MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
71, 1–2 (2019), 104–122
March 2019

research paper
оригинални научни рад

NOTE ON COMBINATORIAL STRUCTURE OF SELF-DUAL SIMPLICIAL COMPLEXES

Marinko Timotijević

Abstract. Simplicial complexes K, in relation to their Alexander dual $\widehat{K}$, can be classified as self-dual ($K = \widehat{K}$), sub-dual ($K \subseteq \widehat{K}$), super-dual ($K \supseteq \widehat{K}$), or transcendent (neither sub-dual nor super-dual). We explore a connection between sub-dual and self-dual complexes providing a new insight into combinatorial structure of self-dual complexes. The *root operator* (Definition 4.3) in Section 4 associates with each self-dual complex K a sub-dual complex $\sqrt{K}$ on a smaller number of vertices. We study the operation of *minimal restructuring* of self-dual complexes and the properties of the associated *neighborhood graph*, defined on the set of all self-dual complexes. Some of the operations and relations, introduced in the paper, were originally developed as a tool for computer-based experiments and enumeration of self-dual complexes.

1. Introduction

The terminology used in this paper is mostly standard and the reader is referred to [16] for all undefined concepts.
We emphasize that, for a given simplicial complex $K \subseteq 2^S$, the set of vertices $Vert(K) = \{v \in S \mid \{v\} \in K\}$ of K can in general be a proper subset of the ambient set S. This distinction is important as visible already in the definition of the Alexander dual of a simplicial complex.

Definition 1.1. The Alexander dual (or simply the dual) of a complex $K \subseteq 2^S$ is the simplicial complex $\widehat{K} \subseteq 2^S$ given by $\widehat{K} = \{S \backslash A \mid A \notin K\}$.

When we want to emphasize the ambient set S, the Alexander dual of the complex K is denoted by $\widehat{K}^S$.
Using the relation "being a subcomplex" we can classify all simplicial complexes in the given ambient in the following way.

2010 Mathematics Subject Classification: 55M05, 05A15, 05E45, 55U10
Keywords and phrases: Alexander dual; self-dual complexes; triangulations; combinatorial classification.

Definition 1.2. Let $K \subseteq 2^S$ be a simplicial complex. We say that the complex K is:
(i) **sub-dual** in the ambient S if $K \subseteq \widehat{K}^S$;

(ii) **super-dual** in the ambient S if $\widehat{K}^S \subseteq K$;

(iii) **self-dual** in the ambient S if $K = \widehat{K}^S$;

(iv) **transcendent** in the ambient S if K and $\widehat{K}^S$ are not comparable.

Self-dual simplical complexes appear in many different branches of mathematics and provide a link between areas as distant as algebraic topology, game theory, hypergraph theory, and combinatorial optimization, to name just a few.
For illustration, the *"Bottleneck theorem"* of Edmonds and Fulkerson [8] is a classical result of optimization theory which implies that for each self-dual complex $K \subset 2^{[n]}$ and each function $f : [n] \to \mathbb{R}$,

$$\max_{A \in \mathcal{C}} \min_{x \in A} \ f(x) = \min_{B \in \mathcal{C}} \max_{y \in B} \ f(y),$$

where $\mathcal{C}$ is the collection of complements P^c of all maximal simplices P in K.
In combinatorial (algebraic) topology, self-dual simplicial complexes provide fundamental examples of triangulated geometrical objects which are not embeddable in an Euclidean space of prescribed dimension. More explicitly, see [16, Section 5], self-dual complexes on n vertices cannot be embedded in an Euclidean space of dimension $n-3$. Moreover, as demonstrated by S. A. Melikhov [17, Theorem 1.12,3], self-dual complexes are subset-minimal examples of simplicial complexes which are not embeddable in $\mathbb{R}^{n-3}$ in the sense that every proper subcomplex of a self-dual subcomplex of $2^{[n]}$ can be embedded in $\mathbb{R}^{n-3}$.

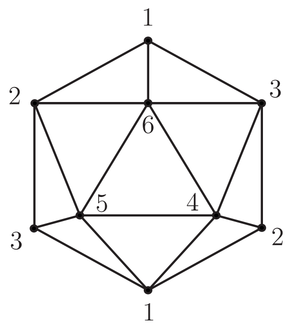

Figure 1: The hemi-icosahedron.

Notable examples of self-dual complexes are the 6-vertex triangulation of the real projective plane (hemi-icosahedron, exhibited in Figure 1) and the unique 9-vertex triangulation of the complex projective plane, see [1, 2, 9, 10].
Motivated in part by algorithmic, computational, and enumerative aspects of the theory of self-dual complexes, we explore their combinatorial structure and study their faithful representation which are as simple as possible. For example we show that each self-dual complex $K \subset 2^{[n]}$ can be uniquely reconstructed from its sub-dual

"root" $\sqrt{K} \subset 2^{[n-1]}$ which will be introduced in Section 4. This in turn allows us a more efficient enumeration and classification of self-dual complexes as illustrated by the following results.

THEOREM 1.3. *The number of self-dual simplicial complexes in the ambient* $[n]$ *is equal to the number of sub-dual complexes in the ambient* $[n-1]$.

THEOREM 1.4. *Self-dual simplicial complexes* K *and* L *in the ambients* S *and* S' *respectively, where* $|S| = |S'|$, *are isomorphic iff there exist vertices* $\{v\} \in K$ *and* $\{w\} \in L$ *such that* $\mathrm{Lk}(\{v\})$ *and* $\mathrm{Lk}(\{w\})$ *are isomorphic.*

In the comment for [17, Theorem 1.11], S. Melikhov observed that new self-dual simplicial complexes can be obtained by exchanging a pair of complementary faces of a given self-dual complex. Using this property, we describe a simple procedure for *minimal restructuring* (minimal modification) of self-dual simplicial complexes. More explicitly, the result of a minimal modification of $K \subset 2^{[n]}$, based on a maximal simplex $A \in K$, is the complex $mm_A(K) := \big(K \backslash \{A\}\big) \cup \{A^c\}$ obtained by removing from K the maximal simplex A and adding the associated minimal non-simplex A^c, where $A^c := [n] \backslash A$.

We say that two self-dual complexes K and L are *neighbors* if one can be obtained from the other by a minimal modification, $L = mm_A(K)$ and $K = mm_{A^c}(L)$. The associated *neighborhood graph* $\mathcal{NG}_n$ (Section 3) provides a convenient ecological niche for the classification and study of self-dual complexes.

It is not difficult to see that the graph $\mathcal{NG}_n$ is always connected. Nevertheless it is instructive to analyze more carefully the paths in this graph (see Propositions 3.1 and 3.2). For illustration each closed cycle in the neighborhood graph NG_n has an even length (Proposition 3.3) which immediately implies that the graph $\mathcal{NG}_n$ is bipartite.

The complex $\Delta^{n-2} := 2^{[n-1]}$ (as a self-dual subcomplex of $2^{[n]}$) is somewhat exceptional. It has only one maximal simplex ($A = [n-1]$) and consequently only one neighbor in $\mathcal{NG}_n$, the complex

$$mm_{[n-1]}(\Delta^{n-2}) = \big(2^{[n-1]} \backslash \big\{[n-1]\big\}\big) \cup \big\{\{n\}\big\} = \partial\Delta^{n-2} \uplus \Delta^0.$$

In Section 4 we focus on paths in the neighborhood graph $\mathcal{NG}_n$, emanating from the simplicial complex Δ^{n-2}. This analysis leads to the construction of the operation $\sqrt{\ } : \mathcal{D}^{[n]} \to \mathcal{SD}^{[n-1]}$, referred to as *the root operator*, where $\mathcal{D}^{[n]}$ (respectively $\mathcal{SD}^{[n-1]}$) is the collection of all self-dual (respectively sub-dual) complexes in $2^{[n]}$ (respectively sub-dual $2^{[n-1]}$). It turns out that the operator $\sqrt{\ }$ is invertible which immediately implies that the number of self-dual simplicial complexes on n vertices is equal to number of sub-dual simplicial complexes on $n-1$ vertices, (Theorem 1.3).

In Section 5 we briefly discuss the relevance of Theorem 1.3 for estimating *Dedekind numbers*. In Section 6 we give a more geometric interpretation of the root operator and its inverse, the *dual upgrade operator* Λ. The central is Proposition 6.2 expressing the root as a link of the vertex n, $\sqrt{K} = \mathrm{Lk}(\{n\})$.

