(x, T^m x). \end{aligned}$$

So

$$d(T^Sx, T^{S+m}x) \leq \lambda^{n_0} d(x,T^mx) \text{ for all } m \in \mathbb{N}. \tag{1}$$

Similarly, we get

$$d(T^{S_1}x, T^{S_1+m}x) \leq \lambda^{n_0+1} d(x,T^mx) \text{ for all } m \in \mathbb{N}. \tag{2}$$

Let $n \in \mathbb{N}$.

1. If $T^nx = T^Sx$ then $d(x,T^nx) \leq l(x)$ and proof is holds.

2. If $T^S x = T^{S_1} x$ then $T^S x = T^{S+1} x$ and $T^S x$ is a fixed point of $T$ and proof is finished. Namely, if $T^S x \neq T^{S+1} x$, we obtain
$$d(T^S x, T^{S+1} x) = d(T^{S_1} x, T^{S_1+1} x) \leq \lambda d(T^S x, T^{S+1} x) < d(T^S x, T^{S+1} x).$$
It is a contradiction.

3. If $T^{S_1} x = x$ then $Tx = x$ and proof is holds. Namely, if $Tx \neq x$ then we have
$$d(x, Tx) = d(T^{S_1} x, T^{S_1+1} x) \leq \lambda^{n_0+1} d(x, Tx) \overset{(2)}{<} d(x, Tx).$$
It is a contradiction.

So, $T^S x$ and $T^{S_1} x$ distinct point and $T^S x, T^{S_1} x \in X \backslash \{T^n x, x\}$. If $n > S$ then there exists an integer $t \geq 0$ such that $tS < n \leq (t+1)S$. From (RbM3), (1) and (2), we obtain
$$\begin{aligned} d(T^n x, x) &\leq s[d(T^{S+(n-S)} x, T^S x) + d(T^S x, T^{S_1} x) + d(T^{S_1} x, x)] \\ &\leq s\left[\lambda^{n_0} d(T^{n-S} x, x) + \lambda^{n_0} d(x, T^{k_{n_0+1}} x) + l(x)\right] \\ &\leq s\left[\frac{1}{2s} d(T^{n-S} x, x) + \frac{1}{2s} l(x) + l(x)\right] \\ &\leq \frac{1}{2} d(T^{n-S} x, x) + \left(\frac{1}{2} + s\right) l(x). \end{aligned}$$
Similarly, we obtain
$$d(T^{n-S} x, x) \leq \frac{1}{2} d(T^{n-2S} x, x) + \left(\frac{1}{2} + s\right) l(x).$$
So,
$$d(T^n x, x) \leq \frac{1}{2^2} d(T^{n-2S} x, Tx) + \left(1 + \frac{1}{2}\right)\left(\frac{1}{2} + s\right) l(x).$$
Continuing in this process we obtain
$$\begin{aligned} d(T^n x, x) &\leq \frac{1}{2^t} d(T^{n-tS} x, Tx) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{t-1}}\right)\left(\frac{1}{2} + s\right) l(x) \\ &\leq \frac{1}{2^t} l(x) + 2\left(\frac{1}{2} + s\right) l(x) \leq 2(1+s) l(x) \end{aligned}$$
and $r(x)$ is finite. □

Theorem 2.2. *Let $(X, d, s)$ be a complete rectangular b-metric space and $T : X \to X$ a mapping satisfying the condition: for each $x \in X$ there exists $k(x) \in \mathbb{N}$ such that*
$$d\left(T^{k(x)} x, T^{k(x)} y\right) \leq \lambda d(x, y), \tag{3}$$
*for all $y \in X$, where $\lambda \in (0, 1)$. Then $T$ has a unique fixed point, say $u \in X$, and $T^n x \to u$ for each $x \in X$.*

*Proof.* Let $x_0 \in X$ be arbitrary. Let $k_1 = k(x_0)$, $x_1 = T^{k_1} x_0$ and inductively

$k_{i+1} = k(x_i)$, $x_{i+1} = T^{k_{i+1}} x_i, i \in \mathbb{N}$. Let $n, p \in \mathbb{N}$. We have

$$\begin{aligned}
d(x_{n+p}, x_n) &= d(T^{k_{n+p}} x_{n+p-1}, T^{k_n} x_{n-1}) \\
&= d(T^{k_{n+p}+k_{n+p-1}} x_{n+p-2}, T^{k_n+k_{n-1}} x_{n-2}) \\
&\vdots \\
&= d(T^{k_{n+p}+k_{n+p-1}+\cdots+k_{p+1}} x_p, T^{k_n+k_{n-1}+\cdots+k_1} x_0) \\
&= d(T^{k_{n+p}+k_{n+p-1}+\cdots+k_{p+1}+k_p} x_{p-1}, T^{k_n+k_{n-1}+\cdots+k_1} x_0) \\
&\vdots \\
&= d(T^{k_{n+p}+k_{n+p-1}+\cdots+k_n+\cdots+k_1} x_0, T^{k_n+k_{n-1}+\cdots+k_1} x_0) \\
&= d(T^{k_n+k_{n-1}+\cdots+k_1}(T^{k_{n+1}+\cdots+k_{n+p}} x_0), T^{k_n+k_{n-1}+\cdots+k_1} x_0) \\
&\leq \lambda^n d(T^{k_{n+1}+\cdots+k_{n+p}} x_0, x_0).
\end{aligned}$$

