

**STRONG CONVERGENCE OF AN INERTIAL-TYPE ALGORITHM
TO A COMMON SOLUTION OF MINIMIZATION AND FIXED
POINT PROBLEMS**

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Abstract. In this paper, we introduce an inertial accelerated iterative algorithm for approximating a common solution of a minimization problem and a fixed point problem for quasi-pseudocontractive mapping in a real Hilbert space. Using the algorithm, we prove a strong convergence theorem for approximating a common solution of a minimization problem and a fixed point problem for quasi-pseudocontractive mapping. Furthermore, we give an application of our main result to solve convexly constrained linear inverse problems, and we also present a numerical example of our algorithm to illustrate its applicability.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H . We shall denote by $F(T)$ the set of fixed points of T .

A nonlinear mapping $T : C \rightarrow C$ is said to be

(i) Nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

(ii) Quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\|, \forall x \in C \text{ and } x^* \in F(T).$$

(iii) Strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \forall x, y \in C.$$

(iv) Demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2, \forall x \in C \text{ and } x^* \in F(T).$$

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DEFINITION 1.1. A mapping $T : C \rightarrow C$ is said to be quasi-pseudocontractive if $F(T) \neq \emptyset$ and $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2 \forall x \in C$ and $x^* \in F(T)$.

DEFINITION 1.2. A mapping $T : C \rightarrow C$ is said to be L -Lipschitzian if there exists some $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in C$.

It is clear that the class of quasi-pseudocontractive mappings includes the class of demicontractive mappings which contains the class of nonexpansive mappings (with nonempty fixed point set) and quasi-nonexpansive mappings.

One of the most important problems in optimization theory and nonlinear analysis is the problem of approximating solutions of Minimization Problem (MP) which is defined as follows: Find $x \in H$ such that

$$f(x) = \min_{y \in H} f(y), \tag{1}$$

where $f : H \rightarrow (-\infty, \infty]$ is a proper and convex function. We denote by $\operatorname{argmin}_{y \in H} f(y)$ the set of all minimizers of f on H . MP is very useful in game theory, convex and nonlinear analysis. Thus, it has attracted the interest of many researchers who have used different methods to solve problem (1) successfully. For instance, in 1970, Martinet [13] introduced the well known Proximal Point Algorithm (PPA) which is a powerful and one of the most popular tools for solving problem (1). Rockafellar [15] further developed the study of the PPA in Hilbert spaces for approximating solutions of problem (1) as follows: Let f be a proper, convex and lower semi-continuous function defined on a real Hilbert space H ; then the PPA is defined for arbitrary $x_1 \in H$ by

$$x_{n+1} = \operatorname{Prox}_{\lambda_n} f(x_n), \forall n \geq 1, \tag{2}$$

where $\lambda_n > 0$ for all $n \geq 1$, and $\operatorname{Prox}_{\lambda} f : H \rightarrow H$ is the Moreau-Yosida resolvent of f in H (also called the proximal operator of f) defined by

$$\operatorname{Prox}_{\lambda} f(x) = \operatorname{argmin}_{y \in H} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

Rockafellar [15] proved that the PPA (2) converges weakly to a minimizer of f provided that the minimizer of f exists and that $\sum_{n=1}^{\infty} \lambda_n = \infty$. He then posed an important question as to whether the PPA converges strongly or not. This question was resolved in the negative by Güler [8] who constructed a counter example showing that the PPA does not necessarily converge strongly. In order to obtain strong convergence of the PPA, Kamimura and Takahashi [9] modified the PPA into Halpern-type PPA so that its strong convergence is guaranteed. Since then, many other authors have also studied different modification of the PPA (see for example [18] and the references therein).

In fixed point theory, constructing iterative schemes with speedy rate of convergence is usually of great interest. For this purpose, Polyak [14] proposed an inertial accelerated extrapolation process to solve the smooth convex minimization problem. Since then, there are growing interests by authors working in this direction. Some latest contributions are: inertial proximal point method [1], modified inertial Mann algorithm and inertial CQ algorithm [4], inertial viscosity method [16], and others.

Most of this works done using the inertial iterative scheme only yielded weak convergence results. For example, in 2007, Mainge [12] introduced the inertial Mann algorithm for solving the fixed point problem for nonexpansive mappings in Hilbert spaces as follows: Take $x_0, x_1 \in H$ and generate the sequence $\{x_n\}$ by

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= y_n + \alpha_n(Ty_n - y_n), \quad n \geq 1, \end{aligned} \tag{3}$$

where T is a nonexpansive mapping on H , $\theta_n \in [0, 1)$ and $\alpha_n \in (0, 1)$. He proved that the sequence generated by iterative scheme (3) converges weakly to a fixed point of T under the following conditions:

- (i) $\theta_n \in [0, \theta)$ where $\theta \in [0, 1)$;
- (ii) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 \leq \infty$;
- (iii) $0 < \inf_{n \geq 1} \alpha_n \leq \sup_{n \geq 1} \alpha_n < 1$.