Section 7 describes a new method for checking combinatorial equivalence (Definition 7.1) of two self-dual simplicial complexes. The main result is Theorem 1.4 stat-

ing that it is sufficient to check combinatorial equivalence of the links of two chosen vertices.
In the concluding section (Section 8), we use the Combinatorial Alexander Duality (Theorem 8.1) to analyze the homology of self-dual upgrades in relation to its root sub-dual root complex and introduce a new technique for constructing self-dual complexes with prescribed homology.

2. Basic properties of self-dual, sub-dual and super-dual complexes

Here are some elementary properties of the duality operator.

LEMMA 2.1. *Let $K, L \subseteq 2^S$ be any simplicial complexes.*
(i) If $K \subseteq L$ then $\widehat{L} \subseteq \widehat{K}$. *(ii)* $\widehat{(\widehat{K})} = K$

Interesting results can be obtained in a more general setting for iterated duality, especially when the ambient for the second dual is increased.
The family of all sub-dual complexes in the ambient S will be denoted by $\mathcal{SD}^S$ and the family of all self-dual complexes in the ambient S will be denoted by $\mathcal{D}^S$. It is immediate from Lemma 2.1 that K is super-dual in the ambient S iff $\widehat{K}^S$ is sub-dual in the ambient S.

EXAMPLE 2.2. Let $\binom{[n]}{k}$ be the $k-1$-skeleton of the complex $\Delta^{n-1} = 2^{[n]}$. Then, by Definition 1.1 its Alexander dual in the ambient $[n]$ is

$$\widehat{\binom{[n]}{k}} = \{[n]\backslash A \mid |A| > k\} = \{A \mid |A| \leq n-k-1\} = \binom{[n]}{n-k-1}.$$

Therefore, the complex $\binom{[n]}{k}$ is sub-dual iff $2k+1 \leq n$, super-dual iff $2k+1 \geq n$, and self-dual iff $2k+1 = n$. Specially, if $k = 2$ we obtain the complex $\binom{[5]}{2} = K_5$, a complete graph on 5 vertices shown in Figure 2.

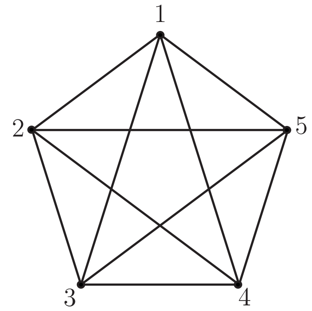

Figure 2: Graph K_5.

Using Lemma 2.1 we conclude that if a given simplicial complex is sub-dual in the ambient S, all of its subcomplexes must also be sub-dual in the ambient S. Therefore, using Example 2.2, we obtain the following proposition.

PROPOSITION 2.3. *A simplicial complex K of dimension k is always sub-dual in the ambient S where $|S| \geqslant 2k+3$.*

The following theorem provides an efficient criterion for verifying sub, super and self-duality of a given simplicial complex.

THEOREM 2.4. *Let $K \subseteq 2^S$ be a simplicial complex. In the ambient S the complex K is:*
(i) sub-dual iff there is no simplex $A \subseteq S$ such that $A \in K$ and $S\backslash A$ belong to K;

(ii) super-dual iff there is no simplex $A \subseteq S$ such that A and $S\backslash A$ are not in K;

(iii) self-dual iff for arbitrary $A \subseteq S$ exactly one of the simplexes A or $S\backslash A$ belongs to K or equivalently $(\forall A \subseteq S) A \in K \iff S\backslash A \notin K$.

Proof. (i)($\Rightarrow$) Let $K \subseteq \widehat{K}$ and let $A \subseteq S$ such that $A, S\backslash A \in K$. Then, because K is a subcomplex of $\widehat{K}$, we have $S\backslash A, A \in \widehat{K}$ which by Definition 1.1 implies that A and $S\backslash A$ do not belong to K contradicting our assumption.
($\Leftarrow$) Suppose there is no simplex $A \subseteq S$ such that A and $S\backslash A$ are in K. Then, for arbitrary $B \in K$, the simplex $S\backslash B$ is not in K which implies that $S\backslash(S\backslash B) = B$ is in $\widehat{K}$. Therefore $K \subseteq \widehat{K}$.
(ii) Following Lemma 2.1, complex K will be super-dual iff $\widehat{K}$ is sub-dual and by statement(i) this will happen iff there is no simplex $A \subseteq S$ such that A and $S\backslash A$ belong to $\widehat{K}$ which, by Definition 1.1 is equivalent to $A, S\backslash A \notin K$.
(iii) Complex K will be self-dual iff it is sub-dual and super-dual. Therefore, for an arbitrary simplex $A \subseteq S$, if $A \in K$ for (i) to be true, the simplex $S\backslash A$ must not be in K. Also, if $A \notin K$, then from (ii) we have $S\backslash A \in A$. □

The following example illustrates how the ambient space affects the duality of simplicial complexes.

EXAMPLE 2.5. The complex $\Delta^{n-1} = 2^{[n]}$ is super-dual in the ambient $[n]$, self-dual in the ambient $[n+1]$ and sub-dual in the ambient $[n+2]$.
Indeed, the following Theorem 2.4, for arbitrary $A \subseteq [n]$, both A and $[n]\backslash A$ are contained in $2^{[n]}$, which confirms (ii). Also, for arbitrary $A \subseteq [n+1]$, the set A does not contain the vertex $\{n+1\}$ iff $[n+1]\backslash A$ contains $\{n+1\}$ or equivalently, $A \in 2^{[n]}$ iff $[n+1]\backslash A \notin 2^{[n]}$. This confirms (iii). Finally, $\Delta^{n-1} = 2^{[n]} \subset 2^{[n+1]}$, so, by Lemma 2.1 we have $\widehat{2^{[n+1]}}^{[n+2]} = 2^{[n+1]} \subset \widehat{2^{[n]}}^{[n+2]}$ and this implies $\Delta^{n-1} \subset \widehat{\Delta^{n-1}}^{[n+2]}$.

This example illustrates that enlargement of the ambient set, in the case of the complex Δ^{n-1}, increases its Alexander dual, which is also true for an arbitrary simplicial complex. Namely, if $K \subseteq 2^{[n]}$ is a simplicial complex then, by Lemma 2.1, we have $\widehat{2^{[n]}}^{[n+1]} \subseteq \widehat{K}^{[n+1]}$ and by Example 2.5 we have $K \subseteq 2^{[n]} \subseteq \widehat{K}^{[n+1]}$. This allows us to conclude that every simplicial complex K in the ambient S is sub-dual in every ambient S' where $S \subset S'$.

Proposition 2.6. *Only simplicial complexes that are self-dual in the ambient S and do not contain all vertices $\{v\} \subset S$ are $2^{S\setminus\{v\}}$.*

Proof. From Example 2.5 we know that $2^{S\setminus\{v\}}$ is self-dual in the ambient S. Suppose $K \subseteq 2^S$ is self-dual and that the vertex $\{v\}$ does not belong to K. Then, by Theorem 2.4 part (iii) the simplex $S\setminus\{v\}$ must be in K. Thus, $K = 2^{S\setminus\{v\}}$. □

So, following Example 2.6, we observe that there are exactly n self-dual simplicial complexes in the ambient $[n]$, which do not contain all singletons as simplices.

Remark 2.7. Theorem 2.4 states that a simplicial complex K is super-dual iff for every partition A, B of the ambient set S into disjoint subsets, at least one of the simplexes A or B belongs to S. The complexes with this property are also called 2-unavoidable, a special case of $r-$unavoidable complexes which where introduced in [5]. In this setting, the self-dual simplicial complexes correspond to minimal $2-$unavoidable complexes. For an in-depth study of combinatorial properties of r-unavoidable complexes see [13, 14].

3. Minimal modification of self-dual complexes

In this section we take a closer look at the method of minimal modifications (minimal restructuring) of self-dual simplicial complexes $K \subset 2^S$ in a fixed ambient set S.
Recall that $A \in K$ is a maximal simplex of a simplicial complex $K \subseteq 2^S$ if A is not a face of any other simplex of the complex K i.e. if $(\forall B \in K)\, A \not\subset B$. Equivalently, a simplex $A \in K$ is maximal iff $K\setminus\{A\}$ is a simplicial complex.
The operation of *minimal modification* $mm_A(K)$ of self-dual simplicial complexes is closely related to the operation of bistellar operations of Bier spheres, described in [16].

