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ON VIABILITY RESULT FOR FIRST-ORDER FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. We prove the existence of solutions, in separable Banach spaces, for the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, T(t)x), & \text{a.e. on } [0,\tau]; \\ x(s) = \varphi(s), & \forall s \in [-a, 0]; \\ x(t) \in C(t), & \forall t \in [0,\tau]; \end{cases}$$

We consider weaker hypotheses on the constraint.

1. Introduction

Let E be a separable Banach space with the norm $\|\cdot\|$. For I a segment in $\mathbb{R}$, we denote by $\mathcal{C}(I, E)$ the Banach space of continuous functions from I to E equipped with the norm $\|x(\cdot)\|_\infty := \sup\{\|x(t)\| \ ; \ t \in I\}$. For a positive number a, we put $\mathcal{C}_a := \mathcal{C}([-a, 0], E)$ and for any $t \in [0, \tau]$, $\tau > 0$, we define the operator $T(t)$ from $\mathcal{C}([-a, \tau], E)$ to $\mathcal{C}_a$ with $(T(t)(x(.)))(s) := (T(t)x)(s) := x(t + s)$, $s \in [-a, 0]$.

The goal of this paper is to prove the existence of solutions to the following functional differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, T(t)x), & \text{a.e. on } [0,\tau]; \\ x(s) = \varphi(s), & \forall s \in [-a, 0]; \\ x(t) \in C(t), & \forall t \in [0,\tau]; \end{cases} \tag{1}$$

where F is a closed-valued multifunction, measurable with respect to the first argument and Lipschitz continuous with respect to the second argument, C is a set-valued map and φ is a given function in $\mathcal{C}_a$.

In [5,6], Haddad first studied functional differential inclusions when the right-hand side is upper semicontinuous with convex and compact values. However, the space of

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state constraints is finite dimensional. The infinite dimensional case was studied by many authors under convex assumption on the set-valued map. In this context, we refer to Syam [9], Gavioli and Malaguti [4] and the reference therein.

For the nonconvex case in separable Banach space Duc Ha has established in [3] the existence of viable solutions to (1) regardless of whether C is fixed and F is a closed-valued multifunction, integrably bounded, measurable with respect to the first argument and Lipschitz continuous with respect to the second argument. The author has established a multi-valued version of Larrieu's work [7]. Lupulescu and Necula [8] have extended Duc Ha's work to functional differential inclusions, but under the same hypotheses on F with C always fixed. They used the same kind of tangential condition. In [1], we extended results which are presented in [3,8]. Indeed, we have got an existence result, in a separable Banach space, for first-order functional differential inclusions, under the same hypotheses on F. The set-valued map $C : [-1, 1] \to 2^E$ is lower semicontinuous with compact graph. The tangency condition is weaker than the one used in [3, 8].

This work extends the last result in [1]. Indeed, we consider weaker growth condition for the right hand side and we suppose simply that the graph of $C : [0, 1] \to 2^E$ is closed. Moreover, in this paper, we use another argument based on Brezis-Browder Theorem.

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, we prove the existence of solutions for (1).

2. Preliminaries and statement of the main result

For measurability purpose, E (resp. $\Omega \subset E$) is endowed with the σ-algebra $B(E)$ (resp. $B(\Omega)$) of Borel subsets for the strong topology and $[0, 1]$ is endowed with Lebesgue measure and the σ-algebra of Lebesgue measurable subsets. For $x \in E$ and $r > 0$ let $B(x, r) := \{y \in E \; ; \; \|y - x\| < r\}$ be the open ball centered at x with radius r and $\overline{B}(x, r)$ be its closure and put $B = B(0, 1)$. For $x \in E$ and for nonempty subsets A, B of E we denote $d_A(x)$ or $d(x, A)$ the real $\inf\{\|y - x\| \; ; \; y \in A\}$, $e(A, B) := \sup\{d_B(x); x \in A\}$ and $H(A, B) = \max\{e(A, B), e(B, A)\}$. A multifunction is said to be measurable if its graph is measurable.

Let us recall the following lemmas that will be used in the sequel.