Here, the inertial is represented by the term $\theta_n(x_n - x_{n-1})$ which is a remarkable tool for improving the performance of algorithms and has some nice convergence properties.

Having reviewed the literature, we noticed that strong convergence results have been proved by very few authors. In this case, the authors employed the inertial type algorithms that require at each step of the iteration process, the computation of the two subsets C_n and Q_n , the computation of their intersection $C_n \cap Q_n$ and the computation of the projection of the initial vector onto this intersection. For example, in 2018, Dong et. al. [4] studied the inertial CQ algorithm for nonexpansive mappings in the framework of a real Hilbert space and proved the following strong convergence result.

THEOREM 1.3 ([4]). *Let $T : H \rightarrow H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^{\infty} \subset [\alpha_1, \alpha_2]$, $\alpha_1 \in (-\infty, 0]$, $\alpha_2 \in [0, \infty)$, $\{\beta_n\}_{n=0}^{\infty} \subset [\beta, 1]$, $\beta \in (0, 1)$. Set $x_0, x_1 \in H$ arbitrarily and define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ y_n = (1 - \beta_n)w_n + \beta_n T w_n; \\ C_n = \{z \in H : \|y_n - z\| \leq \|w_n - z\|\}; \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0 \rangle \leq 0\}; \\ x_{n+1} = P_{C_n \cap Q_n} x_0; \end{cases}$$

for each $n \geq 0$. Then the iterative sequence $\{x_n\}$ converges in norm to $P_{F(T)}(x_0)$.

In general, algorithms that do not involve the construction of C_n and Q_n are more desirable and interesting since they are easy to compute than those that involve these constructions. Thus, it is of practical computational importance to study the strong convergence of inertial type algorithms that do not involve any of the above mentioned computations at each step of the iteration process. In view of this, we introduce an inertial accelerated iterative scheme that does not require construction of any of the subsets used in [4, 5, 16, 17], and prove a strong convergence theorem for approximating a common solution of a minimization problem and a fixed point

problem for quasi-pseudo-contractive mapping in the framework of real Hilbert spaces. Our results extend and complement the results of Suantai et. al. [16], Mainge [12], Dong et. al. [4,5] and a host of other results in this direction.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by “ \rightarrow ” and “ \rightharpoonup ”, respectively and our solution set by $\Theta := F(T) \cap \operatorname{argmin}_{y \in H} f(y)$. The following equality holds in Hilbert spaces.

$$\langle x, y \rangle = \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|x - y\|^2, \quad \forall x, y \in H. \tag{4}$$

DEFINITION 2.1. Let H be a real Hilbert space and C be a nonempty closed and convex subset of H . A mapping $T : C \rightarrow C$ is said to be demiclosed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to x with $\|x_n - Tx_n\| = 0, Tx = x$ holds.

LEMMA 2.2. Let H be a real Hilbert space. Then $\forall x, y \in H$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\|^2 &= \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2; \\ \|x + y\|^2 &\leq \|x\|^2 + 2\langle y, x + y \rangle. \end{aligned}$$

LEMMA 2.3 ([3]). Let H be a real Hilbert space and $T : H \rightarrow H$ be an L -Lipschitzian mapping with $L \geq 1$. Denote $K := (1 - \xi)I + \xi T((1 - \eta)I + \eta T)$ if $0 < \xi < \eta < 1/(1 + \sqrt{1 + L^2})$. Then the following conclusions hold:

- 1) $F(T) = F(T((1 - \eta)I + \eta T)) = F(K)$;
- 2) If T is demiclosed at 0, then K is also demiclosed at 0;
- 3) In addition, if $T : H \rightarrow H$ is quasi-pseudocontractive, then the mapping K is quasi-nonexpansive.

LEMMA 2.4 ([10]). Let H be a real Hilbert space and $f : H \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, we have

$$\frac{1}{2\lambda} \|\operatorname{Prox}_\lambda f(x) - y\|^2 - \frac{1}{2\lambda} \|x - y\|^2 + \frac{1}{2\lambda} \|x - \operatorname{Prox}_\lambda f(x)\|^2 + f(\operatorname{Prox}_\lambda f(x)) \leq f(y).$$

LEMMA 2.5 ([20]). Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n + \gamma_n, n \geq 0$, where $\{\alpha_n\}, \{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \subset [0, 1], \sum_{n=0}^\infty \alpha_n = \infty$,
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$,
- (iii) $\gamma_n \geq 0(n \geq 0), \sum_{n=0}^\infty \gamma_n < \infty$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main results