Proposition 3.1. *Let $K \subseteq 2^S$ be a self-dual simplicial complex and let $A \in K$ be a maximal simplex in K. Then $mm_A(K) := (K\setminus\{A\}) \cup \{S\setminus A\}$ is also a self-dual simplicial complex.*

Proof. $K\setminus\{A\}$ is a simplicial complex because A is maximal. Let $B \subset S\setminus A$ be arbitrary. This implies that $A \subset S\setminus B$, and because A is maximal for K, we have $S\setminus B \notin K$ therefore B must be in K because K is self-dual (Theorem 2.4). Thus, we have shown that $K\setminus\{A\}$ contains all proper faces of the simplex $S\setminus A$, so $(K\setminus\{A\}) \cup \{S\setminus A\}$ is a simplicial complex. Moreover, the complex $(K\setminus\{A\}) \cup \{S\setminus A\}$ clearly satisfies part (iii) of Theorem 2.4. □

We say that the self-dual complex $L = mm_A(K) = (K\setminus\{A\}) \cup \{S\setminus A\}$ is obtained by a minimal modification or minimal restructuring of the self-dual complex K. The following proposition shows that any self-dual complex can be obtained from any other self-dual complex by successive minimal restructurings.

Proposition 3.2. *Let K and L be any pair of self-dual simplicial complexes in the ambient set S. Then the complex L can be obtained from K by successive minimal modifications based on simplexes belonging to the set $K\backslash L$.*

Proof. Let $K\backslash L=\{A_1, A_2, \ldots, A_n\}$ where simplexes A_i are ordered decreasingly by dimension (meaning that $|A_i|\geq|A_{i+1}|$). Let $K_0=K$ and $K_i=(K_{i-1}\backslash\{A_i\}) \cup \{S\backslash A_i\}$. To show that $K_0, K_1, \ldots, K_n$ is a well defined sequence of successive minimal modifications it is sufficient to show that A_i is a maximal simplex of complex K_{i-1}.
First, note that A_1 is a maximal simplex in $K_0=K$. Otherwise, there exists a simplex $B \in K$ such that $A_1 \subset B$ and this simplex will belong to L because $|B| > |A_i|$ for all $i \in [n]$. Since L is a simplicial complex, we have that A_1 is also in L which is not possible. So, A_1 can be used for a minimal modification and by Proposition 3.1 the complex K_1 is self-dual.
Suppose inductively that A_{i-1} is maximal in K_{i-2}. This by Proposition 3.1 implies that K_{i-1} is a self-dual simplicial complex.
If A_i is not maximal in $K_{i-1}=(K\backslash\{A_1, \ldots, A_{i-1}\}) \cup \{S\backslash A_1 \ldots, S\backslash A_{i-1}\}$, then there exists a simplex $B \in K_{i-1}$ such that $A_i \subset B$. We know that $A_i \notin L$ so, B also must not be in L because L is a simplicial complex. Therefore, B is a simplex from $K\backslash L$ such that $|B| > |A_i|$ and, by construction this means that B is one of the simplexes $A_1, \ldots, A_{i-1}$, however these simplices are not in K_{i-1}.
Finally, since K and L are self-dual, by Theorem 2.4 we have:

$$A \in K\backslash L \Leftrightarrow A \in K \wedge A \notin L \Leftrightarrow (S\backslash A) \notin K \wedge (S\backslash A) \in L \Leftrightarrow (S\backslash A) \in L\backslash K.$$

Therefore, $L\backslash K = \{S\backslash A_i \mid i \in [n]\}$ and we have:

$$K_n = (K\backslash\{A_1, \ldots, A_n\}) \cup \{S\backslash A_1 \ldots, S\backslash A_n\} = \big(K\backslash(K\backslash L)\big) \cup (L\backslash K) = L. \qquad \square$$

Propositions 3.1 and 3.2 allow us to introduce a new combinatorial object we will refer to as the restructuring graph $(\mathcal{D}^{[n]}, \mathcal{NG}_n)$ (or the neighborhood graph $\mathcal{NG}_n$ for short). The nodes of this graph are all self-dual simplicial complexes in the ambient $[n]$ and complexes $K, L \in \mathcal{D}^{[n]}$ are neighbors in the graph iff L can be obtained from K by a minimal restructuring, or equivalently if K can be obtained from L by a minimal restructuring (based respectively on simplices A and $[n]\backslash A$).
Note that two self-dual complexes K and L can be connected by a path of length $|K\backslash L|$ and we will show that there are no shorter paths connecting K and L. Moreover, the degree of a node K is equal to the number of its maximal simplices since minimal modifications based on different maximal simplexes yield different self-dual complexes.

Proposition 3.3. *The graph $\mathcal{NG}_n$ has the following properties.*
(i) For an arbitrary path $\{K_0, K_1, \ldots, K_m\}$ connecting complexes K and L we have $m \geq |K\backslash L|$ and $m \equiv |K\backslash L| \mod 2$.

(ii) All the loops in the graph $\mathcal{NG}_n$ are of an even length.

Proof. Let $\{K = K_0, K_1, \ldots, K_m = L\}$ be a sequence of minimal restructurings based on the simplexes $\{A_1, \ldots, A_m\}$. Then, $K_n=(K_0\backslash\{A_1, \ldots, A_m\})\cup\{[n]\backslash A_1 \ldots, [n]\backslash A_m\}$ so the set $K\backslash L$ must be contained in $\{A_1, \ldots, A_m\}$ proving that $|K\backslash L| \leq m$. Also, if for some i the simplex A_i is not one of the simplexes in $K\backslash L$ then the set $\{A_1, \ldots, A_m\}$

must contain $[n]\backslash A_i$ because otherwise $K_m = L$ will also contain $[n]\backslash A_i$ which is not possible. This proves the statement (i) and the statement (ii) is an immediate consequence. □

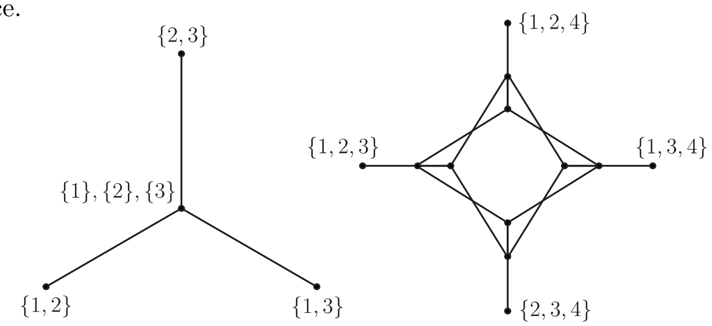

Figure 3: Graphs $\mathcal{NG}_3$ and $\mathcal{NG}_4$, the nodes are labeled with their maximal simplexes

Therefore, we have shown that all the loops in the graph $\mathcal{NG}_n$ have an even length proving the following proposition.

COROLLARY 3.4. *The neighborhood graph $\mathcal{NG}_n$ is bipartite for all $n \in \mathbf{N}$.*

Graphs $\mathcal{NG}_3$ and $\mathcal{NG}_4$ are shown in Figure 3. Studying graph $\mathcal{NG}_n$ can reveal many useful properties of self-dual simplicial complexes, specially the number of different self-dual complexes in the ambient $[n]$.

4. The root operator

In this section we introduce the *root operator*, as our main tool for analyzing self-dual complexes.

Proposition 3.2 says that each pair of self-dual simplicial complexes can be connected in the neighborhood graph $\mathcal{NG}_n$ by a sequence of self-dual simplicial complexes. In this section we focus on such sequences starting from the complex $\Delta^{n-2} = 2^{[n-1]}$.

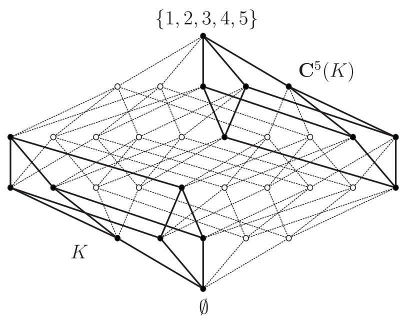

Figure 4: Operator $\mathbf{C}^5$ applied on $2^{[3]}$.

The standard "complement" operator $\mathbf{C}^n : 2^{2^{[n]}} \to 2^{2^{[n]}}$ is defined by

$$\mathbf{C}^n(K) = \{[n]\backslash A \mid A \in K\} \tag{1}$$

If we view $2^{[n]}$ as a family partially ordered by inclusion, then the operator $\mathbf{C}^n$ is naturally interpreted as the symmetry with respect to the center of the poset, as shown on Figure 4.

The following elementary lemma is recorded for the further reference.

LEMMA 4.1. *Suppose that K and L are arbitrary families of sets in the ambient $[n]$. The operator $\mathbf{C}^n$ has the following properties:*
(i) If $K \subseteq L$ then $\mathbf{C}^n(K) \subseteq \mathbf{C}^n(L)$.