Lemma 2.1. *([11]) Let Ω be a nonempty set in E. Assume that $F : [a, b] \times \Omega \to 2^E$ is a multifunction with nonempty closed values satisfying:*

- *For every $x \in \Omega$, $F(\cdot, x)$ is measurable on $[a, b]$;*
- *For every $t \in [a, b]$, $F(t, \cdot)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x(\cdot) : [a, b] \to \Omega$, the multifunction $F(\cdot, x(\cdot))$ is measurable on $[a, b]$.

Lemma 2.2. *([11]) Let $G : [a,b] \to 2^E$ be a measurable multifunction and $y(\cdot) : [a,b] \to E$ a measurable function. Then for any positive measurable function $r(\cdot) : [a,b] \to \mathbb{R}^+$, there exists a measurable selection $g(\cdot)$ of G such that for almost all $t \in [a,b]$ $\|g(t) - y(t)\| \leq d\big(y(t), G(t)\big) + r(t)$.*

Lemma 2.3. *([2]) Let $\preceq$ be a given preorder on the nonempty set $\mathcal{B}$ and let $\phi : \mathcal{B} \to \mathbb{R} \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in $\mathcal{B}$ is majorated in $\mathcal{B}$. Then, for each $x_0 \in \mathcal{B}$, there exists $x_1 \in \mathcal{B}$ such that $x_0 \preceq x_1$ and $\phi(x_1) = \phi(x)$ if $x_1 \preceq x$.*

The above function ϕ is supposed to be finite and bounded from above in [2], but this restriction can be removed by replacing ϕ by the function $x \mapsto \arctan \phi(x)$ (see [10]).

For given measurable functions $v(\cdot) : [0,1] \to E$ and $\rho(\cdot) : [0,1] \to \mathbb{R}^+$, we need the following notation

$$\begin{aligned} S_{v,\rho}(\psi) := \big\{ & f \in L^1([0,1], E) : f(s) \in F(s,\psi) \text{ and} \\ & \|f(s) - v(s)\| \leq d(v(s), F(s,\psi)) + \rho(s) \text{ for all } s \in [0,1] \big\}, \end{aligned}$$

where $\psi \in \mathcal{C}_a$.

We shall use the following hypotheses throughout this paper.

(H1) $C : [0,1] \to 2^E$ is a set-valued map with closed graph and $\mathcal{K} : [0,1] \to \mathcal{C}_a$ is a set-valued map defined by $\mathcal{K}(t) = \{\varphi \in \mathcal{C}_a, \varphi(0) \in C(t)\}$;

(H2) $F : Gr(\mathcal{K}) \to 2^E$ is a set-valued map with nonempty closed values satisfying

- (i) $t \mapsto F(t,\psi)$ is measurable,
- (ii) there exists a function $m(\cdot) \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and $\psi_1, \psi_2 \in \mathcal{K}(t)$
$$H\big(F(t,\psi_1), F(t,\psi_2)\big) \leq m(t)\|\psi_1 - \psi_2\|_\infty,$$
- (iii) There exist $g(\cdot), p(\cdot) \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and $\psi \in \mathcal{K}(t)$
$$\|F(t,\psi)\| := \sup_{y \in F(t,\psi)} \|y\| \leq g(t) + p(t)\|\psi\|_\infty.$$

(H3) (**Tangential condition**) For each measurable function $v(\cdot) : [0,1] \to E$, for all $\rho > 0$, $t \in [0,1]$ and $\psi \in \mathcal{K}(t)$, there exists $f \in S_{v,\rho}(\psi)$ such that
$$\liminf_{h \to 0^+} \frac{1}{h} d\left(\psi(0) + \int_t^{t+h} f(s)ds, C(t+h)\right) = 0.$$

Remark 2.4. If F satisfies the condition (H2), by Lemma 2.1 and Lemma 2.2, the set $S_{v,\rho}(\psi)$ is nonempty.

In the next section, we shall prove the following result.

