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# THE EXISTENCE OF ONE WEAK SOLUTION FOR A SECOND-ORDER IMPULSIVE HAMILTONIAN SYSTEM

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**Abstract**. In this work, we are concerned with the existence of at least one non-trivial weak solution for a second-order impulsive Hamiltonian system. The proof of the main result is based on the critical point theory.

## 1. Introduction

We wish to give sufficient conditions for the existence of a nontrivial weak solution to the second-order impulsive Hamiltonian system

$$\begin{cases} -u''(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \nabla H(u(t)), & \text{a.e. } t \in [0, T], \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), & i = 1, 2, \dots, N, \ j = 1, 2, \dots, p, \\ u(0) - u(T) = u'(0) - u'(T) = 0, & \end{cases} \tag{1}$$

where $N \geq 1$, $p \geq 2$, $u = (u_1, \dots, u_N)$, $T > 0$, $\lambda > 0$ is a parameter, $t_j$, $j = 1, 2, \dots, p$, are the instants at which the impulses occur, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, and $\triangle(u_i'(t_j)) = u_i'(t_j^+) - u_i'(t_j^-) = \lim_{t \to t_j^+} u_i'(t) - \lim_{t \to t_j^-} u_i'(t)$. Without further mention, the following conditions are assumed to hold throughout the remainder of this article. The function $A : [0, T] \to \mathbb{R}^{N \times N}$ is a continuous map from the interval $[0, T]$ to the set of $N \times N$ matrices, such that
(M1) $A(t) = (a_{kl}(t))$, $k = 1, 2, \dots, N$, $l = 1, 2, \dots, N$, is a symmetric matrix with $a_{kl} \in L^\infty([0, T])$ for any $t \in [0, T]$;

(M2) There exists $\delta > 0$ such that $(A(t)\xi, \xi) \geq \delta |\xi|^2$ for any $\xi \in \mathbb{R}^N$ and almost every $t \in [0, T]$.

Note that in (M2) and in the sequel, by $(\cdot, \cdot)$ and $|\cdot|$ we mean the usual inner product and usual norm in $\mathbb{R}^N$, respectively. Also, for the sake of convenience, we define $\mathcal{A} = \{1, 2, \dots, N\}$, $\mathcal{B} = \{1, 2, \dots, p\}$.

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The functions $I_{ij} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with the Lipschitz constants $L_{ij} > 0$, i.e., $|I_{ij}(y_1) - I_{ij}(y_2)| \le L_{ij}|y_1 - y_2|$ for every $y_1, y_2 \in \mathbb{R}$. Also, $\nabla F(t,x)$ is the gradient of the function $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ with respect to $x$; $F(t,x)$ is measurable with respect to $t$ for all $x \in \mathbb{R}^N$ and continuously differentiable in $x$ for almost every $t \in [0,T]$. Moreover, there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T], \mathbb{R}^+)$ such that

$$\max\{|F(t,x)|, |\nabla F(t,x)|\} \le a(|x|)b(t), \tag{2}$$

for all $x \in \mathbb{R}^N$ and almost every $t \in [0,T]$. The function $H : \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, $\nabla H$ is Lipschitz continuous with the Lipschitz constant $L > 0$, i.e.,

$$|\nabla H(\xi_1) - \nabla H(\xi_2)| \le L|\xi_1 - \xi_2| \tag{3}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^N$, $H(0,\dots,0) = 0$ and $\nabla H(0,\dots,0) = 0$.

Remark 1.1. From the previous we deduce that

$$|H(\xi)| \le L|\xi|^2, \qquad L|\xi|^2 \le (\nabla H(\xi), \xi) \le L|\xi|^2, \quad \text{for all} \quad \xi \in \mathbb{R}^N. \tag{4}$$

The study of multiplicity of solutions of Hamiltonian systems, as a special case of dynamical systems, is interesting both from the theoretical and practical point of view. These systems form a natural framework for mathematical models of many natural phenomena in fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, etc. For background, theory, and applications of Hamiltonian systems, we refer to [4, 12]. Inspired by the monographs [11, 13], the existence and multiplicity of weak solutions for Hamiltonian systems have been investigated using variational methods by many authors.