(ii) For arbitrary operation $\diamond \in \{\cup, \cap, \backslash\}$ we have $\mathbf{C}^n(K \diamond L) = \mathbf{C}^n(K) \diamond \mathbf{C}^n(L)$.

(iii) Family K is a simplicial complex iff $(\forall A \in \mathbf{C}^n(K))(\forall B \subseteq [n]) A \subseteq B \Rightarrow B \in \mathbf{C}^n(K)$.

(iv) $\mathbf{C}^n\big(\mathbf{C}^n(K)\big) = K$.

(v) $\mathbf{C}^n(2^{[n]}\backslash K) = 2^{[n]}\backslash \mathbf{C}^n(K)$.

(vi) For $m \geq n$, the Alexander dual $\widehat{K}^{[m]}$ of a simplicial complex K is equal to $\mathbf{C}^m(2^{[m]}\backslash K)$.

Proof. Properties (i) through (iv) are elementary consequences of (1).
For property (v), we deduce from (ii) that $\mathbf{C}^n(2^{[n]}\backslash K) = \mathbf{C}^n(2^{[n]})\backslash \mathbf{C}^n(K)$ and obviously $\mathbf{C}^n(2^{[n]}) = 2^{[n]}$.
For property (vi), from Definition 1.1, we see that the Alexander dual of the complex K in the ambient $[m]$ is equal to $\{[m]\backslash A \mid A \in 2^{[m]}\backslash K\}$ which is by (1) equal to $\mathbf{C}^m(2^{[m]}\backslash K)$. $\square$

Note that, following from the part (v) of Lemma 4.1, the Alexander dual of the complex K in the ambient $[m]$ is also equal to $2^{[m]}\backslash \mathbf{C}^m(K)$.
Proposition 3.2 shows that the symmetrical difference of arbitrary self-dual simplicial complexes K and L in the ambient $[n]$ is equal to $(L\backslash K) \cup \mathbf{C}^n(L\backslash K)$. Instead of the complex L, let us use the complex $\Delta^{n-2} = 2^{[n-1]}$, which is by Example 2.5 self-dual in the ambient $[n]$. We now analyze families of simplexes which arise as the difference $2^{[n-1]}\backslash K$ for some $K \in \mathcal{D}^{[n]}$.

PROPOSITION 4.2. *Let K be a self-dual simplicial complex in the ambient $[n]$. Then, the family of simplexes $\mathbf{C}^{n-1}(2^{[n-1]}\backslash K)$ is a sub-dual simplicial complex in the ambient $[n-1]$.*

Proof. Let $A \in 2^{[n-1]}\backslash K$ be arbitrary and $B \subseteq [n-1]$ such that $A \subseteq B$. Then, since A is not in K and K is a simplicial complex, B also does not belong to K, which implies that $B \in 2^{[n-1]}\backslash K$. This by Lemma 4.1, properties (iii) and (iv), proves that $\mathbf{C}^{n-1}(2^{[n-1]}\backslash K)$ is a simplicial complex.
To prove that this complex is sub-dual, let us check property (i) of Theorem 2.4. Let $A \subseteq [n-1]$ be a simplex such that both A and $[n-1]\backslash A$ belong to $\mathbf{C}^{n-1}(2^{[n-1]}\backslash K)$.

If we apply the operator $\mathbf{C}^{n-1}$ on the inclusion $\{A, [n-1]\backslash A\} \subset \mathbf{C}^{n-1}(2^{[n-1]}\backslash K)$, by properties (i) and (iv) of Lemma 4.1 we get $\{[n-1]\backslash A, A\} \subset 2^{[n-1]}\backslash K$, which implies that both A and $[n-1]\backslash A$ do not belong to K. Because K is a simplicial complex, this means that $A \cup \{n\}$ and $([n-1]\backslash A) \cup \{n\} = [n]\backslash(A \cup \{n\})$ do not belong to K, which in turn contradicts the assumption that K is self-dual in the ambient $[n]$. $\square$

Summarizing, we can associate a sub-dual complex in the ambient $[n-1]$ to every self-dual simplicial complex in the ambient $[n]$.

Definition 4.3. The root operator is a map $\surd : \mathcal{D}^{[n]} \to \mathcal{SD}^{[n-1]}$ given by

$$\surd K = \mathbf{C}^{n-1}(2^{[n-1]}\backslash K). \tag{2}$$

Notice that, in light of the property (vi) of Lemma 4.1, the root complex of a self-dual simplicial complex in the ambient $[n]$ can be understood as its Alexander dual in a smaller ambient $[n-1]$. It is obvious from (2) that the root operator is injective. To prove that this operator is bijective we describe its inverse.

In light of Proposition 3.2, the complement set of an arbitrary sub-dual simplicial complex is supposed to be the difference $2^{[n]}\backslash K$ for some complex $K \in \mathcal{D}^{[n]}$. This observation leads to the following proposition.

Proposition 4.4. *For a given sub-dual simplicial complex K in the ambient $[n-1]$, the family $L = \big(2^{[n-1]}\backslash \mathbf{C}^{n-1}(K)\big) \cup \mathbf{C}^n\big(\mathbf{C}^{n-1}(K)\big)$ is a self-dual simplicial complex in the ambient $[n]$.*

Proof. As in the proof of Proposition 3.2, since $2^{[n-1]}$ is self dual in the ambient $[n]$, it is sufficient to show that L can be obtained from $2^{[n-1]}$ by successive minimal restructurings based on the simplexes belonging to $\mathbf{C}^{n-1}(K)$.

First, note that $\mathbf{C}^n\big(\mathbf{C}^{n-1}(\{A\})\big) = \{[n]\backslash([n-1]\backslash A)\} = \{A \cup \{n\}\}$. Let $\mathbf{C}^n(K) = \{[n-1]\backslash A_1, \ldots, [n-1]\backslash A_k\}$ where $|A_{i+1}| \leq |A_i|$ for all $i \in [k-1]$. Let $K_0 = 2^{[n-1]}$ and $K_i = \big(K_{i-1}\backslash\{[n-1]\backslash A_i\}\big) \cup \big\{A_i \cup \{n\}\big\}$. Thus we have

$$K_i = \big(K\backslash\{[n-1]\backslash A_1, \ldots, [n-1]\backslash A_i\}\big) \cup \big\{A_1 \cup \{n\}, \ldots, A_i \cup \{n\}\big\}.$$

Let us show that the simplex $[n-1]\backslash A_i$ is maximal in K_{i-1}. Note that, $[n-1]\backslash A_i$ is a simplex of K_{i-1}. Since $A_1 = \emptyset$, we have $[n-1]\backslash A_1 = [n-1]$ and this simplex is maximal in K_0. Suppose $[n-1]\backslash A_i$ is not maximal K_{i-1} meaning that there exists $B \in K_i$ such that $[n-1]\backslash A_i \subset B$. Because K is a simplicial complex, by Lemma 4.1 property (iii), all simplexes from $2^{[n-1]}$ containing A_i are in $\{[n-1]\backslash A_1, \ldots, [n-1]\backslash A_{i-1}\}$ and we know that $|B| > |[n]\backslash A_i| \geq |[n]\backslash A_j|$ for every $j \in [i-1]$. Therefore $B \notin 2^{[n-1]}$ so B must be one of the simplexes $A_1 \cup \{n\}, \ldots, A_{i-1} \cup \{n\}$. If $[n]\backslash A_i \subset A_j \cup \{n\}$ for $j < i$, we get that $A_i \supset [n]\backslash(A_j \cup \{n\})$ and since $n \notin A_j$ we have $A_i \supset [n-1]\backslash A_j$. Because $A_i \in K$, this implies that $[n-1]\backslash A_j$ must also be in K. Therefore both A_j and $[n-1]\backslash A_j$ belong to K, but this by Theorem 2.4 contradicts the assumption that K is sub-dual.

Therefore, A_i is maximal in K_{i-1} which by Proposition 3.1 proves that all complexes K_i are self-dual in the ambient $[n]$ including $K_k = L$. $\square$

The following "upgrade operator" will turn out to be the inverse of the root operator.