Theorem 2.5. *If assumptions (H1)–(H3) are satisfied, then there exists $\tau > 0$ such that for all $(x_0, \varphi) \in C(0) \times \mathcal{C}_a$, $\varphi(0) = x_0$, there exists an absolutely continuous function $x(\cdot) : [0,\tau] \to E$ such that $x(\cdot)$ is a solution of (1).*

3. Proof of the main result

Throughout the paper, fix $\varphi \in \mathcal{C}_a$ such that $\varphi(0) = x_0 \in C(0)$. Let $\tau_1, \tau_2, \tau_3 > 0$ be such that

$$\int_0^{\tau_1} m(t)\,dt < 1, \quad \int_0^{\tau_2} g(t)\,dt < 1 \text{ and } \int_0^{\tau_3} p(t)\,dt < \frac{1}{2}. \tag{2}$$

Put $\tau = \inf\{\tau_1, \tau_2, \tau_3, 1\}$. For $\varepsilon > 0$ set

$$\eta(\varepsilon) := \sup\Big\{\rho \in]0,\varepsilon] : \Big|\int_{t_1}^{t_2} (g(s) + Mp(s))\,ds\Big| < \varepsilon \text{ and}$$
$$|\varphi(t_1) - \varphi(t_2)| < \varepsilon, \text{ if } |t_1 - t_2| \le \rho\Big\} \tag{3}$$

where $M = 4\|\varphi\|_\infty + 2a + 2$.

For all $0 < \varepsilon < a$ and $v(\cdot) \in L^1([0,1], E)$, set $\mathcal{B}(\varepsilon, v(\cdot))$ for the set of all 4-tuples $(f, x, \theta, u)_d$ where $d \in]0, \tau]$, $f(\cdot), u(\cdot) \in L^1([0,d], E)$, $x(\cdot) : [-a, d] \to E$ is a continuous mapping and $\theta(\cdot) : [0,d] \to [0,d]$ is a step function such that

(i) $x(t) = x_0 + \int_0^t (u(s) + f(s))\,ds$ for all $t \in [0,d]$;

(ii) $f(t) \in F(t, T(\theta(t))x)$, $u(t) \in \varepsilon B$, $0 \le t - \theta(t) \le \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $x(\theta(t)) \in C(\theta(t))$ for all $t \in [0, d[$;

(iii) $x(d) \in C(d)$;

(iv) $\|f(t) - v(t)\| \le d\big(v(t), F(t, T(\theta(t))x)\big) + \varepsilon$ for all $t \in [0, d[$;

(v) $\big\|x(t) - x_0 - \int_0^t f(\tau) d\tau\big\| \le \varepsilon t$ for all $t \in [0, d]$.

Proposition 3.1. *If the assumptions (H1)–(H3) are satisfied, then for all $0 < \varepsilon < a$, and $v(\cdot) \in L^1([0,1], E)$, there exists at least one $(f, x, \theta, u)_\tau \in \mathcal{B}(\varepsilon, v(\cdot))$.*

Proof. Let $0 < \varepsilon < a$ and $v(\cdot) \in L^1([0,1], E)$ be fixed. Put $x(t) = \varphi(t)$, $\forall t \in [-a, 0]$. By the tangential condition, there exist $f_0 \in S_{v,\varepsilon}(T(0)x)$ and $h_0 \in]0, \inf\{\tau, \frac{1}{4}\eta(\frac{\varepsilon}{4})\}]$, such that

$$\frac{1}{h_0} d\Big(x_0 + \int_0^{h_0} f_0(s)\,ds, C(h_0)\Big) \le \frac{\varepsilon}{2}.$$

Then there exists $x_1 \in C(h_0)$ such that

$$\frac{1}{h_0}\Big\|x_1 - x_0 - \int_0^{h_0} f_0(s)\,ds\Big\| \le \varepsilon.$$

Set
$$u_0 = \frac{1}{h_0}\Big(x_1 - x_0 - \int_0^{h_0} f_0(s)\,ds\Big).$$

Hence, we get $x_1 = x_0 + h_0 u_0 + \int_0^{h_0} f_0(s)\,ds$. We take $d_0 = h_0$, $u_0(s) = u_0$ and $x_0(t) = x_0 + \int_0^t (u_0(s) + f_0(s))\,ds$, $\forall t \in [0, d_0]$. Then one has, for all $t \in [0, d_0]$,

$$\Big\|x_0(t) - x_0 - \int_0^t f_0(s)\,ds\Big\| = \Big\|\int_0^t u_0(s)\,ds\Big\| \le \varepsilon t.$$

