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# REMARKS ON ALMOST $\eta$-SOLITONS

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**Abstract**. A general definition of almost soliton is considered and the particular cases of almost $\eta$-Ricci, -Einstein and -Yamabe solitons are stretched. Conditions of existence of conjugate solitons are also provided and examples from paracontact geometry are constructed. In the gradient case, two inequalities are deduced and a Bochner-type formula is obtained.

## 1. Introduction

*Solitons* are stationary solutions of geometric flows with applications in different branches of physics. They are basically defined on a pseudo-Riemannian manifold by a vector field (that generates the flow with respect to the metric) and a tensor field that encodes its geometrical meaning. The most studied solitons in Riemannian geometry are Ricci solitons, Einstein solitons and Yamabe solitons, where the Ricci tensor (and the scalar curvature) plays a definitory rôle. Nevertheless, solitons can be considered in a more general context, not necessary Riemannian, by fixing a vector field, a linear connection and an arbitrary tensor field. Precisely, for a couple $(\nabla, J)$ consisting of a linear connection $\nabla$ and a $(1,1)$-tensor field $J$ on a smooth manifold $M$, Crasmareanu introduced in [3] the $(\nabla, J)$-*soliton* as being the data $(\nabla, J, \xi, \lambda)$ satisfying $\nabla\xi + J + \lambda I = 0$, where $\xi$ is a vector field on $M$ and $\lambda$ is a real constant. If the manifold $M$ carries a Riemannian metric $g$ and the vector field $\xi$ is of gradient type, $(\nabla, J, \xi, \lambda)$ is called a *gradient* $(\nabla, J)$-*soliton*.

Generalizing the definition of a $(\nabla, J)$-soliton, Crasmareanu [3] also defined the $(\nabla, J, \eta)$-*soliton* on $(M, g)$ as being the data $(\nabla, J, \xi, \lambda, \mu)$ which satisfy:

$$\nabla\xi + J + \lambda I + \mu\eta \otimes \xi = 0, \tag{1}$$

where $\xi$ is a vector field on $M$, $\eta$ is the $g$-dual 1-form of $\xi$ and $\lambda$, $\mu$ are real constants. If the vector field $\xi$ is of gradient type, $(\nabla, J, \xi, \lambda, \mu)$ is called a *gradient* $(\nabla, J, \eta)$-*soliton*. More general, if $\lambda$ and $\mu$ are smooth functions on $M$, we say that $(\nabla, J, \xi, \lambda, \mu)$ is a

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*(gradient) almost* $(\nabla, J, \eta)$*-soliton* which will be called *steady* if $\lambda = 0$, *shrinking* if $\lambda < 0$ or *expanding* if $\lambda > 0$.

From the soliton equation (1), the relation between the curvature $R_\nabla$ and the torsion $T_\nabla$ of $\nabla$ can be obtained:

$$R_\nabla(\cdot,\cdot)\xi + \lambda T_\nabla(\cdot,\cdot) + d^\nabla J + [d\lambda, I] + ([d\mu, \eta] + \mu d\eta) \otimes \xi + \mu[\eta, J + \lambda I] = 0, \quad (2)$$

where $(d^\nabla J)(X,Y) := (\nabla_X J)Y - (\nabla_Y J)X + J(T_\nabla(X,Y))$ and $[\alpha, T] := \alpha \otimes T - T \otimes \alpha$, for $\alpha$ being a 1-form and $T$ being a $(1,1)$-tensor field. If $\eta$ is closed (in particular, for gradient solitons) and $\nabla$ is a flat (i.e. $R_\nabla = 0$), torsionless (i.e. $T_\nabla = 0$) and $J$-special connection (i.e. $d^\nabla = 0$), from (2) we get $\mu[\eta, J + \lambda I] + [d\lambda, I] + [d\mu, \eta] \otimes \xi = 0$ which in the particular case when $\lambda$ and $\mu$ are constant ($\mu \neq 0$) yields $[\eta, J + \lambda I] = 0$ which is equivalent to $[\eta, \nabla\xi] = 0$.