Definition 4.5. The upgrade operator is a map $\Lambda : \mathcal{D}^{[n-1]} \to \mathcal{D}^{[n]}$ given by

$$\Lambda K = \big(2^{[n-1]}\backslash \mathbf{C}^{n-1}(K)\big) \cup \mathbf{C}^n\big(\mathbf{C}^{n-1}(K)\big).$$

The complex ΛK will be also referred to as the dual upgrade of the simplicial complex K. Later, we will give another, possibly more elegant descriptions of operators $\surd$ and Λ, see Propositions 6.4 and 6.2.
Let us show that $\Lambda : \mathcal{SD}^{[n-1]} \to \mathcal{D}^{[n]}$ is the inverse of the root operator $\surd$.
Let $K \in \mathcal{D}^{[n]}$ be arbitrary. Using Lemma 4.1 we have the following array of equalities:

$$\begin{aligned} &\Lambda \circ \surd(K) = \Lambda\big(\mathbf{C}^{n-1}(2^{[n-1]}\backslash K)\big)\\ &=\big[2^{[n-1]}\backslash \mathbf{C}^{n-1}\big(\mathbf{C}^{n-1}(2^{[n-1]}\backslash K)\big)\big] \cup \mathbf{C}^n\big[\mathbf{C}^{n-1}\big(\mathbf{C}^{n-1}(2^{[n-1]}\backslash K)\big)\big]\\ &=\big(2^{[n-1]}\backslash(2^{[n-1]}\backslash K)\big) \cup \mathbf{C}^n(2^{[n-1]}\backslash K) = (K \cap 2^{[n-1]}) \cup \mathbf{C}^n(2^{[n-1]}\backslash K)\end{aligned}$$

Since $2^{[n-1]}$ and K are self-dual complexes in the ambient $[n]$, by Theorem 2.4 we have

$$\begin{aligned} A \in \mathbf{C}^n(2^{[n-1]}\backslash K) &\iff [n]\backslash A \in 2^{[n-1]}\backslash K \Leftrightarrow ([n]\backslash A \in 2^{[n-1]} \wedge [n]\backslash A \notin K)\\ &\iff (A \notin 2^{[n-1]} \wedge A \in K) \Leftrightarrow A \in K\backslash 2^{[n-1]}\end{aligned}$$

Therefore $\mathbf{C}^n(2^{[n-1]}\backslash K) = K\backslash 2^{[n-1]}$ and we get $\Lambda\circ\surd(K) = (K\cap 2^{[n-1]})\cup(K\backslash 2^{[n-1]}) = K$. So, the composition $\Lambda \circ \surd$ is the identity map. Before we analyze the composition $\surd \circ \Lambda$, let us introduce few more properties of the complement operator $\mathbf{C}^n$.

Lemma 4.6. *Let $K \subseteq 2^{[n]}$ be an arbitrary family of sets. Then for every $m \geq n$ we have: (i) $\mathbf{C}^m \circ \mathbf{C}^n(K) = \{A \cup ([m]\backslash[n]) \mid A \in K\}$; (ii) $\mathbf{C}^n \circ \mathbf{C}^m \circ \mathbf{C}^n = \mathbf{C}^n$.*

Proof. (i) $\mathbf{C}^m\big(\mathbf{C}^n(K)\big)=\{[m]\backslash([n]\backslash A) | A \in K\}=\{[m]\backslash\big([m]\backslash\big[(A \cup ([m]\backslash[n])\big]\big) | A \in K\}$ $=\{A \cup ([m]\backslash[n]) | A \in K\}$
(ii) $\mathbf{C}^n \circ \big(\mathbf{C}^m \circ \mathbf{C}^n(K)\big)=\{[n]\backslash\big(A\cup([m]\backslash[n])\big) | A \in K\}=\{([n]\backslash A)\cap\big([n]\backslash([m]\backslash[n])\big) | A \in K\}=\{([n]\backslash A) \cap [n] | A \in K\}=\{[n]\backslash A | A \in K\}=\mathbf{C}^n(K)$. □

Now, let $K \in \mathcal{SD}^{[n-1]}$ be an arbitrary simplicial complex. Using Lemmas 4.1 and 4.6 we have

$$\begin{aligned} \surd \circ \Lambda(K) &= \mathbf{C}^{n-1}\big(2^{[n-1]}\backslash\Lambda(K)\big) = 2^{[n-1]}\backslash \mathbf{C}^{n-1}\big(\Lambda(K)\big)\\ &= 2^{[n-1]}\backslash \mathbf{C}^{n-1}\big[\big(2^{[n-1]}\backslash \mathbf{C}^{n-1}(K)\big) \cup \mathbf{C}^n\big(\mathbf{C}^{n-1}(K)\big)\big]\\ &= 2^{[n-1]}\backslash\big[\mathbf{C}^{n-1}\big(2^{[n-1]}\backslash \mathbf{C}^{n-1}(K)\big) \cup \mathbf{C}^{n-1}\big(\mathbf{C}^n(\mathbf{C}^{n-1}(K))\big)\big]\\ &= 2^{[n-1]}\backslash\big[2^{[n-1]}\backslash \mathbf{C}^{n-1}\big(\mathbf{C}^{n-1}(K)\big) \cup \mathbf{C}^{n-1}(K)\big]\\ &= 2^{[n-1]}\backslash\big[(2^{[n-1]}\backslash K) \cup \mathbf{C}^{n-1}(K)\big]\\ &= \big[2^{[n-1]}\backslash(2^{[n-1]}\backslash K)\big] \cap \big[2^{[n-1]}\backslash\big(\mathbf{C}^{n-1}(K)\big)\big] = K \cap \widehat{K}^{[n-1]} = K.\end{aligned}$$

The last equality holds because by Definition 1.2 the complex K is a subset of $\widehat{K}^{[n-1]}$. Summarizing, we have established the following proposition.

Proposition 4.7. *The root operator $\surd : \mathcal{D}^{[n]} \to \mathcal{SD}^{[n-1]}$ is invertible and its inverse is the upgrade operator $\Lambda : \mathcal{SD}^{[n-1]} \to \mathcal{D}^{[n]}$.*

Therefore $\surd : \mathcal{D}^{[n]} \to \mathcal{SD}^{[n-1]}$ is a bijection and thus we have proved Theorem 1.3.

5. Duality and Dedekind numbers

In this section we examine the relevance of Theorem 1.3 for the study of Dedekind numbers.

Recall that the Dedekind number $\mathbb{D}(n)$ was introduced in [7] by Richard Dedekind as the number of different monotone Boolean functions on n variables. A formula for computing $\mathbb{D}(n)$ was discovered by Kisielewicz in [15]. Despite this discovery, the exact value of $\mathbb{D}(n)$ is known only for values $n \leq 8$ and many efforts have been made to estimate $\mathbb{D}(n)$ for greater values of n.

It is easy to see that each monotone Boolean function on n variables corresponds to a unique simplicial complex in the ambient $[n]$. Thus, $\mathbb{D}(n)$ can be calculated as a number of simplicial complexes with at most n vertices.

Let $\mathcal{K}^{[n]}$ be a family of all simplicial complexes in the ambient set $[n]$. Also, let $\mathcal{SPD}^{[n]}$ be the family of all-super dual complexes in the ambient $[n]$ and $\mathcal{T}^{[n]}$ a family of all transcendent complexes in the ambient $[n]$.

n	$\lvert\mathcal{SD}^{[n]}\rvert$	$\lvert\mathcal{D}^{[n]}\rvert$	$\lvert\mathcal{T}^{[n]}\rvert$	$\mathbb{D}(n)$
1	2	1	0	3
2	4	2	0	6
3	12	4	0	20
4	81	12	18	168
5	2646	81	2370	7581

Table 1: Dedekind numbers

Since every complex $K \in \mathcal{K}^{[n]}$ is either sub-dual, super-dual or transcendent, and transcendent complexes cannot be super or self-dual, we have a following formula:

$$|\mathcal{K}^{[n]}| = |\mathcal{SD}^{[n]}| + |\mathcal{SPD}^{[n]}| - |\mathcal{SD}^{[n]} \cap \mathcal{SPD}^{[n]}| + |\mathcal{T}^{[n]}|.$$

By Lemma 2.1 we see that the operator $\widehat{\_}^{[n]} : \mathcal{SD}^{[n]} \to \mathcal{SPD}^{[n]}$ is its own inverse, hence a bijection, implying that $|\mathcal{SD}^{[n]}|$ is equal to $|\mathcal{SPD}^{[n]}|$. Also, we know by Definition 1.2 that the complex is self-dual iff it is sub-dual and super-dual. This, together with Theorem 1.3 shows that,

$$\mathbb{D}(n) = 2|\mathcal{D}^{[n+1]}| - |\mathcal{D}^{[n]}| + |\mathcal{T}^{[n]}| \tag{3}$$

Table 1, obtained by an elementary computer assisted calculation, illustrates the use of the formula (3) for small values of n.