Set $\theta_0(t) = 0$ for all $t \in [0, d_0]$. It is clear that $(f_0, x_0, \theta_0, u_0)_{d_0} \in \mathcal{B}(\varepsilon, v(\cdot))$. Thus $\mathcal{B}(\varepsilon, v(\cdot)) \neq \emptyset$. Now, consider the following preorder:

$$\begin{aligned} &(f_1, x_1, \theta_1, u_1)_{d_1} \preceq (f_2, x_2, \theta_2, u_2)_{d_2} \\ \Leftrightarrow\ & d_1 \leq d_2,\ f_1 = f_2|_{[0,d_1]},\ x_1 = x_2|_{[0,d_1]},\ \theta_1 = \theta_2|_{[0,d_1]},\ u_1 = u_2|_{[0,d_1]} \end{aligned}$$

and let $\phi : \mathcal{B}(\varepsilon, v(\cdot)) \to \mathbb{R}$ be the function defined by $\phi((f, x, \theta, u)_d) = d$ for all $(f, x, \theta, u)_d \in \mathcal{B}(\varepsilon, v(\cdot))$. We remark that ϕ is increasing on $\mathcal{B}(\varepsilon, v(\cdot))$.

Now, if $\big((f_i, x_i, \theta_i, u_i)_{d_i}\big)_{i\in\mathbb{N}}$ is an increasing sequence in $\mathcal{B}(\varepsilon, v(\cdot))$, we construct a majorant of $\Big((f_i, x_i, \theta_i, u_i)_{d_i}\Big)_{i\in\mathbb{N}}$ as follows:

$$d = \lim_i d_i,\ f(t) = f_i(t),\ \theta(t) = \theta_i(t),\ u(t) = u_i(t),\ \forall t \in [0, d_i]$$

and
$$x(t) = x_0 + \int_0^t (u(s) + f(s))\, ds,\ \forall t \in [0, d].$$

We claim that $(f, x, \theta, u)_d \in \mathcal{B}(\varepsilon, v(\cdot))$. Indeed, for all $i \in \mathbb{N}$, we have $x(d_i) = x_i(d_i) \in C(d_i)$. Since the graph of C is closed, we get $x(d) \in C(d)$. The other assertions are obvious.

Next, for applying Lemma 2.3, we need the following proposition.

Proposition 3.2. *For all $(f, x, \theta, u)_d \in \mathcal{B}(\varepsilon, v(\cdot))$ with $d < \tau$, there exists $(\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}} \in \mathcal{B}(\varepsilon, v(\cdot))$ such that $(f, x, \theta, u)_d \preceq (\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}}$ and $\phi((f, x, \theta, u)_d) < \phi((\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}})$.*

Proof. Let $(f, x, \theta, u)_d \in \mathcal{B}(\varepsilon, v(\cdot))$ with $d < \tau$. For $x(d) \in C(d)$, by the tangential condition, there exist $\tilde{f} \in S_{v,\varepsilon}(T(d)x)$ and $h \in]0, \inf\{\tau - d, \frac{1}{4}\eta(\frac{\varepsilon}{4})\}]$, such that

$$\frac{1}{h} d\bigg(x(d) + \int_d^{d+h} \tilde{f}(s)\, ds, C(d+h)\bigg) \leq \frac{\varepsilon}{2}.$$

Then there exists $x_1 \in C(d+h)$ such that

$$\frac{1}{h}\bigg\|x_1 - x(d) - \int_d^{d+h} \tilde{f}(s)\, ds\bigg\| \leq \varepsilon.$$

Put
$$u_1 = \frac{1}{h}\bigg(x_1 - x(d) - \int_d^{d+h} \tilde{f}(s)\, ds\bigg).$$