Examples of almost $(\nabla, J, \eta)$-solitons are provided by semi-symmetric metric connections $\nabla$ [7] which can be expressed in terms of the Levi-Civita connection on $(M, g)$ $\nabla = \nabla^g + I \otimes \eta - g \otimes \xi$, for $\eta$ the $g$-dual 1-form of the vector field $\xi$. In this case, $(\nabla, J, \xi, \lambda, \mu)$ defines a shrinking almost $(\nabla, J, \eta)$-soliton for $J := -\nabla^g\xi$, $\lambda := -|\xi|^2$ and $\mu := 1$.

After a brief description of almost $(\nabla, J, \eta)$-solitons in the special case when the vector field is torse-forming, we give some examples of such solitons on a para-Kenmotsu and para-Sasakian manifold and introduce the *conjugate solitons* specifying existence conditions in almost tangent an almost product geometries. The main results consist in proving a double inequality for the gradient almost $(\nabla^g, J, \eta)$-soliton on a Riemannian manifold $(M, g)$ and deducing a Bochner-type formula for this case, both of the results being followed by some remarks.

## 2. Solitons with torse-forming vector fields. Conjugate solitons

In this section we shall treat the case when the potential vector field $\xi$ of an almost $(\nabla, J, \eta)$-soliton on $(M, g)$ is torse-forming. Also, we shall determine the conditions such that in different geometries (tangent, product and paracontact), the $J$-conjugate connection of $\nabla$ to define an almost soliton, too, which we shall call *conjugate soliton*.

*Torse-forming vector fields* were introduced by Yano [6] and appear in many branches of differential geometry and physics. They are natural generalizations of concircular vector fields. Particular cases of torse-forming vector fields naturally arise in different geometries (para-Kenmotsu, para-Sasaki etc.) where the function $f$ is constant $\pm 1$ and $\pm\gamma$ is the $g$-dual of $\xi$.

Assume that $\xi$ is a torse-forming vector field (i.e. $\nabla\xi = fI + \gamma \otimes \xi$, $f$ a smooth function and $\gamma$ a 1-form on $M$) and remark that:

(i) $(\nabla, 0, \xi, -f, 1)$ defines an almost $(\nabla, 0, -\gamma)$-soliton on $(M, g)$ if and only if $-\gamma$ is the $g$-dual of $\xi$;

(ii) for $J := -(f + \lambda)I - (\gamma + \mu\eta) \otimes \xi$, with $\lambda$ and $\mu$ smooth functions on $M$ and $\eta$ the $g$-dual 1-form of $\xi$, $(\nabla, J, \xi, \lambda, \mu)$ defines an almost $(\nabla, J, \eta)$-soliton on $(M, g)$. In this case $J^2 = (f + \lambda)^2 I + [2(f + \lambda) + \gamma(\xi) + \mu|\xi|^2](\gamma + \mu\eta) \otimes \xi$, hence

$J^2\xi = [f+\lambda+\gamma(\xi)+\mu|\xi|^2]^2\xi$ so $J$ cannot be an almost complex structure.

EXAMPLE 2.1. Let $\nabla$ be a linear connection on the Riemannian manifold $(M,g)$, $\xi$ a vector field on $M$ such that $\nabla\xi = fI + \gamma\otimes\xi$, $f$ a smooth function, $\gamma$ a 1-form on $M$, $J := -(f+\lambda)I - (\gamma+\mu\eta)\otimes\xi$, with $\lambda$ and $\mu$ smooth functions on $M$ and $\eta$ the $g$-dual 1-form of $\xi$.