The equation (3) implies that the number of self-dual complexes on n vertices provides a lower bound to $\mathbb{D}(n)$. Also, since all complexes in the ambient $[n]$ are sub-dual in the ambient $[n+1]$, Theorem 1.3 allows us to give an upper bound to the number $\mathbb{D}(n)$ as follows,

$$2|\mathcal{D}^{[n+1]}| - |\mathcal{D}^{[n]}| \leq \mathbb{D}(n) < |\mathcal{D}^{[n+2]}| \tag{4}$$

Studying the neighborhood graph $\mathcal{NG}_n$ using tools developed in Section 3 can reveal the number of different self-dual simplicial complexes which in turn, using (4), can give an estimation of $D(n)$. In order to fully determine $D(n)$ using this method, one needs to further analyze combinatorial properties of transcendent simplicial complexes.

6. Geometrical descriptions of operators $\surd$ and Λ

In this section we return to the study of operators $\surd$ and Λ, this time emphasizing a different, more geometrical, point of view.

Definition 6.1. For a given simplicial complex $K \subseteq 2^S$, the link of a simplex $A \subseteq S$, labeled $\mathrm{Lk}(A)$, is defined by $\mathrm{Lk}(A) = \{B \in K \mid A \not\subseteq B, B \cup A \in K\}$.

It is easy to show that $\mathrm{Lk}(A)$ is a subcomplex of the simplicial complex K. If $A \notin K$, then $\mathrm{Lk}(A) = \emptyset$.

Proposition 6.2. *For an arbitrary self-dual complex K in the ambient $[n]$, $\sqrt{K} = \mathrm{Lk}(\{n\})$.*

Proof. Let $K \subseteq 2^{[n]}$ be a self-dual simplicial complex and let $A \in \sqrt{K}$ be an arbitrary simplex. Then, by (2), $A \in \mathbf{C}^{n-1}(2^{[n-1]} \backslash K)$. Moreover, $A = [n-1] \backslash B$ where $B \subset [n-1]$ and $B \notin K$. Since K is self-dual in the ambient $[n]$ and $B \notin K$, by Theorem 2.4 we have that $[n] \backslash B \in K$ and because K is a simplicial complex, $[n-1] \backslash B = A \in K$. Since, $A \subseteq [n-1]$ we have $\{n\} \not\subseteq A$. So, the simplex A has the property that $\{n\} \not\subseteq A$ and $A \cup \{n\} = ([n-1] \backslash B) \cup \{n\} = [n] \backslash B \in K$. This by Definition 6.1 implies that $A \in \mathrm{Lk}(\{n\})$ and we have $\sqrt{K} \subseteq \mathrm{Lk}(\{n\})$.
Let $A \in \mathrm{Lk}(\{n\})$ be arbitrary. This, by Definition 6.1 means that $A \in K$ satisfies $\{n\} \not\subseteq A$ and $A \cup \{n\} \in K$. Since K is self-dual in the ambient $[n]$, by Theorem 2.4 the simplex $[n] \backslash (A \cup \{n\}) = [n-1] \backslash A$ does not belong to K. Thus, $[n-1] \backslash A \in 2^{[n-1]} \backslash K$ and by (1) we have $A \in \mathbf{C}^{n-1}(2^{[n-1]} \backslash K) = \sqrt{K}$. Therefore $\mathrm{Lk}(\{n\}) \subseteq \sqrt{K}$. □

Definition 6.3. Let $K \subseteq 2^S$ and $L \subseteq 2^{S'}$ be simplicial complexes. The join of complexes K and L is a simplicial complex in the ambient $S \uplus S'$ given by $K * L = \{A \uplus B \mid A \in K, B \in L\}$.

Note, if the ambient sets S and S' for complexes K and L are disjoint, then the complex $K * L$ is obtained by taking the union of all pairs (A, B) of simplexes where $A \in K$ and $B \in L$. Specially, when the complex L is point-complex $L = \big\{\emptyset, \{v\}\big\}$ then $L * K$ is called *the cone of* K, and denoted by $\mathrm{C}K$.

Proposition 6.4. *If K is an arbitrary sub-dual simplicial complex in the ambient $[n-1]$ then $\Lambda K = \widehat{K}^{[n-1]} \cup \mathrm{C}K$.*

Proof. Let K be an arbitrary sub-dual simplicial complex in the ambient $[n-1]$. Then, by Definition 4.5 we have that $\Lambda K = (2^{[n-1]} \backslash \mathbf{C}^{n-1}(K)) \cup \mathbf{C}^n(\mathbf{C}^{n-1}(K))$. Following Lemma 4.1 (property (vi)) the first part of the union $2^{[n-1]} \backslash \mathbf{C}^{n-1}(K)$ is $\widehat{K}^{[n-1]}$, the Alexander dual of complex K in the ambient $[n-1]$. The second part of the union, $\mathbf{C}^n(\mathbf{C}^{n-1}(K))$ is by Lemma 4.6 equal to $\{A \cup \{n\} \mid A \in K\}$. Finally, since K is sub-dual in the ambient $[n-1]$ meaning $K \subset \widehat{K}^{[n-1]}$, the self-dual complex ΛK can be expressed as:

$$\begin{aligned}\Lambda K &= \widehat{K}^{[n-1]} \cup \{A \cup \{n\} \mid A \in K\} \cup K \\ &= \widehat{K}^{[n-1]} \cup \{A \cup \{n\} \mid A \in K\} \cup \{A \cup \emptyset \mid A \in K\} \\ &= \widehat{K}^{[n-1]} \cup K * \{\emptyset, \{n\}\}. \qquad \square\end{aligned}$$

We conclude that the operator Λ also has a simple form. Note that since $K \subseteq \widehat{K}$, the dual upgrade of the complex K is a simplicial complex $\widehat{K}^{[n-1]} \cup \mathrm{C}K$, which is homotopic to the factor space $\widehat{K}/K$.

7. Combinatorial structure of self-dual complexes

In this section we record for the future reference more invariant versions of the fundamental relations from Propositions 6.4 and 6.2, which are sometimes more convenient for immediate applications.

Definition 7.1. Let $K \subseteq 2^S$ and $L \subseteq 2^{S'}$ be simplicial complexes. We say that complexes K and L are isomorphic (or combinatorially equivalent) if there exists a bijection $\sigma : S \to S'$ such that $(\forall A \subseteq S)\, A \in K \Leftrightarrow \sigma(A) \in L$.

From Definition 7.1 we see that ambient sets of isomorphic simplicial complexes must be of the same cardinality. For convenience, the ambient sets are supposed to be minimal.

Let K be an arbitrary simplicial complex in the ambient S where $|S| = n$. Then, for any vertex $\{v\} \in S$ there exists a bijection $\sigma : S \to [n]$ which sends v to n. Thus, the simplicial complex $\sigma(K) = \{\sigma(A) \mid A \in K\}$ is a complex in the ambient $[n]$, isomorphic to the complex K. If we suppose that the complex K is sub-dual (self-dual) in the ambient S, then the simplicial complex $\sigma(K)$ is also sub-dual (self-dual) in the ambient $[n]$. This simple observation allows us to extend all the results from Section 4, about the complex $\sigma(K)$ to the complex K. In particular, any vertex $v \in S$ can play the role of the exceptional vertex n.

Corollary 7.2. *Let K be a self-dual simplicial complex in the ambient S. Then, for any vertex $\{v\} \subset S$, $\mathrm{Lk}(\{v\})$ is a sub-dual simplicial complex in the ambient $S \backslash \{v\}$.*

Corollary 7.3. *If K is a sub-dual simplicial complex in the ambient S then $\widehat{K}^S \cup \mathrm{C}K$ is a self-dual simplicial complex in the ambient $S \uplus \{v\}$ where $\{v\}$ is the vertex of $\mathrm{C}K$.*

By Proposition 4.7 we know that the root and upgrade operators are inverse to each other. This is true in any ambient set S which allows us to describe the main structural property of self dual simplicial complexes.

Corollary 7.4. *For every self-dual simplicial complex K in the ambient S and every vertex $\{v\} \subset S$ we have $K = \widehat{\mathrm{Lk}(\{v\})}^{S\backslash\{v\}} \cup \mathrm{C}\,\mathrm{Lk}(\{v\})$ where $\mathrm{C}\,\mathrm{Lk}(\{v\}) = Lk(\{v\}) * \big\{\emptyset, \{v\}\big\}$.*

The following example serves as an illustration of Corollary 7.4.