Then, we have $x_1 = x(d) + hu_1 + \int_d^{d+h} \tilde{f}(s)\, ds$. Next, set $\bar{d} = d + h$, $\tilde{x}(t) = x(d) + (t - d)u_1 + \int_d^t \tilde{f}(s)\, ds$, $\tilde{u}(t) = u_1$ and $\tilde{\theta}(t) = d$ for all $t \in [d, \bar{d}]$. We define $\bar{f}$, $\bar{x}$ and $\bar{\theta}$ as follows:

$$\bar{f}(t) = f(t),\ \bar{x}(t) = x(t),\ \bar{\theta}(t) = \theta(t),\ \bar{u}(t) = u(t),\quad \text{for all } t \in [0, d]$$

and
$$\bar{f}(t) = \tilde{f}(t),\ \bar{x}(t) = \tilde{x}(t),\ \bar{\theta}(t) = \tilde{\theta}(t),\ \bar{u}(t) = \tilde{u}(t),\quad \text{for all } t \in]d, \bar{d}].$$

We can easily show that, for all $t \in [0, \bar{d}]$,

$$\bar{x}(t) = x_0 + \int_0^t (\bar{u}(s) + \bar{f}(s))\, ds.$$

Then for all $t \in [0, \bar{d}]$,

$$\left\| \bar{x}(t) - x_0 - \int_0^t \bar{f}(s)\, ds \right\| \leq \varepsilon t.$$

Finally, we conclude that $(\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}} \in \mathcal{B}(\varepsilon, v(\cdot))$, $(f, x, \theta, u)_d \preceq (\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}}$ and $\phi((f, x, \theta, u)_d) < \phi((\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}})$. □

Now, we are ready to complete the proof of Proposition 3.1. From Lemma 2.3, there exists $(f, x, \theta, u)_d \in \mathcal{B}(\varepsilon, v(\cdot))$ such that $\phi((f, x, \theta, u)_d) = \phi((\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}})$ and $(f, x, \theta, u)_d \preceq (\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}}$ for all $(\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}} \in \mathcal{B}(\varepsilon, v(\cdot))$. Moreover, if $\phi((f, x, \theta, u)_d) < \tau$, by the Proposition 3.2, there exists $(\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}} \in \mathcal{B}(\varepsilon, v(\cdot))$ such that $(f, x, \theta, u)_d \preceq (\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}}$ and $\phi((f, x, \theta, u)_d) < \phi((\bar{f}, \bar{x}, \bar{\theta}, \bar{u})_{\bar{d}})$. Hence $\phi((f, x, \theta, u)_d) = \tau$. □

Now, we are prepared to prove our Theorem 2.5. Let $(\varepsilon_n)_{n\geq 1}$ be a strictly decreasing sequence of positive scalars such that $0 < \varepsilon_n < a$, $n \geq 1$, and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In view of Proposition 3.1, we can define inductively sequences $(f_n(\cdot))_{n\geq 1} \subset L^1([0, \tau], E)$, $(x_n(\cdot))_{n\geq 1} \subset \mathcal{C}([-a, \tau], E)$, and $(\theta_n(\cdot))_{n\geq 1}, \subset S([0, \tau], [0, \tau])$, where $S([0, \tau], [0, \tau])$ denotes the space of step functions from $[0, \tau]$ into $[0, \tau]$ such that

(A1) $x_n(\cdot) \in \mathcal{C}^1([0, \tau]), f_n(t) \in F(t, T(\theta_n(t))x_n), x_n(\theta_n(t)) \in C(\theta_n(t)), 0 \leq t - \theta_n(t) \leq \frac{1}{4}\eta(\frac{\varepsilon_n}{4})$ for all $t \in [0, \tau[$ and $x_n \equiv \varphi$ on $[-a, 0]$;

(A2) $x_n(0) = x_0$ and $x_n(\tau) \in C(\tau)$;

(A3) $\|f_{n+1}(t) - f_n(t)\| \leq d\big(f_n(t), F(t, T(\theta_{n+1}(t))x_{n+1}\big) + \varepsilon_{n+1}$ for all $t \in [0, \tau[$;

(A4) $\left\| x_n(t) - x_0 - \int_0^t f_n(\tau) d\tau \right\| \leq \varepsilon_n t$ for all $t \in [0, \tau]$.

In the sequel, we need the following propositions.