LEMMA 3.1. *Let H be a real Hilbert space and $f : H \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $0 < \lambda < \mu$ and $x \in H$, we have*

$$\|\text{Prox}_{\lambda f}(x) - x\| \leq \|\text{Prox}_{\mu f}(x) - x\|.$$

Proof. For $x, y \in H$, we obtain from the definition of the resolvent of f that

$$f(\text{Prox}_{\mu f}(x)) + \frac{1}{2\mu} \|\text{Prox}_{\mu f}(x) - x\|^2 \leq f(y) + \frac{1}{2\mu} \|y - x\|^2.$$

In particular, we have that

$$f(\text{Prox}_{\mu f}(x)) + \frac{1}{2\mu} \|\text{Prox}_{\mu f}(x) - x\|^2 \leq f(\text{Prox}_{\lambda f}(x)) + \frac{1}{2\mu} \|\text{Prox}_{\lambda f}(x) - x\|^2. \quad (5)$$

Similarly, we obtain

$$f(\text{Prox}_{\lambda f}(x)) + \frac{1}{2\lambda} \|\text{Prox}_{\lambda f}(x) - x\|^2 \leq f(\text{Prox}_{\mu f}(x)) + \frac{1}{2\lambda} \|\text{Prox}_{\mu f}(x) - x\|^2. \quad (6)$$

Adding (5) and (6), we obtain that

$$\|\text{Prox}_{\lambda f}(x) - x\|^2 - \frac{\lambda}{\mu} \|\text{Prox}_{\lambda f}(x) - x\|^2 \leq \|\text{Prox}_{\mu f}(x) - x\|^2 - \frac{\lambda}{\mu} \|\text{Prox}_{\mu f}(x) - x\|^2,$$

that is,

$$\left(1 - \frac{\lambda}{\mu}\right) \|\text{Prox}_{\lambda f}(x) - x\|^2 \leq \left(1 - \frac{\lambda}{\mu}\right) \|\text{Prox}_{\mu f}(x) - x\|^2.$$

Since $0 < \lambda < \mu$, then $1 - \frac{\lambda}{\mu} > 0$. Thus, we obtain that

$$\|\text{Prox}_{\lambda f}(x) - x\| \leq \|\text{Prox}_{\mu f}(x) - x\|. \quad \square$$

THEOREM 3.2. *Let H be a real Hilbert space and $f : H \rightarrow (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let $T : H \rightarrow H$ be an L -Lipschitzian and quasi-pseudocontractive mapping with $L \geq 1$ such that T is demiclosed at 0. Suppose that $\Theta := F(T) \cap \text{argmin}_{y \in H} f(y)$ is nonempty; then the sequence $\{x_n\}$ generated iteratively for an arbitrary $x_0, x_1, u \in H$ by*

$$\begin{cases} u_n = \text{Prox}_{\lambda_n f}(x_n + \theta_n(x_n - x_{n-1})), \\ w_n = \alpha_n u_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n T((1 - \eta_n)I + \eta_n T))u_n, \\ x_{n+1} = (1 - \beta_n)w_n + \beta_n u, \quad n \geq 1, \end{cases} \quad (7)$$

converges strongly to a point $p = P_{\Theta}u$, where P_{Θ} is the metric projection of H onto Θ , $\lambda_n > \lambda > 0$, $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
- (ii) $0 < a < \xi_n < \eta_n < b < 1/(1 + \sqrt{1 + L^2}) \forall n \geq 1$;
- (iii) $0 < \liminf \alpha_n \leq \limsup \alpha_n < 1$;
- (iv) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$.

Proof. Let $p = P_{\Theta}u$, $y_n = x_n + \theta_n(x_n - x_{n-1})$ and $K_n := (1 - \alpha_n)((1 - \xi_n)I + \xi_n T((1 - \eta_n)I + \eta_n T)) \forall n \geq 1$. Then, from (7), Lemma 2.2 and Lemma 2.3, we obtain that

$$\begin{aligned} \|w_n - p\|^2 &= \|\alpha_n u_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n T((1 - \eta_n)I + \eta_n T))u_n - p\|^2 \\ &= \|\alpha_n(u_n - p) + (1 - \alpha_n)(K_n u_n - p)\|^2 \\ &= \alpha_n \|u_n - p\|^2 + (1 - \alpha_n) \|K_n u_n - p\|^2 - \alpha_n(1 - \alpha_n) \|K_n u_n - u_n\|^2 \\ &\leq \|u_n - p\|^2 - \alpha_n(1 - \alpha_n) \|K_n u_n - u_n\|^2 \leq \|\text{Prox}_{\lambda_n f} y_n - p\|^2 \leq \|y_n - p\|^2. \end{aligned} \tag{8}$$

Since $y_n = x_n + \theta_n(x_n - x_{n-1})$, we have that

$$\begin{aligned} \|y_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| = \|(x_n - p) + \theta_n(x_n - x_{n-1})\| \\ &\leq \|x_n - p\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{9}$$