In the recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems (IBVPs), by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics, and physics phenomena are described. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [10]. Some classical tools or techniques have been used to study (IBVPs). These classical techniques include the method of upper and lower solutions and some fixed point theorems. On the other hand, in the last twelve years, some researchers have used critical point theory to study the existence of solutions for (IBVPs). We refer to [1, 2, 9]. Very recently, a great deal of work has been done on the existence of multiple solutions to second-order impulsive Hamiltonian systems. In [5,14] the existence of multiple solutions to second-order impulsive Hamiltonian systems based on variational methods and critical point theory was established. We also refer to [6,8,16,17] in which second-order Hamiltonian systems with impulsive effects have been examined.

The results presented here were motivated by the recent papers [3, 7].

## 2. Preliminaries

We recall some basic concepts that will be used in the following text. Set

$E = \{u : [0,T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, u(0) = u(T), u' \in L^2([0,T], \mathbb{R}^N)\}$ with the inner product and the corresponding norm defined by

$$\langle u, v\rangle_E = \int_0^T \left[\big(u'(t), v'(t)\big) + \big(u(t), v(t)\big)\right] dt, \quad \text{for all} \quad u, v \in E,$$

$$\|u\|_E = \left(\int_0^T \big(|u'(t)|^2 + |u(t)|^2\big)\, dt\right)^{\frac{1}{2}} \quad \text{for all} \quad u \in E.$$

For every $u, v \in E$, we define

$$\langle u, v\rangle = \int_0^T \left[\big(u'(t), v'(t)\big) + \big(A(t)u(t), v(t)\big)\right] dt,$$

and we observe that conditions (M1) and (M2) ensure that this defines an inner product in $E$. Then $E$ is a reflexive Banach space with the norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$ for all $u \in E$. A simple computation shows that for every $t \in [0,T]$ and $\xi \in \mathbb{R}^N$

$$(A(t)\xi, \xi) = \sum_{k,l=1}^{N} a_{kl}(t)\xi_k\xi_l \le \sum_{k,l=1}^{N} \|a_{kl}\|_{L^\infty}|\xi|^2. \tag{5}$$

Along with (M2), this implies $\sqrt{C_1}\|u\|_E \le \|u\| \le \sqrt{C_2}\|u\|_E$, where $C_1 = \min\{1, \delta\}$ and $C_2 = \max\{1, \sum_{k,l=1}^{N}\|a_{kl}\|_\infty\}$, which means the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$. Since $(E, \|.\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [11]), there exists a positive constant $C$ such that $\|u\|_\infty \le C\|u\|$, where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$ and $C = \sqrt{\frac{2}{C_1}} \max\{\frac{1}{\sqrt{T}}, \sqrt{T}\}$ (see [5]).

Now, consider the following function space: $\widetilde{E} = \{u \in E : \int_0^T u(t)\, dt = 0\}$. Then, we have the Wirtinger's and Sobolev's inequalities respectively

$$\|u\|_{L^2}^2 \le \frac{T^2}{4\pi^2}\|u'\|_{L^2}^2, \quad \text{for all} \quad u \in \widetilde{E},$$

$$\|u\|_\infty^2 \le \frac{T}{12}\|u'\|_{L^2}^2, \quad \text{for all} \quad u \in \widetilde{E}, \quad \text{(see [11])}.$$

By Wirtinger's inequality we have

$$\|w\|_E^2 \le \frac{4\pi^2 + T^2}{4\pi^2}\|w'\|_{L^2}^2 \quad \text{for all} \quad w \in \widetilde{E}, \tag{6}$$

and from Sobolev's inequality it follows that

$$|w(t)| \le \|w\|_\infty \le \frac{\sqrt{3T}}{6}\|w'\|_{L^2} \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad w \in \widetilde{E}, \tag{7}$$

which combining $\|w'\|_{L^2} \le \|w\|_E$ yields that

$$|w(t)| \le \frac{\sqrt{3T}}{6}\|w\|_E \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad w \in \widetilde{E}. \tag{8}$$

In view of (7), for every $i \in \mathcal{A}$, $t \in [0,T]$ and $w \in \widetilde{E}$, we have

$$|w_i(t)| \le |w(t)| \le \frac{\sqrt{3T}}{6}\|w'\|_{L^2}. \tag{9}$$

For every $t \in [0,T]$, $w \in \widetilde{E}$ and $x \in \mathbb{R}^N$, it follows from (8) that

$$|x + w(t)| \le |x| + |w(t)| \le |x| + \frac{\sqrt{3T}}{6}\|w\|_E. \tag{10}$$

Next, we define what we mean by a solution of (1).