1. If $J$ is an almost tangent structure on $M$ (i.e. $J^2=0$), then $(\nabla, J, \xi, \lambda, \mu)$ defines an almost $(\nabla, J, \eta)$-soliton on $(M,g)$, for $\lambda = -f-\gamma(\xi)-\mu|\xi|^2$.
2. If $J$ is an almost product structure on $M$ (i.e. $J^2=I$), then $(\nabla, J, \xi, \lambda, \mu)$ defines an almost $(\nabla, J, \eta)$-soliton on $(M,g)$, for $\lambda = -f-\gamma(\xi)-\mu|\xi|^2 \pm 1$.

EXAMPLE 2.2. Let $(M, J, \xi, \eta, g)$ be a para-Kenmotsu manifold [5]. Then $(\nabla^g, J, \xi, -1, \mu)$ and $(\nabla^g, J, \xi, -\mu, \mu)$ with $\mu$ a smooth function on $M$, define almost $(\nabla^g, J, \eta)$-solitons on $(M,g)$.

EXAMPLE 2.3. Let $(M, J, \xi, \eta, g)$ be a para-Sasakian manifold [4]. Then $(\nabla^g, J, \xi, -\mu, \mu)$ with $\mu$ a smooth function on $M$, defines an almost $(\nabla^g, J, \eta)$-soliton on $(M,g)$.

For the almost $(\nabla, J, \eta)$-soliton given by $(\nabla, J, \xi, \lambda, \mu)$, with $\nabla\xi = fI+\gamma\otimes\xi$, computing $\nabla J$ from (1) we obtain:

$$\begin{aligned}(\nabla_X J)Y :=& \nabla_X JY - J(\nabla_X Y) = -X(f+\lambda)Y - f[\gamma(Y)+\mu\eta(Y)]X\\ &- [X(\gamma(Y)) + X(\mu)\eta(Y) + \gamma(X)\gamma(Y) + \mu\gamma(X)\eta(Y)\\ &+ \mu X(\eta(Y)) - \mu\eta(\nabla_X Y) - \gamma(\nabla_X Y)]\xi.\end{aligned}$$

Observe that if $\xi$ is a concurrent vector field (i.e. $f=1$ and $\gamma=0$) and $\nabla$ is a $J$-connection (i.e. $\nabla J = 0$), then $\operatorname{grad}(\lambda+\mu|\xi|^2)$ is $g$-orthogonal to $\xi$. In particular, for $\lambda$ and $\mu$ constants we get either $\mu=0$ or $\xi(|\xi|^2)=0$.

Assume now $\nabla_\xi\xi = 0$ [which is true also for torse-forming vector fields with $f=-\gamma(\xi)$] and let $(\nabla, J, \xi, \lambda, \mu)$ be an almost $(\nabla, J, \eta)$-soliton on $(M,g)$. In this case, $h := -(\lambda+\mu|\xi|^2)$ is an eigenfunction of $J$ corresponding to the eigenvector $\xi$.

Consider the $J$-conjugate connection $\nabla^{(J)}$ of $\nabla$ defined by:

$$\nabla^{(J)}_X Y := \nabla_X Y + J((\nabla_X J)Y) = (I-J^2)(\nabla_X Y) + J(\nabla_X JY).$$

Then

$$\begin{aligned}&\nabla^{(J)}_X \xi - J^3X + (h-\lambda)J^2X + (h\lambda+1)JX + \lambda X\\ &+ [(h^2+1)\mu\eta - hdh](X)\xi - \mu\eta(X)J^2\xi + J(\nabla_X\nabla_\xi\xi) = 0\end{aligned}$$

and the following proposition holds.