Using (8) and (9), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \beta_n)(w_n - p) + \beta_n(u - p)\| \leq (1 - \beta_n) \|w_n - p\| + \beta_n \|u - p\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|u - p\| \\ &= (1 - \beta_n) [\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|] + \beta_n \|u - p\| \\ &= (1 - \beta_n) \|x_n - p\| + \theta_n(1 - \beta_n) \|x_n - x_{n-1}\| + \beta_n \|u - p\| \\ &\leq \max\{\|x_n - p\|, \|u - p\|\} + \theta_n \|x_n - x_{n-1}\| \\ &\leq \max\{\max\{\|x_{n-1} - p\|, \|u - p\|\} + \theta_{n-1} \|x_{n-1} - x_{n-2}\|, \|u - p\|\} + \theta_n \|x_n - x_{n-1}\| \\ &= \max\{\|x_{n-1} - p\|, \|u - p\|\} + \theta_{n-1} \|x_{n-1} - x_{n-2}\| + \theta_n \|x_n - x_{n-1}\| \end{aligned}$$

Let $M = \sum_{i=1}^n \theta_i \|x_i - x_{i-1}\| < \infty$. Since $\sum_{i=1}^n \theta_i \|x_i - x_{i-1}\| < \infty$, we have that $\|x_{n+1} - p\| \leq \max\{\|x_1 - p\|, \|u - p\|\} + M$. Therefore $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ and $\{w_n\}$ are bounded. From (7) and (4), we obtain

$$\begin{aligned} \|y_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\ &= \|x_n - p\|^2 + \theta_n (-\|x_{n-1} - p\|^2 + \|x_n - p\|^2 + \|x_n - x_{n-1}\|^2) + \theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - p\|^2 + \theta_n (\|x_n - p\|^2 - \|x_{n-1} - p\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2. \end{aligned} \tag{10}$$

From (7), (8), (10) and Lemma 2.3, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \beta_n)(w_n - p) + \beta_n(u - p)\|^2 \\ &\leq (1 - \beta_n) \|w_n - p\|^2 + \beta_n \|u - p\|^2 + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &\leq (1 - \beta_n) [\|u_n - p\|^2 - \alpha_n(1 - \alpha_n) \|K_n u_n - u_n\|^2] + \beta_n \|u - p\|^2 \\ &\quad + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &= (1 - \beta_n) \|u_n - p\|^2 - \alpha_n(1 - \alpha_n)(1 - \beta_n) \|K_n u_n - u_n\|^2 + \beta_n \|u - p\|^2 \\ &\quad + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &\leq (1 - \beta_n) \|y_n - p\|^2 - \alpha_n(1 - \alpha_n)(1 - \beta_n) \|K_n u_n - u_n\|^2 + \beta_n \|u - p\|^2 \\ &\quad + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &\leq (1 - \beta_n) [\|x_n - p\|^2 + \theta_n (\|x_n - p\|^2 - \|x_{n-1} - p\|^2) + 2\theta_n \|x_n - x_{n-1}\|^2] \end{aligned}$$

$$\begin{aligned}
& -\alpha_n(1-\alpha_n)(1-\beta_n)\|K_n u_n - u_n\|^2 + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle \\
& = (1-\beta_n)\|x_n - p\|^2 + (1-\beta_n)\theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) \\
& \quad + 2(1-\beta_n)\theta_n\|x_n - x_{n-1}\|^2 - \alpha_n(1-\alpha_n)(1-\beta_n)\|K_n u_n - u_n\|^2 \\
& \quad + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle. \tag{11}
\end{aligned}$$

Now set $\Gamma_n := \|x_n - p\|^2$ for all $n \in \mathbb{N}$. We now consider the following two cases.

Case 1: Suppose that there exists a natural number \mathbb{N} such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq \mathbb{N}$. In this case, $\{\Gamma_n\}$ is convergent. Therefore, we have from (11) that

$$\begin{aligned}
& \alpha_n(1-\alpha_n)(1-\beta_n)\|K_n u_n - u_n\|^2 \leq (1-\beta_n)\Gamma_n - \Gamma_{n+1} + (1-\beta_n)\theta_n(\Gamma_n - \Gamma_{n-1}) \\
& \quad + 2(1-\beta_n)\theta_n\|x_n - x_{n-1}\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle + \beta_n\|u - p\|^2
\end{aligned}$$

From conditions (i), (ii) and (iv) of Theorem 3.2, we obtain that

$$\lim_{n \rightarrow \infty} \|K_n u_n - u_n\| = 0. \tag{12}$$

Also from (7), $\|w_n - u_n\| = \|\alpha_n u_n + (1-\alpha_n)K_n u_n - u_n\| \leq (1-\alpha_n)\|K_n u_n - u_n\|$, which implies from (12) that

$$\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0. \tag{13}$$

From Lemma 2.4, we obtain that

$$\frac{1}{2\lambda_n}\|u_n - p\|^2 + \frac{1}{2\lambda_n}\|x_n - p\|^2 + \frac{1}{2\lambda_n}\|x_n - u_n\|^2 \leq f(y) - f(u_n).$$