Definition 2.1. By a weak solution of the problem (1), we mean any $u \in E$ such that

$$\begin{aligned} -\int_0^T &\big[(u'(t), v'(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\big]\, dt \\ &- \sum_{j=1}^p \sum_{i=1}^N I_{ij}(u_i(t_j))v_i(t_j) + \lambda \int_0^T \big(\nabla F(t, u(t)), v(t)\big)\, dt = 0 \end{aligned}$$

for every $v \in E$.

An important relationship between a weak solution and a classical solution of (1) is given in the next lemma.

Lemma 2.2 ([6, Lemma 2.2]). *If $u \in E$ is a weak solution of* (1)*, then $u$ is a classical solution of* (1).

Define the functional $\Phi_\lambda : E \to \mathbb{R}$ by

$$\Phi_\lambda(u) = -\frac{1}{2}\|u\|^2 - \sum_{j=1}^p \sum_{i=1}^N \int_0^{u_i(t_j)} I_{ij}(y)\, dy + \int_0^T H(u(t))\, dt + \lambda \int_0^T F(t, u(t))\, dt,$$

for every $u \in E$.
For convenience, if we define the function $g : E \to \mathbb{R}$ by

$$g(u) = -\sum_{j=1}^p \sum_{i=1}^N \int_0^{u_i(t_j)} I_{ij}(y)\, dy, \text{ for every } u \in E,$$

then $$\Phi_\lambda(u) = -\frac{1}{2}\|u\|^2 + g(u) + \int_0^T H(u(t))\, dt + \lambda \int_0^T F(t, u(t))\, dt, \text{ for every } u \in E.$$

Remark 2.3. For any $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and $w = (w_1, w_2, \ldots, w_N) \in \widetilde{E}$, it follows that $g(x + w) = -\sum_{j=1}^p \sum_{i=1}^N \int_0^{x_i + w_i(t_j)} I_{ij}(y)\, dy$. Putting $w = 0$ in the previous equation we get $g(x) = -\sum_{j=1}^p \sum_{i=1}^N \int_0^{x_i} I_{ij}(y)\, dy$.

Lemma 2.4 ([3, Lemma 5]). *If $I_{ij}(y)$ is nonincreasing in $y \in \mathbb{R}$ for all $i \in \mathcal{A}$ and $j \in \mathcal{B}$, then $g(x)$ is convex in $x \in \mathbb{R}^N$.*

It is well-known that $\Phi_\lambda$ is a Gateaux differentiable functional whose Gateaux derivative at the point $u \in E$ is the functional $\Phi'_\lambda(u) \in E^*$ given by

$$\begin{aligned} \Phi'_\lambda(u)(v) = -\int_0^T &\big[(u'(t), v'(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\big]\, dt \\ &- \sum_{j=1}^p \sum_{i=1}^N I_{ij}(u_i(t_j))v_i(t_j) + \lambda \int_0^T \big(\nabla F(t, u(t)), v(t)\big)\, dt, \end{aligned} \tag{11}$$

for every $v \in E$.

By (11) and Definition 2.1, we have the following corollary.

COROLLARY 2.5. *The weak solutions of problem* (1) *are precisely the critical points of* $\Phi_\lambda$.

Inspired by the notations, addressed above, assume that

$$K = C^2\Big(2LT + \sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}\Big),$$

and
$$a_w(r) = \frac{\int_0^T \max_{|\xi|\le c(\frac{2r}{1-K})^{\frac{1}{2}}} F(t,\xi)\,dt - \int_0^T F(t,w(t))\,dt}{r - \frac{1}{2}(1+K)\|w\|^2},$$

for a given $r \ge 0$ and a given $w \in E$ with $r \ne \frac{1}{2}(1+K)\|w\|^2$. Then Graef et al. proved the following two main theorems in [7], under the extra conditions $I_{ij}(0) = 0$ for all $i \in \mathcal{A}$, $j \in \mathcal{B}$ and $F(t,0,\dots,0) = 0$ for any $t \in [0,T]$.