Proposition 3.3. *For all $n \in \mathbb{N}^*$, we have $\|x_n\|_\infty \leq M$.*

Proof. By (A4), for $t \in [0, \tau]$, we have

$$\|x_n(t)\| \leq \|x_0\| + a + \int_0^t \|f_n(s)\|\, ds \leq \|x_0\| + a + \int_0^\tau g(s)\, ds + \int_0^\tau p(s)\|T(\theta_n(s))x_n\|_\infty\, ds.$$

Since

$$\begin{aligned}\|T(\theta_n(s))x_n\|_\infty &= \sup_{-a\leq t\leq 0} \|x_n(\theta_n(s) + t)\| \leq \sup_{-a\leq t\leq \tau} \|x_n(t)\| \\ &\leq \sup_{-a\leq t\leq 0} \|x_n(t)\| + \sup_{0\leq t\leq \tau} \|x_n(t)\| \leq \|\varphi\|_\infty + \sup_{0\leq t\leq \tau} \|x_n(t)\|,\end{aligned}$$

we get

$$\sup_{0\leq t\leq \tau} \|x_n(t)\| \leq \|x_0\| + a + \int_0^\tau (g(s) + p(s)\|\varphi\|_\infty)\, ds + \sup_{0\leq t\leq \tau} \|x_n(t)\| \int_0^\tau p(s)\, ds,$$

hence

$$\begin{aligned}\sup_{0\leq t\leq \tau} \|x_n(t)\| &\leq \frac{1}{1 - \int_0^\tau p(s)\, ds}\Big(\|x_0\| + a + \int_0^\tau (g(s) + p(s)\|\varphi\|_\infty)\, ds\Big) \\ &\leq 2\Big(\|\varphi\|_\infty + a + 1 + \|\varphi\|_\infty\Big) = M.\end{aligned}$$

Consequently, we obtain $\|x_n\|_\infty = \sup_{-a\le t\le\tau} \|x_n(t)\| \le M$. $\square$

PROPOSITION 3.4. *For all $n \in \mathbb{N}^*$ and $t \in [0,\tau]$, we have $\|f_n(t)\| \le g(t) + Mp(t)$.*

Proof. Let $t \in [0,\tau]$. Since $f_n(t) \in F(t, T(\theta_n(t))x_n)$, by (H2) and the above proposition, we have

$$\|f_n(t)\| \le g(t) + p(t)\|T(\theta_n(t))x_n\|_\infty \le g(t) + p(t)\|x_n\|_\infty \le g(t) + Mp(t). \qquad \square$$

PROPOSITION 3.5. *For all $n \in \mathbb{N}^*$ and $t \in [0,\tau]$, we have*

$$\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty \le\| x_n - x_{n+1} \|_\infty + \frac{10\varepsilon_n}{4}.$$

Proof. We have

$$\begin{aligned}
&\|(T(\theta_n(t))x_n)(s) - (T(\theta_{n+1}(t))x_{n+1})(s)\| = \|x_n(\theta_n(t)+s) - x_{n+1}(\theta_{n+1}(t)+s)\| \\
&\quad \le \|x_n(\theta_n(t)+s) - x_{n+1}(\theta_n(t)+s)\| + \|x_{n+1}(\theta_n(t)+s) - x_{n+1}(\theta_{n+1}(t)+s)\| \\
&\quad \le \|x_n - x_{n+1}\|_\infty + \|x_{n+1}(\theta_n(t)+s) - x_{n+1}(\theta_{n+1}(t)+s)\|.
\end{aligned}$$

Then $\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty$

$$\le \sup_{s\in[-a,0]} \|x_{n+1}(\theta_n(t)+s) - x_{n+1}(\theta_{n+1}(t)+s)\| + \|x_n - x_{n+1}\|_\infty.$$

Let us denote the modulus of continuity of a function ψ defined on interval I of $\mathbb{R}$ by

$$\omega(\psi, I, \varepsilon) := \sup\big\{\|\psi(t) - \psi(s)\| \;;\; s,t \in I, \mid s-t \mid< \varepsilon\big\}, \varepsilon > 0.$$