Since $f(p) \leq f(u_n)$ for all $n \geq 1$, we obtain

$$\|u_n - p\|^2 \leq \|y_n - p\|^2 - \|y_n - u_n\|^2. \tag{14}$$

From (7), (10) and (14), we have

$$\begin{aligned}
& \|x_{n+1} - p\|^2 = \|(1-\beta_n)(w_n - p) + \beta_n(u - p)\|^2 \\
& \leq (1-\beta_n)\|w_n - p\|^2 + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle \\
& \leq (1-\beta_n)\|u_n - p\|^2 + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle \\
& \leq (1-\beta_n)[\|y_n - p\|^2 - \|y_n - u_n\|^2] + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle \\
& \leq (1-\beta_n)[\|x_n - p\|^2 + \theta_n(\|x_n - p\|^2 - \|x_{n-1} - p\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2 - \|y_n - u_n\|^2] \\
& \quad + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
& (1-\beta_n)\|y_n - u_n\|^2 \leq (1-\beta_n)\Gamma_n - \Gamma_{n+1} + (1-\beta_n)\theta_n(\Gamma_n - \Gamma_{n-1}) \\
& \quad + 2(1-\beta_n)\theta_n\|x_n - x_{n-1}\|^2 + \beta_n\|u - p\|^2 + 2\beta_n(1-\beta_n)\langle w_n - p, u - p \rangle.
\end{aligned}$$

Thus, from conditions (i) and (iv) of Theorem 3.2, we obtain

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \tag{15}$$

Also, we obtain from condition (iv) of Theorem 3.2 that

$$\|y_n - x_n\| = \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| = \theta_n\|x_n - x_{n-1}\| \rightarrow 0, \quad n \rightarrow \infty. \tag{16}$$

From (15) and (16), we obtain

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{17}$$

From (13) and (17), we obtain

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \tag{18}$$

Again, from (7), we have $\|x_{n+1} - w_n\| = \|(1 - \beta_n)w_n + \beta_n u - w_n\| = \beta_n \|u - w_n\|$, which implies from condition (i) of Theorem 3.2 that $\lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| = 0$. Thus, we obtain from (18) that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Since $\lambda_n > \lambda > 0$, we obtain from Lemma 3.1 that $\|\text{Prox}_{\lambda f} y_n - y_n\| \leq \|\text{Prox}_{\lambda_n f} y_n - y_n\|$, which implies from (15) that

$$\lim_{n \rightarrow \infty} \|\text{Prox}_{\lambda f} y_n - y_n\| = 0. \tag{19}$$

Since $\{x_n\}$ is bounded, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to x^* . It follows from (17) and (16) that subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ converge weakly to x^* respectively. Since T is demiclosed at 0, we obtain from Lemma 2.3 and (12) that $x^* \in F(T)$. Also, since $\text{Prox}_{\lambda f}$ is nonexpansive, it follows from the demiclosedness principle, and (19) that $x^* \in F(\text{Prox}_{\lambda f})$. Therefore, $x^* \in \Theta$.

Now, we need to show that $\limsup_{n \rightarrow \infty} \langle w_n - p, u - p \rangle \leq 0$. By (18), there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^*$. Thus, we obtain that

$$\limsup_{n \rightarrow \infty} \langle w_n - p, u - p \rangle = \lim_{k \rightarrow \infty} \langle w_{n_k} - p, u - p \rangle = \langle x^* - p, u - p \rangle \leq 0. \tag{20}$$

Now, from (7), (8) and (8), we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \beta_n)(w_n - p) + \beta_n(u - p)\|^2 \\ &\leq (1 - \beta_n)^2 \|w_n - p\|^2 + \beta_n^2 \|u - p\|^2 + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &\leq (1 - \beta_n) \|y_n - p\|^2 + \beta_n^2 \|u - p\|^2 + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &\leq (1 - \beta_n) (\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|)^2 + \beta_n^2 \|u - p\|^2 + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &\leq (1 - \beta_n) (\|x_n - p\|^2 + 2\theta_n \|x_n - x_{n-1}\| \|x_n - p\| + \theta_n^2 \|x_n - x_{n-1}\|^2) \\ &\quad + \beta_n^2 \|u - p\|^2 + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle. \end{aligned} \tag{21}$$

Put $M_1 = \sup_{n \in \mathbb{N}} \|x_n - p\|$, then (21) implies

$$\begin{aligned} \Gamma_{n+1} &\leq (1 - \beta_n) \Gamma_n + 2\theta_n \|x_n - x_{n-1}\| M_1 + \theta_n \|x_n - x_{n-1}\|^2 + \beta_n^2 \|u - p\|^2 \\ &\quad + 2\beta_n(1 - \beta_n) \langle w_n - p, u - p \rangle \\ &= (1 - \beta_n) \Gamma_n + \beta_n (\beta_n \|u - p\|^2 - 2(1 - \beta_n) \langle w_n - p, u - p \rangle) \\ &\quad + 2\theta_n \|x_n - x_{n-1}\| M_1 + \theta_n \|x_n - x_{n-1}\|^2. \end{aligned}$$

By applying Lemma 2.5, (20) and conditions (i) and (iv) of Theorem 3.2, we conclude that $\Gamma_n = \|x_n - p\|^2 \rightarrow 0$ and hence $x_n \rightarrow p$ as $n \rightarrow \infty$.