EXAMPLE 7.5. We know from Example 2.2 that $\binom{[2k+1]}{k}$ is a self-dual simplicial complex in the ambient $[2k+1]$. For this complex, by Definition 6.1 we have that

$$\mathrm{Lk}(\{2k+1\}) = \{A \subset [2k] \mid |A \cup \{2k+1\}| \leq k\} = \binom{[2k]}{k-1}.$$

Example 2.2 also shows that $\widehat{\binom{[2k]}{k-1}}^{[2k]} = \binom{[2k]}{2k-(k+1)-1} = \binom{[2k]}{k}$. For $\mathrm{Lk}(\{2k+1\}) * \{\emptyset, \{2k+1\}\}$ we get $\binom{[2k]}{k-1} \cup \{A \cup \{2k-1\} \mid A \subset [2k], |A| < k-1\} = \binom{[2k]}{k-1} \cup \{A \in \binom{[2k+1]}{k} \mid \{2k+1\} \in A\}$. Therefore

$$\mathrm{Lk}(\widehat{\{2k+1\}})^{[2k]} \cup \mathrm{C}\,\mathrm{Lk}(\{2k+1\}) = \binom{[2k+1]}{k}.$$

An interesting consequence of Corollary 7.4 is that any self-dual simplicial complex is completely determined by the link of any of its simplexes.

Proof (Theorem 1.4). Let K and L be self-dual complexes in the ambients S and S' respectively and let $|S| = |S'|$.
($\Rightarrow$) Let $\pi : S \to S'$ be an isomorphism of complexes K and L. Then, for an arbitrary $v \in S$, the map π is an isomorphism of complexes $\mathrm{Lk}(\{v\}) \subseteq K$ and $\pi\big(\mathrm{Lk}(\{v\})\big) \subseteq L$. Using Definitions 7.1 and 6.1 we get the following array of equivalences:

$$\begin{aligned} \pi(A) \in \pi\big(\mathrm{Lk}(\{v\})\big) &\iff A \in K \wedge v \notin A \wedge A \cup \{v\} \in K \\ &\iff \pi(A) \in L \wedge \pi(v) \notin \pi(A) \wedge \pi(A) \cup \pi(\{v\}) \in L \\ &\iff \pi(A) \in \mathrm{Lk}\,\big(\pi(\{v\})\big). \end{aligned}$$

Therefore $\pi\big(\mathrm{Lk}(\{v\})\big) = \mathrm{Lk}\,\big(\pi(\{v\})\big)$ which proves that simplicial complexes $\mathrm{Lk}(\{v\})$ and $\mathrm{Lk}\,\big(\pi(\{v\})\big)$ are isomorphic.
($\Leftarrow$) Let $\{v\} \in K$ and $\{w\} \in L$ and let $\pi : S\backslash\{v\} \to S'\backslash\{w\}$ be an isomorphism of complexes $\mathrm{Lk}(\{v\}) \subseteq K$ and $\mathrm{Lk}(\{w\}) \subseteq L$. Then, by Corollary 7.4 we have that $K = \widehat{\mathrm{Lk}(\{v\})}^{S\backslash\{v\}} \cup \big\{A \cup \{v\} \mid A \in \mathrm{Lk}(\{v\})\big\}$ and $L = \widehat{\mathrm{Lk}(\{w\})}^{S'\backslash\{w\}} \cup \big\{A \cup \{w\} \mid A \in \mathrm{Lk}(\{w\})\big\}$.
We define a bijection $\Lambda\pi : S \to S'$ with

$$\Lambda\pi(s) = \begin{cases} \pi(s), & s \neq v, \\ w, & s = v. \end{cases}$$

We will show that $\Lambda\pi$ is an isomorphism of complexes K and L.
Let $A \in K$ be arbitrary. If $A \in \widehat{\mathrm{Lk}(\{v\})}^{S\backslash\{v\}}$ then $(S\backslash\{v\})\backslash A$ is not in $\mathrm{Lk}(\{v\})$. Because π is an isomorphism of $\mathrm{Lk}(\{v\})$ and $\mathrm{Lk}(\{w\})$, we have that $\pi\big((S\backslash\{v\})\backslash A\big) = (S'\backslash\{w\})\backslash\pi(A) \notin \mathrm{Lk}(\{w\})$ implying that $\pi(A) \in \widehat{\mathrm{Lk}(\{w\})}^{S'\backslash\{w\}}$. Since $v \notin A$, we have shown that $\Lambda\pi(A) = \pi(A)$ belongs to the complex L.
If $A \in \big\{A \cup \{v\} \mid A \in \mathrm{Lk}(\{v\})\big\}$ we have $A = B \cup \{v\}$ for some $B \in \mathrm{Lk}(\{v\})$. Then, $\pi(B) \in \mathrm{Lk}(\{w\})$ which proves that simplex $\Lambda\pi(A) = \Lambda\pi(B \cup \{v\}) = \pi(B) \cup \{w\}$ belongs to $\big\{A \cup \{w\} \mid A \in \mathrm{Lk}(\{w\})\big\} \subseteq L$.

Thus, we have shown that $\Lambda\pi$ is a simplicial map of complexes K and L. Similarly, $(\lambda\pi)^{-1}$ is a simplicial map of complexes L and K. $\square$

Theorem 1.4 significantly simplifies the combinatorial classification of self dual complexes. Indeed, in order to establish a combinatorial equivalence one has to find a bijection of ambient sets which satisfies Definition 7.1. If the cardinality of the ambient set is n there are $n!$ such bijections. Since the link of vertex $\{v\}$ does not contain $\{v\}$, Theorem 1.4 reduces the number of potential bijections to $(n-1)!$. Of course, it is sufficient to find a vertex of the given self-dual complex whose link has the smallest number of vertices and compare it to the vertex-link from the second complex with the same number of vertices. This typically simplifies the classification problem.

8. Homology and cohomology of dual upgrades

In this section we study the relationship between the homology and cohomology of a given simplicial complex and its self-dual upgrade, described in Corollary 7.3. For an introduction into simplicial (co)homology theory the reader is referred to [12, Chapters 2 and 3].

The following theorem, originally introduced in [18], is known as *the Combinatorial Alexander Duality*. For a simplified and transparent proof the reader is referred to [4].

Theorem 8.1. *Let K be a simplicial complex in the ambient S where $|S| = n$. Then $H_i(K) = H^{n-3-i}(\widehat{K})$ where H_i and H^j represent the reduced homology and cohomology groups over integers.*

Let K be a simplicial complex in the ambient S. By Example 2.5 we may assume that K is sub-dual in S since sub-duality can be achieved by enlarging the ambient S. Let us consider the dual upgrade $\Lambda K = \widehat{K} \cup \mathrm{C}K$ of the complex K.

By Corollary 7.3 we know that the dual upgrade of a simplicial complex is self-dual in the ambient $S \cup \{v\}$, so as a consequence of Theorem 8.1 we obtain the following corollary.

Corollary 8.2. *Let K be a sub-dual simplicial complex in the ambient S where $|S| = n$. Then, for its dual-upgrade ΛK we have the following relation, $H_i\big(\Lambda(K)\big) = H^{n-i-2}\big(\Lambda(K)\big)$.*

We now turn our attention to the pair $(\Lambda(K), \widehat{K})$. Our goal is to describe the homology of the complex ΛK in terms of the homology of K. From the long exact sequence of reduced homology groups we have:

$$\cdots \to H_k\big(\widehat{K}\big) \xrightarrow{i_*} H_k\big(\Lambda K\big) \xrightarrow{q_*} H_k\big(\Lambda K, \widehat{K}\big) \xrightarrow{\partial} H_{k-1}\big(\widehat{K}\big) \to \cdots \tag{5}$$

Since a simplicial complex and its subcomplex always form a good pair, we know that the group $H_k\big(\Lambda K, \widehat{K}\big)$ is isomorphic to $H_k\big(\Lambda K/\widehat{K}\big)$ where the factor space $\Lambda K/\widehat{K}$ is actually $\big(\widehat{K} \cup \mathrm{C}K\big)/\widehat{K}$. Note that, by Corollary 7.3, $\widehat{K} \cap \mathrm{C}K = K$. So, the factor space $\Lambda K/\widehat{K}$ is homeomorphic to $\mathrm{C}K/K$ and this space is homotopy equivalent to $\mathrm{S}K$, the suspension of the simplicial complex K. It is a well know that the group $H_k(\mathrm{S}K)$ is isomorphic to $H_{k-1}(K)$, hence by replacing $H_k\big(\Lambda(K),\widehat{K}\big)$ by $H_{k-1}(K)$ in (5) we get

$$\cdots \to H_k\big(\widehat{K}\big) \xrightarrow{i_*} H_k\big(\Lambda K\big) \xrightarrow{q'_*} H_{k-1}(K) \xrightarrow{\partial'} H_{k-1}\big(\widehat{K}\big) \to \cdots \tag{6}$$