Since $t - \theta_n(t) < \frac{1}{2}\eta(\frac{\varepsilon_n}{4})$, $t - \theta_{n+1}(t) < \frac{1}{2}\eta(\frac{\varepsilon_{n+1}}{4})$ and $\eta(\frac{\varepsilon_{n+1}}{4}) \le \eta(\frac{\varepsilon_n}{4})$, it follows $|\theta_{n+1}(t) - \theta_n(t)| < \eta(\frac{\varepsilon_n}{4})$. Then we have

$$\begin{aligned}
\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty &\le \|x_n - x_{n+1}\|_\infty + \omega(x_{n+1}, [-a,\tau], \eta(\frac{\varepsilon_n}{4})) \\
&\le \|x_n - x_{n+1}\|_\infty + \omega(\varphi, [-a,0], \eta(\frac{\varepsilon_n}{4})) + \omega(x_{n+1}, [0,\tau], \eta(\frac{\varepsilon_n}{4})).
\end{aligned}$$

Now, let $t, t' \in [0,\tau]$ such that $0 \le t - t' < \eta(\frac{\varepsilon_n}{4})$. One has

$$\begin{aligned}
\|x_{n+1}(t) - x_{n+1}(t')\| \le& \Big\|x_{n+1}(t) - x_0 - \int_0^t f_{n+1}(s)\,ds\Big\| \\
&+ \Big\|x_{n+1}(t') - x_0 - \int_0^{t'} f_{n+1}(s)\,ds\Big\| + \int_{t'}^t \|f_{n+1}(s)\|\,ds \\
\le& \varepsilon_{n+1}t + \varepsilon_{n+1}t' + \int_{t'}^t (g(s) + Mp(s))\,ds \le 2\varepsilon_{n+1} + \frac{\varepsilon_n}{4} \le \frac{9\varepsilon_n}{4}.
\end{aligned}$$

So

$$\omega(x_{n+1}, [0,\tau], \eta(\frac{\varepsilon_n}{4})) \le \frac{9\varepsilon_n}{4}. \tag{4}$$

Also, for $t, t' \in [-a,0]$ such that $|t' - t \mid < \eta(\frac{\varepsilon_n}{4})$, we get $\|\varphi(t) - \varphi(t')\| < \frac{\varepsilon_n}{4}$. Then $\omega(\varphi, [-a,0], \eta(\frac{\varepsilon_n}{4})) \le \frac{\varepsilon_n}{4}$. Consequently, we have

$$\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty \le \|x_n - x_{n+1}\|_\infty + \frac{10\varepsilon_n}{4}. \qquad \square$$

From (A1), (A3) and Proposition 3.5, we deduce for all $t \in [0, \tau[$

$$\begin{aligned}\|f_{n+1}(t) - f_n(t)\| &\leq H\Big(F(t, T(\theta_n(t))x_n), F(t, T(\theta_{n+1}(t))x_{n+1})\Big) + \varepsilon_{n+1} \\ &\leq m(t)\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty + \varepsilon_{n+1} \\ &\leq m(t)\Big(\|x_n - x_{n+1}\|_\infty + \frac{10\varepsilon_n}{4}\Big) + \varepsilon_{n+1}. \end{aligned} \tag{5}$$

Now, relations (2) and (A4) yield for all $t \in [0, \tau]$,

$$\begin{aligned}\|x_{n+1}(t) - x_n(t)\| \leq & \left\|x_{n+1}(t) - x_0 - \int_0^t f_{n+1}(s)\,ds\right\| \\ & + \int_0^t \|f_{n+1}(s) - f_n(s)\|\,ds + \left\|x_n(t) - x_0 - \int_0^t f_n(s)\,ds\right\| \\ \leq & \varepsilon_{n+1}t + \varepsilon_n t + \|x_n(\cdot) - x_{n+1}(\cdot)\|_\infty \int_0^t m(s)\,ds + \frac{10\varepsilon_n}{4}\int_0^t m(s)\,ds + t\varepsilon_{n+1} \\ \leq & 2\varepsilon_n t + \|x_n(\cdot) - x_{n+1}(\cdot)\|_\infty \int_0^\tau m(s)\,ds + \frac{10\varepsilon_n}{4} + t\varepsilon_n \\ \leq & \frac{11\varepsilon_n}{2} + \|x_n(\cdot) - x_{n+1}(\cdot)\|_\infty \int_0^\tau m(s)\,ds. \end{aligned}$$