Case 2: Assume that $\{\|x_n - p\|\}$ is not monotonically decreasing sequence. Let $\tau : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by $\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}$. Clearly, τ is a non-decreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$. It follows from (11) that

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \leq \|x_{\tau(n)+1} - p\|^2 - (1 - \beta_{\tau(n)}) \|x_{\tau(n)} - p\|^2 \\ &\leq (1 - \beta_{\tau(n)}) \theta_{\tau(n)} (\|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)-1} - p\|^2) + 2(1 - \beta_{\tau(n)}) \theta_{\tau(n)} \|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \end{aligned}$$

$$-\alpha_{\tau(n)}(1-\alpha_{\tau(n)})(1-\beta_{\tau(n)})\|K_{\tau(n)}u_{\tau(n)}-u_{\tau(n)}\|^2 + \beta_{\tau(n)}\|u-p\|^2 + 2\beta_{\tau(n)}(1-\beta_{\tau(n)})\langle w_{\tau(n)}-p, u-p \rangle.$$

That is,

$$\begin{aligned} & \alpha_{\tau(n)}(1-\alpha_{\tau(n)})(1-\beta_{\tau(n)})\|K_{\tau(n)}u_{\tau(n)}-u_{\tau(n)}\|^2 \\ \leq & (1-\beta_{\tau(n)})\theta_{\tau(n)}(\|x_{\tau(n)}-p\|^2 - \|x_{\tau(n)-1}-p\|^2) + 2(1-\beta_{\tau(n)})\theta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|^2 \\ & + \beta_{\tau(n)}\|u-p\|^2 + 2\beta_{\tau(n)}(1-\beta_{\tau(n)})\langle w_{\tau(n)}-p, u-p \rangle \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

By the same argument as in (12) to (15) in **Case 1**, we conclude that $\{x_{\tau(n)}\}, \{u_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ converge weakly to $p \in \Theta$. Now for all $n \geq n_0$

$$\begin{aligned} 0 \leq & \|x_{\tau(n)+1}-p\|^2 - \|x_{\tau(n)}-p\|^2 \leq (1-\beta_{\tau(n)})\|x_{\tau(n)}-p\|^2 \\ & + (1-\beta_{\tau(n)})[\theta_{\tau(n)}(\|x_{\tau(n)}-p\|^2 - \|x_{\tau(n)-1}-p\|^2) + 2\theta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|^2] \\ & + \beta_{\tau(n)}\|u-p\|^2 + 2\beta_{\tau(n)}(1-\beta_{\tau(n)})\langle w_{\tau(n)}-p, u-p \rangle - \|x_{\tau(n)}-p\|^2 \\ = & (1-\beta_{\tau(n)})[\theta_{\tau(n)}(\Gamma_{\tau(n)} - \Gamma_{\tau(n)-1}) + 2\theta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|^2] \\ & + \beta_{\tau(n)}(\|u-p\|^2 + 2(1-\beta_{\tau(n)})\langle w_{\tau(n)}-p, u-p \rangle - \|x_{\tau(n)}-p\|^2). \end{aligned}$$

Therefore,

$$\|x_{\tau(n)}-p\|^2 \leq (1-\beta_{\tau(n)})\theta_{\tau(n)}(\Gamma_{\tau(n)} - \Gamma_{\tau(n)-1}) + 2(1-\beta_{\tau(n)})\theta_{\tau(n)}\|x_{\tau(n)}-x_{\tau(n)-1}\|^2 + \beta_{\tau(n)}\|u-p\|^2 + 2\beta_{\tau(n)}(1-\beta_{\tau(n)})\langle w_{\tau(n)}-p, u-p \rangle.$$

Thus, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e. $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. Consequently, for all $n \geq n_0$, $0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$. So $\lim_{n \rightarrow \infty} \Gamma_n = 0$, which implies that $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converge strongly to $p \in \Theta$. \square

REMARK 3.3. The main result of this work (Theorem 3.2) extends and improves many important results in the following ways:

- (i) In [4], the authors obtained their results using nonexpansive mappings but our result holds for quasi-pseudocontractive mappings which are more general.
- (ii) The results of [4,5] were proved by construction of the subsets C_n and Q_n . These were dispensed within our result.
- (iii) We remove the compactness condition imposed on the map used in [3].
- (iv) We prove a strong convergence theorem which makes our work extends the works of authors working in this direction in literature (see [1,11,12] and references contained therein).