PROPOSITION 2.4. *For $(\nabla, J, \xi, \lambda, \mu)$ an almost $(\nabla, J, \eta)$-soliton on $(M,g)$ with $\nabla_\xi\xi = 0$:*

*(i) if $J$ is an almost tangent structure, then $(\nabla^{(J)}, J, \xi, \lambda, \mu)$ defines an almost $(\nabla^{(J)}, J, \eta)$-soliton on $(M,g)$ if and only if $d\lambda = \frac{1-\lambda^2}{|\xi|^2}\eta$ and $\mu = \frac{\lambda^2-1}{\lambda|\xi|^2}$; in particular, if $\lambda$ is constant, then the necessary and sufficient condition is $(\lambda,\mu)\in\{(-1,0),(1,0)\}$;*

*(ii) if $J$ is an almost product structure, then $(\nabla^{(J)}, J, \xi, \lambda, \mu)$ defines an almost $(\nabla^{(J)}, J, \eta)$-soliton on $(M,g)$ if and only if $(\lambda,\mu)\in\{(-1,1),(1,1)\}$.*

COROLLARY 2.5. *Under the hypotheses of Proposition 2.4, there exists no conjugate steady almost $(\nabla^{(J)}, J, \eta)$-solitons.*

Example 2.6. Also, if $(\nabla, J, \xi, \lambda, \mu)$ is an almost $(\nabla, J, \eta)$-soliton on $(M, g)$ with $\nabla_\xi \xi = 0$, different other almost solitons can be constructed if $h := -(\lambda + \mu|\xi|^2)$ satisfies certain conditions.

1. If $J$ is an almost tangent structure, then $(\nabla^{(J)}, J', \xi, \lambda, 1)$ defines an almost $(\nabla^{(J)}, J', \eta')$-soliton on $(M, g)$ if and only if $h \cdot \mathrm{grad}\,(h) = [(1 + h^2)\mu - 1]\xi$, where $J' =: (h\lambda + 1)J$ and $\eta' =: (h^2 + 1)\mu\eta - hdh$;
2. If $J$ is an almost product structure, then $(\nabla^{(J)}, J', \xi, h, 1)$ defines an almost $(\nabla^{(J)}, J', \eta')$-soliton on $(M, g)$ if and only if $h \cdot \mathrm{grad}\,(h) = (h^2\mu - 1)\xi$, where $J' =: h\lambda J$ and $\eta' =: h(h\mu\eta - dh)$.

Remark 2.7. If we consider an almost $(\nabla, J, \eta)$-soliton $(\nabla, J, \xi, \lambda, \mu)$ on $(M, g)$ with $\nabla\xi = -\gamma(\xi)I + \gamma \otimes \xi$, for $\gamma$ a 1-form, then we can check that $(\nabla^{(J)}, 0, \xi, \lambda', \mu')$ is an almost $(\nabla^{(J)}, 0, \eta)$-soliton if and only if $\gamma = \frac{1}{3}\{hdh - [\mu(h^2 + 4) - h - 1]\eta\}$, where $h := -(\lambda + \mu|\xi|^2)$, with $\lambda'$ and $\mu'$ depending on $|\xi|^2$, $\lambda$ and $\mu$.

Example 2.8. Let $(M, J, \xi, \eta, g)$ be an almost paracontact metric manifold [8] and $\nabla$ a linear connection on $M$. If $(\nabla, J, \xi, \lambda, \mu)$ defines an almost $(\nabla, J, \eta)$-soliton on $(M, g)$, then $(\nabla^{(J)}, 0, \xi, 0, \lambda + \mu)$ defines a steady almost $(\nabla^{(J)}, 0, \eta)$-soliton on $(M, g)$.

## 3. Gradient solitons

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. Remark that in the gradient case, for $\xi = \mathrm{grad}\,(u)$ with $u \in C^\infty(M)$, if $\nabla = \nabla^g$ is the Levi-Civita connection of $g$, from the soliton equation (1) we get:

$$\mathrm{Hess}\,(u) + g(\cdot, J\cdot) + \lambda g + \mu du \otimes du = 0 \tag{3}$$

with $g(\cdot, J\cdot) =: \Omega$ a symmetric $(0, 2)$-tensor field, therefore the gradient solitons will be considered only for tensor fields $J$ with $g(JX, Y) = g(X, JY)$.