THEOREM 2.6 ([7, Theorem 3.1]). *Assume that there exist constants $r_1 \ge 0$ and $r_2 > 0$, and a function $w \in E$ such that*
*(A1)* $(\frac{2r_1}{1-K})^{\frac{1}{2}} < \|w\| < (\frac{2r_2}{1+K})^{\frac{1}{2}}$;
*(A2)* $a_w(r_2) < a_w(r_1)$.
*Then, for each* $\lambda \in \big(\frac{1}{a_w(r_1)}, \frac{1}{a_w(r_2)}\big)$, *problem* (1) *has a non-trivial weak solution* $u^* \in E$ *such that* $r_1 < \frac{1}{2}\|u^*\|^2 - g(u^*) - \int_0^T H(u^*(t))\,dt < r_2$.

THEOREM 2.7 ([7, Theorem 3.5]). *Assume that there exist a constant $\bar{r} > 0$ and a function $\bar{w}$ with $\frac{2\bar{r}}{1-K} < \|\bar{w}\|^2$ such that*
*(B1)* $\int_0^T \max_{|\xi|\le c(\frac{2\bar{r}}{1-K})^{\frac{1}{2}}} F(t,\xi)\,dt < \int_0^T F(t,\bar{w}(t))\,dt$;
*(B2)* $\limsup_{|\xi|\to+\infty} \frac{F(t,\xi)}{|\xi|^2} \le 0$, *uniformly for* $t \in [0,T]$.
*Then, for each* $\lambda \in (\frac{1}{a_{\bar{w}}(\bar{r})}, +\infty)$, *problem* (1) *has a non-trivial weak solution* $\bar{u} \in E$ *such that* $\frac{1}{2}\|\bar{u}\|^2 - g(\bar{u}) - \int_0^T H(\bar{u}(t))\,dt > \bar{r}$.

For the reader's convenience, we now recall the critical point theorem obtained in [16]. It will be our main tool.

THEOREM 2.8 ([16, Theorem 1.1]). *Suppose that $V$ and $W$ are reflexive Banach spaces, $\Phi \in C^1(V \times W, \mathbb{R})$, $\Phi(v,\cdot)$ is weakly upper semi-continuous for all $v \in V$ and $\Phi(.,w) : V \to \mathbb{R}$ is convex for all $w \in W$, that is $\Phi(\lambda v_1 + (1-\lambda)v_2, w) \le \lambda\Phi(v_1,w) + (1-\lambda)\Phi(v_2,w)$, for all $\lambda \in [0,1]$ and $v_1, v_2 \in V$, $w \in W$, and $\Phi'$ is weakly continuous. Assume that $\Phi(0,w) \to -\infty$, as $\|w\| \to +\infty$, and, for every $M > 0$, $\Phi(v,w) \to +\infty$, as $\|v\| \to +\infty$ uniformly for $\|w\| \le M$. Then $\Phi$ has at least one critical point.*

## 3. Main results

For convenience, we introduce some assumptions:

(H1) $F(t,x)$ is convex in $x \in \mathbb{R}^N$ for almost every $t \in [0,T]$;

(H2) $H(x)$ is convex in $x \in \mathbb{R}^N$;

(H3) $-(A(t)x,x)$ is convex in $x \in \mathbb{R}^N$ for almost every $t \in [0,T]$;

(H4) For $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$,

$$-\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{x_i} I_{ij}(y)\,dy + \lambda\int_0^T F(t,x)\,dt \to +\infty, \quad \text{as} \quad |x| \to +\infty;$$

(H5) There exist $\sigma \in L^1([0,T];\mathbb{R}^+)$ and $\beta \in L^1([0,T];\mathbb{R})$ such that

$$F(t,x) \le \sigma(t)|x|^2 + \beta(t) \quad \text{for every} \quad x \in \mathbb{R}^N \quad \text{and almost every} \quad t \in [0,T];$$