COROLLARY 3.4. *Let H be a real Hilbert space and $f : H \rightarrow (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let $T : H \rightarrow H$ be a nonexpansive mapping, and assume that $\Theta := F(T) \cap \operatorname{argmin}_{y \in H} f(y)$ is nonempty. Then the sequence $\{x_n\}$ generated iteratively for an arbitrary $x_0, x_1, u \in H$ by*

$$\begin{cases} u_n = \operatorname{Prox}_{\lambda_n f}(x_n + \theta_n(x_n - x_{n-1})), \\ w_n = \alpha_n u_n + (1 - \alpha_n)T u_n, \\ x_{n+1} = (1 - \beta_n)w_n + \beta_n u, \quad n \geq 1 \end{cases}$$

converges strongly to a point $p = P_{\Theta}u$, where P_{Θ} is the metric projection of H onto Θ , $\lambda_n > \lambda > 0$ for some λ , $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (iii) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$.

4. Application

Convexly constrained linear inverse problem

Consider the following convexly constrained linear inverse problem (see [6]),

$$\begin{cases} Ax = b, \\ x \in C, \end{cases} \tag{22}$$

where A is a bounded linear operator from a real Hilbert space H_1 to another real Hilbert space H_2 , and $b \in H_2$. To solve (22), we consider the following convexly constrained minimization problem:

$$\min_{x \in C} f(x) = \min_{x \in C} \frac{1}{2} \|Ax - b\|^2. \tag{23}$$

In general, every solution to (22) is a solution to (23). However, a solution to (23) may not necessarily be a solution to (22). Moreover, if a solution set of problem (22) is nonempty, then it follows from [19, Lemma 4.2] that $C \cap (\nabla f)^{-1} \neq \emptyset$. It is well known that the projected Landweber method (see [7]) given by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C[x_n - \lambda_n A^*(Ax_n - b)], n \geq 1, \end{cases}$$

where A^* is the adjoint of A and $0 < \lambda < 2\alpha$ with $\alpha = \frac{1}{\|A\|^2}$, converges weakly to a solution of (22). In what follows, we present an iterative algorithm for approximating the solution of fixed point problem and problem (22) for quasi-pseudocontractive mapping. We use the following theorem to prove a strong convergence theorem.

COROLLARY 4.1. *Let C be a nonempty, closed and convex subset of a Hilbert space H . Suppose that the convexly constrained linear inverse problem (22) is consistent and let Θ denote its solution set. Let $T : H \rightarrow H$ be an L -Lipschitzian and quasi-pseudo-contractive mapping with $L \geq 1$, $F(T) \neq \emptyset$ and T is demiclosed at 0. Let $\Gamma := F(T) \cap \Theta$ be nonempty and λ_n be a sequence in $(0, \frac{2}{\|A\|^2})$. Suppose the sequence $\{x_n\}$ is generated iteratively for an arbitrary $x_1 \in C$ by*

$$\begin{cases} u_n = P_C(x_n - \lambda_n A^*(Ax_n - b)); \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n T((1 - \eta_n)I + \eta_n T))u_n; \end{cases}$$

such that the following conditions are satisfied:

- (i) $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{2}{\|A\|^2}$,

(ii) $0 < a < \xi_n < \eta_n < b < 1/(1 + \sqrt{1 + L^2}) \forall n \geq 1$;

(iii) $0 < \liminf \alpha_n \leq \limsup \alpha_n < 1$.

Then, $\{x_n\}$ converges strongly to an of Γ .

5. Numerical examples

In this section, we give a numerical example of the algorithm (7) to illustrate its performance. Let $H = \mathbb{R}^2$ be endowed with the Euclidean norm and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x_1, x_2) = -\left(\frac{2\alpha+1}{2}\right)(x_1, x_2)$, $\forall \alpha > \frac{1}{2}$. Then, T is a quasi-pseudocontractive mapping.

Indeed, observe that $F(T) = \{0\}$, and for $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that

$$\|Tx - 0\|^2 = \|x - 0\|^2 + \left(\frac{4\alpha^2 + 4\alpha - 3}{4}\right) \|x - 0\|^2 \tag{24}$$

and
$$\|x - Tx\|^2 = \left(\frac{2\alpha + 3}{2}\right)^2 \|x - 0\|^2. \tag{25}$$

Using (24) and (25), we obtain

$$\|Tx - 0\|^2 = \|x - 0\|^2 + \left(\frac{4\alpha^2 + 4\alpha - 3}{(2\alpha + 3)^2}\right) \|x - Tx\|^2 \leq \|x - 0\|^2 + \|x - Tx\|^2.$$