Remark 3.1. If $(\nabla^g, J, \xi, \lambda, \mu)$ defines a gradient almost $(\nabla^g, J, \eta)$-soliton on $(M, g)$, then it is a gradient almost $\eta$-Ricci soliton if $J := Q$ with $Q$ the Ricci operator $g(QX, Y) := Ric(X, Y)$, a gradient almost $\eta$-Einstein soliton if $J := Q - \frac{\mathrm{scal}}{2} \cdot I$ and a gradient almost $\eta$-Yamabe soliton if $J := -\mathrm{scal} \cdot I$.

In the geometry of gradient almost solitons two inequalities will be further obtained inspired by the inequalities in the gradient Ricci soliton case discussed by M. Crasmareanu in [3], but using a slightly different argument.

Theorem 3.2. *If* (3) *defines a gradient almost* $(\nabla^g, J, \eta)$*-soliton on the* $n$*-dimensional Riemannian manifold* $(M, g)$ *and* $\eta = du$ *is the* $g$*-dual of the gradient vector field* $\xi := \mathrm{grad}\,(u)$*, then:*

$$\begin{aligned}|\mathrm{Hess}(u)|^2 + \mu^2|\xi|^4 + \mu\xi(|\xi|^2) - \frac{(\Delta(u) + \mu|\xi|^2)^2}{n} &\leq |\Omega|^2 \\ &\leq |\mathrm{Hess}(u)|^2 + \mu^2|\xi|^4 + \mu\xi(|\xi|^2) + \frac{(\mathrm{trace}(\Omega))^2}{n}.\end{aligned} \tag{4}$$

*Proof.* From (3) we obtain:

$$|\mathrm{Hess}\,(u)|^2 = |\Omega|^2 + \lambda^2 n + 2\lambda \mathrm{trace}\,(\Omega) - \mu^2|\xi|^4 - \mu\xi(|\xi|^2)$$

and $$|\Omega|^2 = |\mathrm{Hess}\,(u)|^2 + \lambda^2 n + 2\lambda(\Delta(u) + \mu|\xi|^2) + \mu^2|\xi|^4 + \mu\xi(|\xi|^2). \tag{5}$$

Note that the conditions for an existence of a solution (in $\lambda$) are:

$$(\mathrm{trace}\,(\Omega))^2 - n[|\Omega|^2 - |\mathrm{Hess}\,(u)|^2 - \mu^2|\xi|^4 - \mu\xi(|\xi|^2)] \geq 0$$

and $$(\Delta(u) + \mu|\xi|^2)^2 - n[|\mathrm{Hess}\,(u)|^2 - |\Omega|^2 + \mu^2|\xi|^4 + \mu\xi(|\xi|^2)] \geq 0,$$

which just imply the double inequality from the conclusion. □

REMARK 3.3. (i) In [1] we obtained the corresponding inequalities for the particular case of gradient almost $\eta$-Ricci soliton, which generalizes the case of gradient Ricci soliton treated by Crasmareanu in [3]. A similar estimation holds for gradient almost $\eta$-Einstein solitons [1]: the lower and the upper bound of $|\mathrm{Ric}|^2$ are the lefthand side of (4), but in the righthand side term of (4), $\mu\cdot\mathrm{scal}\cdot|\xi|^2$ will be supplementary added.
(ii) If $\xi$ is of constant length, $|\xi|^2 =: k$, then (4) simplifies to:

$$|\mathrm{Hess}\,(u)|^2 + \mu^2k^2 - \frac{(\Delta(u) + \mu k)^2}{n} \leq |\Omega|^2 \leq |\mathrm{Hess}\,(u)|^2 + \mu^2k^2 + \frac{(\mathrm{trace}\,(\Omega))^2}{n}.$$

(iii) The simultaneous equalities hold for $(\mathrm{trace}\,(\Omega))^2 = -(\Delta(u) + \mu|\xi|^2)^2$ $(= 0)$ i.e. for steady gradient almost soliton ($\lambda = 0$) with $\mathrm{trace}\,(\Omega) = 0$ and $\Delta(u) = -\mu|\xi|^2$. In this case, if $|\xi|^2 =: k$ is constant, then $|\Omega|^2 = |\mathrm{Hess}\,(u)|^2 + \mu^2k^2$.