Therefore, $d(x_{n+p}, x_n) \leq \lambda^n r(x_0)$.

1. If $r(x_0)$ is not finite, from Lemma 2.1. we conclude that T has the fixed point and the proof is finished.

2. If $r(x_0) < +\infty$ we infer that $(x_n)$ is Cauchy. From the completeness of $(X, d, s)$ we have $x_n \to u$, for some $u \in X$. Now, we shall show that $Tu = u$. For this $u$ there is $k(u) \in \mathbb{N}$ such that $d(T^{k(u)}u, T^{k(u)}x_n) \leq \lambda d(x_n, u)$. Hence,

$$\lim_{n\to\infty} d(T^{k(u)} x_n, T^{k(u)} u) = 0. \tag{4}$$

Now, from (3) we have

$$d(T^{k(u)} x_n, x_n) = d(T^{k(u)+k_{n-1}} x_{n-1}, T^{k_{n-1}} x_{n-1}) \leq \lambda d(T^{k(u)} x_{n-1}, x_{n-1})$$

and it follows that

$$d(T^{k(u)} x_n, x_n) \leq \lambda^n d(T^{k(u)} x_0, x_0) \leq \lambda^n r(x_0).$$

From Lemma 2.1 we obtain

$$\lim_{n\to\infty} d(T^{k(u)} x_n, x_n) = 0. \tag{5}$$

From triangle inequality (RbM3) we obtain

$$d(T^{k(u)} u, u) \leq s[d(T^{k(u)} u, T^{k(u)} x_n) + d(T^{k(u)} x_n, x_n) + d(x_n, u)]$$

and together with (4) and (5) we obtain $d(T^{k(u)}u, u) = 0$. By (3), $u$ is the unique fixed point for $T^{k(u)}$. Then $Tu = T(T^{k(u)}) = T^{k(u)}(Tu)$ implies that $Tu = u$. But then $u$ is the unique fixed point of $T$ □

EXAMPLE 2.3. The space $l^p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}, p \in (0,1)$, together with the function $d : l^p \times l^p \to \mathbb{R}$,

$$d(x, y) = \left( \sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = (x_n), y = (y_n) \in l^p$, is a rectangular $b$-metric space with $s = 2^{2+\frac{2}{p}}$. Indeed, by an elementary calculation we obtain $d(x,y) \leq 2^{2+\frac{2}{p}}[d(x,u)+d(u,v)+d(v,y)]$, for all $x,y \in l^p$ and all distinct points $u,v \in l^p \backslash \{x,y\}$. Let $T : l^p \to l^p$ be a mapping defined by

$$T(x_1,x_2,x_3,x_4\ldots) = \big(0, x_1, \frac{x_2}{2}, \frac{x_3}{2}, \frac{x_4}{2}, \ldots\big)$$

has a unique fixed point $(0,0,0,\ldots)$. Then,

$$\begin{aligned}
T^2(x_1,x_2,x_3,x_4\ldots) &= \big(0,0,\frac{x_1}{2},\frac{x_2}{2^2},\frac{x_3}{2^2},\frac{x_4}{2^2},\ldots\big),\\
T^3(x_1,x_2,x_3,x_4,\ldots) &= \big(0,0,0,\frac{x_1}{2^2},\frac{x_2}{2^3},\frac{x_3}{2^3},\frac{x_4}{2^3},\ldots\big),\\
&\;\;\vdots\\
T^n(x_1,x_2,x_3,x_4\ldots) &= \big(\underbrace{0,\ldots,0}_{n},\frac{x_1}{2^{n-1}},\frac{x_2}{2^n},\frac{x_3}{2^n},\frac{x_4}{2^n},\ldots\big).
\end{aligned}$$

Further, for fixed $x \in l^p$ and any $y \in l^p$, we have

$$\begin{aligned}
d(T^n x, T^n y) &= \left(\frac{|x_1-y_1|^p}{2^{p(n-1)}} + \frac{|x_2-y_2|^p}{2^{pn}} + \frac{|x_3-y_3|^p}{2^{pn}} + \cdots\right)^{\frac{1}{p}}\\
&\leq \left[\frac{1}{2^{p(n-1)}}\left(|x_1-y_1|^p + |x_2-y_2|^p + |x_3-y_3|^p + \cdots\right)\right]^{\frac{1}{p}}\\
&\leq \frac{1}{2^{n-1}} d(x,y).
\end{aligned}$$