Also, it is easy to see that T is L -Lipschitzian with $L = \left(\frac{2\alpha+1}{2}\right)^2$, $\alpha > \frac{1}{2}$

Now, define $f : \mathbb{R}^2 \rightarrow (-\infty, \infty]$ by $f(x) = \frac{1}{2}\|B(x) - b\|^2$, where $B(x) = (2x_1 + x_2, x_1 + 3x_2)$ and $b = (0, 0)$. Then, f is a proper convex and lower semi-continuous function, since B is a continuous linear mapping (see [13]). Let $\lambda_n = 1 \forall n \geq 1$; then

$$\begin{aligned} \text{Prox}_{1f}(x) &= \operatorname{argmin}_{y \in \mathbb{R}^2} \left[f(y) + \frac{1}{2} \|y - x\|^2 \right] = (I + B^T B)^{-1}(x + B^T b^T) \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= \left(\frac{11x_1 - 5x_2}{41}, \frac{-5x_1 + 6x_2}{41} \right). \end{aligned}$$

Now, take $\beta_n = \frac{1}{n+2}$, $\alpha_n = \frac{n}{2n+3}$, $\eta_n = 1/(2 + (\frac{2\alpha+1}{2})^4)$ and $\xi_n = \frac{8}{(2\alpha+1)^4}$ for all $n \geq 1$, $\alpha > \frac{1}{2}$. Then, the conditions of Theorem 3.2 are satisfied. Hence, for arbitrary $x_0, x_1, u \in \mathbb{R}^2$, algorithm (7) becomes:

$$\begin{cases} u_n = \text{Prox}_{1f}(x_n + \theta_n(x_n - x_{n-1})), \\ w_n = \frac{n}{2n+3}u_n + \frac{n+3}{2n+3}((1 - \xi_n)I + \xi_n T((1 - \eta_n)I + \eta_n T))u_n, \\ x_{n+1} = \frac{n+1}{n+2}w_n + \frac{u}{n+2}, \quad n \geq 1, \end{cases}$$

Case I $x_0 = (0.1, 2)^T$, $x_1 = (0.5, 3)^T$, $u = (1, 2)^T$, $\alpha = 2$ and $\theta_n = \frac{1}{4n^2+1}$.

Case II $x_0 = (0.5, 3)^T$, $x_1 = (0.1, 2)^T$, $u = (1, 2)^T$, $\alpha = 5$ and $\theta_n = \frac{1}{n^3+1}$.

Case III $x_0 = (0.1, 2)^T$, $x_1 = (0.5, 3)^T$, $u = (0.5, 3)^T$, $\alpha = 2$ and $\theta_n = \frac{1}{4n^2+1}$.

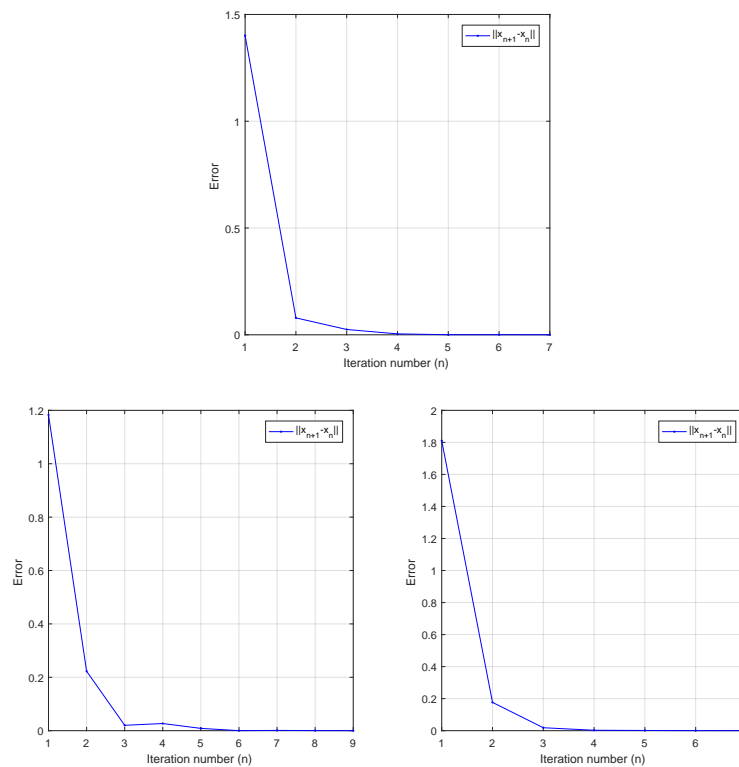


Figure 1: Errors vs Iteration numbers(n): **Case I** (top); **Case II** (bottom left); **Case III** (bottom right).

REMARK 5.1. (i) From the graph, we can see the error in the algorithm using different starting points in the three cases.

(ii) With the different starting points, we saw that the sequence still converge to the fixed point, suggesting that choosing arbitrary starting points is good enough.

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