Thus,

$$\|x_n(\cdot) - x_{n+1}(\cdot)\|_\infty \quad \leq \quad \frac{11\varepsilon_n}{2(1-L)} \tag{6}$$

where $L = \int_0^\tau m(s)\,ds$. Therefore we have, $\|x_m(\cdot) - x_n(\cdot)\|_\infty \leq \frac{11}{2(1-L)}\sum_{i=n}^{m-1}\varepsilon_i$, for $n < m$. So the sequence $\{x_n(\cdot)\}_{n=1}^\infty$ is a Cauchy sequence, hence it converges uniformly on $[0, \tau]$ to a function $x(\cdot)$. Since all functions $x_n(\cdot)$ agree with φ on $[-a, 0]$, we can obviously say that $x_n(\cdot)$ converges uniformly to $x(\cdot)$ on $[-a, \tau]$, if we extend $x(\cdot)$ in such a way that $x(\cdot) \equiv \varphi$ on $[-a, 0]$. Also, by (4) and the following inequality

$$\|x_n(\theta_n(t)) - x(t)\| \leq \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\|,$$

we deduce that $x_n(\theta_n(\cdot))$ converges uniformly to $x(\cdot)$ on $[0, \tau]$. By construction, we have $x_n(\theta_n(t)) \in C(\theta_n(t))$ for every $t \in [0, \tau]$, and since the graph of C is closed, we get $x(t) \in C(t)$ for all $t \in [0, \tau]$. In addition, by (3) and (4), we have

$$\begin{aligned}\|T(\theta_n(t))x_n - T(t)x_n\|_\infty &= \sup_{-a\leq s\leq 0}\|x_n(\theta_n(t)+s) - x_n(t+s)\| \leq \omega(x_n, [-a, \tau], \eta(\frac{\varepsilon_n}{4})) \\ &\leq \omega(\varphi, [-a, 0], \eta(\frac{\varepsilon_n}{4})) + \omega(x_n, [0, \tau], \eta(\frac{\varepsilon_n}{4})) \leq \frac{\varepsilon_n}{4} + \frac{9\varepsilon_n}{4} = \frac{10\varepsilon_n}{2}\end{aligned}$$

hence $\|T(\theta_n(t))x_n - T(t)x_n\|_\infty$ converges to 0 as $n \to \infty$. Therefore, since the uniform convergence of x_n to x on $[-a, \tau]$ implies that $T(t)x_n$ converges to $T(t)x$ uniformly on $[-a, 0]$, we deduce that

$$T(\theta_n(t))x_n \text{ converges to } T(t)x \text{ in } \mathcal{C}_a. \tag{7}$$

Now, we return to the relation (5). By the relation (6) we get

$$\|f_{n+1}(t) - f_n(t)\| \leq \Big(m(t)\Big(\frac{5}{2} + \frac{11}{2(1-L)}\Big) + 1\Big)\varepsilon_n.$$

This implies (as above) that $\{f_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence and $(f_n(t))_n$ converges to $f(t)$. Further, since $\|f_n(t)\| \leq g(t) + Mp(t)$, by (A4) and by the dominated convergence theorem $x(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}(x_0 + \int_0^t f_n(s)\,ds) = x_0 + \int_0^t f(s)\,ds$. Hence $\dot{x}(t) = f(t)$. Finally, observe that by (A1),

$$\begin{aligned} d(f(t), F(t, T(t)x)) \leq & \|f(t) - f_n(t)\| + H\bigg(F(t, T(\theta_n(t))x_n)), F(t, T(t)x)\bigg) \\ \leq & \|f(t) - f_n(t)\| + m(t)\|T(\theta_n(t))x_n - T(t)x\|_\infty. \end{aligned}$$

Since $f_n(t)$ converges to $f(t)$ and by (7) the last term converges to 0. So that $\dot{x}(t) = f(t) \in F(t, T(t)x)$ a.e on $[0, \tau]$. The proof is complete.

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