By Theorem 8.1, and the Universal Coefficient Theorem, connecting the homology and the cohomology of a given simplicial complex, the groups $H_k\big(\widehat{K}\big)$ are easily determined. More explicitly, to each $\mathbb{Z}$ summand of $H_k(K)$ corresponds a $\mathbb{Z}$ summand of $H_{n-3-i}\big(\widehat{K}\big)$ and to each $\mathbb{Z}_p$ summand of $H_k(K)$ there is a corresponding $\mathbb{Z}_p$ summand of $H_{n-4-k}\big(\widehat{K}\big)$.
It follows that for the determination of $H_k\big(\Lambda(K)\big)$ it is sufficient to know the homomorphisms q'_* and ∂'. By construction these homomorphisms are closely related and can be recovered from the homomorphsims q_* and ∂ from (5).
However, there is a more direct and simpler description of q'_* and ∂'. Indeed, let us consider the long exact sequence for a pair $(\widehat{K},K)$.

$$\cdots \to H_k\big(\widehat{K}\big) \xrightarrow{q^o_*} H_k\big(\widehat{K},K\big) \xrightarrow{\partial^o} H_{k-1}(K) \xrightarrow{i^o_*} H_{k-1}\big(\widehat{K}\big) \to \cdots \tag{7}$$

Here, $H_k\big(\widehat{K},K\big)$ is isomorphic to $H_k\big(\widehat{K}/K\big)$ and the factor space $\widehat{K}/K$ has the same homotopy type as $\widehat{K}\cup \mathrm{C}K$, which is precisely ΛK. Therefore, by comparing (6) and (7), we conclude that ∂' is induced by the inclusion $i^0 : K \to \widehat{K}$ and q'_* is induced by the boundary operator ∂^o.

Example 8.3. Let K be a pentagonal cycle shown in Figure 5. Since K is one-dimensional, it is by Proposition 2.3 sub-dual in the ambient [5], so its dual upgrade is a self-dual simplcial complex in the ambient [6]. Since minimal non simplices of K are diagonals of K, the maximal simplexes of $\widehat{K}$ are their complements. Therefore, $\widehat{K}$ is a triangulation of the Möebious band with boundary K, as shown in Figure 5.

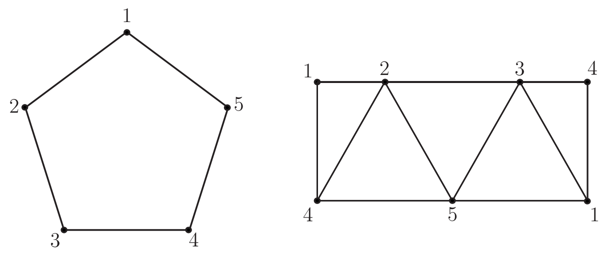

Figure 5: Pentagonal cycle and its dual, the Möebius band.

Since $H_i(K) \approx H_{5-i-3}(\widehat{K})$ and these groups are trivial for $k \neq 1$ and $\mathbb{Z}$ for $k=1$, by using (6) we obtain that $H_k\big(\Lambda(K)\big)=0$ for $k\neq 1$. Since the boundary of $\widehat{K}$ wraps two times around the cycle generating $H_1\big(\widehat{K}\big)$, we conclude that i^o_* is the multiplication by 2. Therefore, the only non-trivial part of (6) is $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{i_*} H_1\big(\Lambda(K)\big) \to 0$. Since i_* is a surjection, the group $H_1\big(\Lambda(K)\big)$ is isomorphic to $\mathrm{Im} i_*/\mathrm{Ker} i_* = \mathrm{Im} i_*/\mathrm{Im} i^o_*$ $= \mathbb{Z}/2\mathbb{Z} \approx \mathbb{Z}_2$.

We conclude that ΛK is a two dimensional simplicial complex on 6 vertices with $\mathbb{Z}_2$ homology in dimension 1, which is in agreement with the fact that it represents a self-dual triangulation of the real projective plane.

This example also shows that (6) can be used to construct self-dual simplicial complexes with prescribed homology groups. Of course, the homology groups need to have the structure described in Theorem 8.1.

THEOREM 8.4. *Let K be a simplicial complex of dimension k in the ambient S where $|S| \geq 2k+3$. Then ΛK has the same homology and cohomology groups as the space $\widehat{K} \vee \mathrm{S}K$ where $\vee$ is the wedge sum of spaces.*

Proof. By Proposition 2.3 the complex K is self dual in the ambient S.
Moreover, since the dimension of K is k, all groups $H_i(K)$ are trivial for $i > k$. Also, if $|S| = n$ and $n \geq 2k+3$, then by Theorem 8.1 and the universal coefficient theorem, the only possibly non trivial homology groups of $\widehat{K}$ are in dimensions $n-3, n-4, \ldots, n-k-3$ (note that $H_k(K)$ torsion-free). Since $n-k-3 \geq 2k+3-k-3 = k$, we conclude that in the long exact sequence (6) for the pair $(\Lambda(K), \widehat{K})$ groups $H_i\big(\widehat{K}\big)$ and $H_{i-1}\big(\widehat{K}\big)$ are trivial or $H_i(K)$ and $H_{i-1}(K)$ are trivial. This implies that $H_i\big(\Lambda K\big)$ is isomorphic to $H_{i-1}(K)$ or $H_i\big(\widehat{K}\big)$, respectively which completes the proof. $\square$

ACKNOWLEDGEMENT. I would like to thank Prof. Rade Živaljević, my Ph.D thesis adviser, for his kind support and encouragement, valuable remarks and useful suggestions.
This research was supported by the Grant 174034 of the Ministry of Education, Science and Technological Development of the Republic of Serbia.

REFERENCES

[1] B. Bagchi, B. Datta, *A short proof of the uniqueness of Kühnel's 9-vertex complex projective plane,* Adv. Geom. **1** (2001), 157–163.

[2] J. Bagchi, B. Datta, *On 9-vertex complex projective plane,* Geom. Dedicata **50** (1994), 1–13.

[3] T.F. Banchoff, W. Kühnel, *The 9-vertex complex projective plane,* Math. Intell. **5(3)** (1983), 11–22.

[4] A. Björner, M. Tancer, *Combinatorial Alexander duality – a short and elementary proof,* Discrete Comput. Geom. **42(4)** (2009), 586–593.

[5] P. Blagojević, F. Frick, G. Ziegler, *Tverberg plus constraints,* Bull. Lond. Math. Soc. **46** (2014), 953–967.

[6] U. Brehm, W. Kühnel, *15-vertex triangulations of an 8-manifold,* Math. Ann. **294** (1992), 167–193.

[7] R. Dedekind, *Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler*, GW **2** (1897), 103–148.

[8] J. Edmonds, D. R. Fulkerson, *Bottleneck Extrema*, J. Combin. Theory **8** (1970), 299–306.

[9] J. Eells and N.H. Kuiper, *Manifolds which are like projective plane*, Publ. Math. I.H.E.S. **14** (1962), 181–222.

[10] P. Galashin, G. Panina, *Simple game induced manifolds*, arXiv:1311.6966 [math.GT].

[11] D. Gorodkov, *A 15-vertex triangulation of the quaternionic projective plane*, arXiv:1603.05541 [math.AT].

[12] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[13] M. Jelić, D. Jojić, M. Timotijević, S.T. Vrećica, R.T. Živaljević, *Combinatorics of unavoidable complexes*, arXiv:1612.09487 [math.AT].

[14] D. Jojić, W. Marzantowicz, S.T. Vrećica, R.T. Živaljević, *Topology and combinatorics of 'unavoidable complexes'*, arXiv:1603.08472 [math.AT].

[15] A. Kisielewicz, *A solution of Dedekind's problem on the number of isotone Boolean functions*, J. Reine Angew. Math. **386** (1988), 139–144.

[16] J. Matoušek, *Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry.* Universitext, Springer-Verlag, Heidelberg, 2003 (Corrected 2nd printing 2008).

[17] S.A. Melikhov, *Combinatorics of Embeddings*, arXiv:1103.5457 [math.AT].

[18] R. P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math. **68** (1982), 175–193.

(received 06.07.2018; in revised form 01.12.2018; available online 09.12.2018)

University of Kragujevac, Faculty of Science, Radoja Domanovića 12, Kragujevac, Serbia
E-mail: timotijevicmarinko@yahoo.com