(H6) For any $i \in \mathcal{A}$, $j \in \mathcal{B}$, there exist constants $a_{ij} > 0$, $b_{ij} > 0$ and $\gamma_{ij} \in [0,1]$ (among which $\gamma_{ij} = 1$ for $(i,j) \in \mathcal{C} \subseteq \mathcal{A}\times\mathcal{B}$ and $\gamma_{ij} \in [0,1)$ for $(i,j) \in (\mathcal{A}\times\mathcal{B})\setminus\mathcal{C}$ such that $I_{ij}(y) \ge -a_{ij} - b_{ij}y^{\gamma_{ij}}$, $\forall y \ge 0$ and $I_{ij}(y) \le a_{ij} + b_{ij}(-y)^{\gamma_{ij}}$, $\forall y < 0$.

(H7) Under the conditions stated in (H5) and (H6),

$$\sum_{(i,j)\in\mathcal{C}} b_{ij} + 2\lambda\int_0^T \sigma(t)\,dt < \frac{12}{T} \quad \text{and} \quad \sum_{(i,j)\in\mathcal{C}} b_{ij} = 0 \quad \text{if} \quad \mathcal{C} = \emptyset.$$

Theorem 3.1. *Suppose that (H1)–(H7) hold, and $I_{ij}(y)$ is nonincreasing in $y \in \mathbb{R}$ for all $i \in \mathcal{A}$, $j \in \mathcal{B}$. Then problem* (1) *has at least one weak solution in $E$, for every $0 < \lambda < 1$.*

*Proof.* Let the spaces $V$ and $W$ of Theorem 2.8 be $\mathbb{R}^N$ and $\widetilde{E}$, respectively, and $0 < \lambda < 1$. We complete the proof in six steps:

Step 1. Assumption (2) implies that $-\frac{1}{2}\int_0^T |u'(t)|^2 dt + \lambda\int_0^T F(t,u(t))\,dt \in C^1(E,\mathbb{R})$ (see [11, pp. 13]). Moreover, it follows from the continuity of all $I_{ij}$ that $g \in C^1(E,\mathbb{R})$. Also, by (5), we have that $-\frac{1}{2}\int_0^T (A(t)u(t),u(t))\,dt \in C^1(E,\mathbb{R})$. In addition, since $H$ is continuously differentiable, $\int_0^T H(u(t))\,dt \in C^1(E,\mathbb{R})$.

Step 2. The function $\Phi_\lambda(x+w)$ is weakly upper semi-continuous on $w \in \widetilde{E}$ for all $x \in \mathbb{R}^N$ (see [11, pp. 13] and [6]).

Step 3. By (H1), $F(t,x+w(t))$ is convex in $x \in \mathbb{R}^N$ for every $w \in \widetilde{E}$ and almost every $t \in [0,T]$. Then $\lambda\int_0^T F(t,x+w(t))\,dt$ is convex in $x \in \mathbb{R}^N$ for every $w \in \widetilde{E}$. Also, by (H2) and (H3), $\int_0^T H(x+w(t))\,dt$ and $-\frac{1}{2}\int_0^T (A(t)(x+w(t)),x+w(t))\,dt$ are convex, respectively, in $x \in \mathbb{R}^N$ for every $w \in \widetilde{E}$. Moreover, it follows from Lemma 2.4 that $g(x+w)$ is convex in $x \in \mathbb{R}^N$ for every $w \in \widetilde{E}$. Thus, for every $w \in \widetilde{E}$

$$\Phi_\lambda(x+w) = -\frac{1}{2}\int_0^T |w'(t)|^2 dt - \frac{1}{2}\int_0^T \big(A(t)(x+w(t)),x+w(t)\big)\,dt$$

$$+ g(x+w) + \int_0^T H(x+w(t))\,dt + \lambda \int_0^T F(t, x+w(t))\,dt, \qquad (12)$$

is convex in $x \in \mathbb{R}^N$.

Step 4. Let $(u_k)$ be a weakly convergent sequence to $u$ in $E$. Then $(u_k)$ converges uniformly to $u$ on $[0,T]$ [11, Proposition 1.2]. In view of (11), assumptions (2), (3) and the continuity of all $I_{ij}$, we have that $\Phi'_\lambda$ is weakly continuous.