A Bochner-type formula will be obtained for a gradient almost $(\nabla^g, J, \eta)$-soliton.

THEOREM 3.4. *If (3) defines a gradient almost $(\nabla^g, J, \eta)$-soliton on the $n$-dimensional Riemannian manifold $(M, g)$ and $\eta = du$ is the $g$-dual of the gradient vector field $\xi := \mathrm{grad}\,(u)$, then:*

$$\frac{1}{2}(\Delta + \mu\nabla^g_\xi)(|\xi|^2) = |\mathrm{Hess}\,(u)|^2 - \xi(\lambda) - \mu\Delta(u)\xi(\mu) - \mathrm{div}\,(\Omega)(\xi). \tag{6}$$

*Proof.* First note that $\mathrm{trace}\,(\mu\eta\otimes\eta) = \mu|\xi|^2$ and $\mathrm{div}\,(\mu\eta\otimes\eta) = \frac{\mu}{2}d(|\xi|^2) + \mu\Delta(u)du + d\mu(\xi)du$. Taking the trace of the equation (3), we obtain $\Delta(u) + \mathrm{trace}\,(\Omega) + n\lambda + \mu|\xi|^2 = 0$ and by differentiating it:

$$d(\Delta(u)) + d(\mathrm{trace}\,(\Omega)) + nd\lambda + \mu d(|\xi|^2) + |\xi|^2 d\mu = 0. \tag{7}$$

Now taking the divergence of the same equation, we get:

$$\mathrm{div}\,(\mathrm{Hess}\,(u)) + \mathrm{div}\,(\Omega) + d\lambda + \frac{\mu}{2}d(|\xi|^2) + \mu\Delta(u)du + d\mu(\xi)du = 0. \tag{8}$$

We obtain (6) by subtracting the relations (8) and (7) computed in $\xi$ and using [2]:

$$\mathrm{div}\,(\mathrm{Hess}\,(u)) = d(\Delta(u)) + i_{Q\xi}g,$$

$$(\mathrm{div}\,(\mathrm{Hess}\,(u)))(\xi) = \frac{1}{2}\Delta(|\xi|^2) - |\mathrm{Hess}\,(u)|^2.$$ □

REMARK 3.5. Denoting the diffusion operator by $\Delta_u := \Delta - \nabla^g_\xi$, for $\mu = -1$ in Theorem 3.4, we get $\frac{1}{2}\Delta_u(|\xi|^2) = |\mathrm{Hess}\,(u)|^2 - \xi(\lambda) - \mathrm{div}\,(\Omega)(\xi)$. Under the assumption $\mathrm{div}\,(\Omega)(\xi) \leq -\xi(\lambda)$ we get $\Delta_u(|\xi|^2) \geq 0$ and from the maximum principle it follows that $|\xi|^2$ is constant in a neighborhood of any local maximum. If $|\xi|$ achieves its maximum, then $\operatorname{div}(\Omega)(\xi) = -\xi(\lambda)$ and $\operatorname{Hess}(u) = 0$ which implies from the soliton equation that $\Omega = -\lambda g + du \otimes du$. Therefore, $\operatorname{div}(\Omega) = -d\lambda$ and $\xi(\lambda) = 0$.

REMARK 3.6. For $M$ compact, $\xi$ of constant length $|\xi|^2 =: k$ and $\lambda$ and $\mu$ real constants, from (6) we get $\operatorname{div}(\Omega)(\xi) = |\operatorname{Hess}(u)|^2$ and by integrating (5):

$$\int_M |\Omega|^2 = \int_M |\operatorname{Hess}(u)|^2 + [(n-1)\lambda^2 + (\lambda + \mu k)^2] \cdot \operatorname{vol}(M) \geq \int_M \operatorname{div}(\Omega)(\xi).$$

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