Hence, for any fixed $\lambda \in [0,1)$ and every $x \in l^p$ there exists $k(x) \in \mathbb{N}$ such that for every $y \in l^p$

$$d\left(T^{k(x)}x, T^{k(x)}y\right) \leq \lambda d\,(x,y)\,.$$

On the other hand, $T$ is not a contraction. For $x = (1,0,0,\ldots)$ and $y = (2,0,0,\ldots)$, we have $Tx = (0,1,0,0,\ldots)$, $Ty = (0,2,0,0,\ldots)$, $d(x,y) = 1$, $d(Tx,Ty) = 1$. So, $d(Tx,Ty) \leq \lambda d(x,y)$ implies $\lambda \geq 1$.

From Theorem 2.2 we obtain the following variant of Banach and theorem in rectangular $b$-metric spaces.

COROLLARY 2.4. *Let $(X,d)$ be a complete rectangular $b$-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying $d(Tx,Ty) \leq \alpha d(x,y)$, for all $x,y \in X$, where $\alpha \in [0,1)$. Then $T$ has a unique fixed point.*

## 3. Sehgal-Guseman theorem in $b$-metric spaces

LEMMA 3.1. *If $(X,d)$ is a $b$-metric space with coefficient $s$, then $(X,d)$ is a rectangular $b$-metric space with coefficient $s^2$.*

*Proof.* Let $(X,d)$ be a $b$-metric space with coefficient $s$. Let $u$ and $v$ be distinct points such that $u, v \in X \backslash \{x, y\}$. Then we have

$$\begin{aligned} d(x,y) &\leq s[d(x,u) + d(u,y)] \\ &\leq s[d(x,u) + s[d(u,v) + d(v,y)]] \leq s^2[d(x,u) + d(u,v) + d(v,y)]. \end{aligned}$$

So, $(X, d)$ is a rectangular $b$-metric space with coefficient $s^2$. □

From Lemma 3.1 and Theorem 2.2 we obtain the next result in $b$-metric space.

Theorem 3.2. *Let $(X, d, s)$ be a complete $b$-metric space and $T : X \to X$ a mapping satisfying the condition: for each $x \in X$ there exists $k(x) \in \mathbb{N}$ such that*

$$d\left(T^{k(x)}x, T^{k(x)}y\right) \leq \lambda d(x,y),$$

*for all $y \in X$, where $\lambda \in (0,1)$. Then $T$ has a unique fixed point, say $u \in X$, and $T^n x \to u$ for each $x \in X$.*

Note that, from Theorem 3.2. we obtain the Banach contraction principle in $b$-metric spaces.

Theorem 3.3. *[3, Theorem 2.1] Let $(X, d, s)$ be a complete $b$-metric space and let $T : X \to X$ be a map such that for all $x, y \in X$ and some $\lambda \in [0, 1)$,*

$$d(Tx, Ty) \leq \lambda d(x, y).$$

*Then $T$ has a unique fixed point $u$ and $\lim_{n \to \infty} T^n x = u$ for all $x \in X$.*

Remark 3.4. Corollary 2.4 provides a complete solution to an open problem 1 raised by George, Radenović, Reshma and Shukla in [4].

References

[1] I. A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., **30** (1989), 26–37.

[2] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., **1** (1993), 5–11.

[3] N. V. Dung, V. T. L. Hang, *On relaxations of contraction constants and Caristi's theorem in b-metric spaces*, J. Fixed Point Theory Appl. **18**, (2016), 267–284.

[4] R. George, S. Radenović, K. P. Reshma, S. Shukla, *Rectangular b-metric space and contraction principles*, J. Nonlinear Sci. Appl. 8 (2015), 1005–1013.

[5] L. F. Guseman, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Am. Math. Soc. **26**, (1970), 615–618.

[6] Z. Mustafa, V. Parvaneh, J. R. Roshan, Z. Kadelburg, *b2-Metric spaces and some fixed point theorems*, Fixed Point Theory Appl. 2014,2014:144.

[7] J. R. Roshan, V. Parvaneh, Z. Kadelburg, N. Hussain, *New fixed point results in b-rectangular metric spaces*, Nonlinear Anal., Model. Control, **21**, No. 5, (2016), 614–634.

[8] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. **23** (1969), 631–634.

(received 09.04.2017; in revised form 13.07.2017; available online 10.11.2017)

University of Banja Luka, Faculty of Electrical Engineering, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina

*E-mail*: zmitrovic@etfbl.net, zoran.mitrovic@etf.unibl.org