Step 5. Owing to (H6), we have that

$$\int_0^z I_{ij}(y)\,dy \geq -a_{ij}z - \frac{b_{ij}}{\gamma_{ij}+1} z^{\gamma_{ij}+1} = -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1}, \quad \forall z \geq 0,$$
$$\int_z^0 I_{ij}(y)\,dy \leq -a_{ij}z - \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij}+1} z^{\gamma_{ij}+1} = a_{ij}|z| + \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1}, \quad \forall z < 0.$$

Therefore, for any $i \in \mathcal{A}$, $j \in \mathcal{B}$ and $z \in \mathbb{R}$, we have

$$\int_0^z I_{ij}(y)\,dy \geq -a_{ij}|z| - \frac{b_{ij}}{\gamma_{ij}+1}|z|^{\gamma_{ij}+1},$$

which combining with (9) yields that

$$\begin{aligned}\int_0^{w_i(t_j)} I_{ij}(y)\,dy &\geq -a_{ij}|w_i(t_j)| - \frac{b_{ij}}{\gamma_{ij}+1}|w_i(t_j)|^{\gamma_{ij}+1} \\ &\geq -a_{ij}\frac{\sqrt{3T}}{6}\|w'\|_{L^2} - \frac{b_{ij}}{\gamma_{ij}+1}\left(\frac{\sqrt{3T}}{6}\|w'\|_{L^2}\right)^{\gamma_{ij}+1},\end{aligned}$$

for every $i \in \mathcal{A}$, $j \in \mathcal{B}$ and $w \in \widetilde{E}$. Thus for every $w \in \widetilde{E}$, we have

$$g(w) \leq \sum_{j=1}^{p}\sum_{i=1}^{N} a_{ij}\frac{\sqrt{3T}}{6}\|w'\|_{L^2} + \sum_{j=1}^{p}\sum_{i=1}^{N}\frac{b_{ij}}{\gamma_{ij}+1}\left(\frac{\sqrt{3T}}{6}\|w'\|_{L^2}\right)^{\gamma_{ij}+1}. \qquad (13)$$

We deduce from (H5), (7) and $\sigma \in L^1([0,T];\mathbb{R}^+)$ that, for every $w \in \widetilde{E}$ and almost every $t \in [0,T]$, $F(t, w(t)) \leq \sigma(t)|w(t)|^2 + \beta(t) \leq \sigma(t)\frac{T}{12}\|w'\|^2_{L^2} + \beta(t)$, which implies that

$$\int_0^T F(t, w(t))\,dt \leq \frac{T}{12}\|w'\|^2_{L^2}\int_0^T \sigma(t)\,dt + \int_0^T \beta(t)\,dt. \qquad (14)$$

For every $w \in \widetilde{E}$, we deduce from (M2), (4), (13) and (14) that

$$\begin{aligned}\Phi_\lambda(w) =& -\frac{1}{2}\int_0^T |w'(t)|^2 dt - \frac{1}{2}\int_0^T (A(t)w(t), w(t))\,dt \\ &+ g(w) + \int_0^T H(w(t))\,dt + \lambda\int_0^T F(t, w(t))\,dt \\ \leq& \left[-\frac{1}{2} + \lambda\frac{T}{12}\int_0^T \sigma(t)\,dt\right]\|w'\|^2_{L^2} + \lambda\int_0^T \beta(t)\,dt + (L - \frac{1}{2}\delta)\|w\|^2_{L^2} \\ &+ \sum_{j=1}^{p}\sum_{i=1}^{N} a_{ij}\frac{\sqrt{3T}}{6}\|w'\|_{L^2} + \sum_{j=1}^{p}\sum_{i=1}^{N}\frac{b_{ij}}{\gamma_{ij}+1}\left(\frac{\sqrt{3T}}{6}\|w'\|_{L^2}\right)^{\gamma_{ij}+1}. \qquad (15)\end{aligned}$$

In view of (6), for every $w \in \widetilde{E}$, $\|w\|_E \to +\infty$ implies $\|w'\|_{L^2} \to +\infty$. Then we have the following result. If $\mathcal{C} = \emptyset$, then $\gamma_{ij} \in [0,1)$ for all $i \in \mathcal{A}$, $j \in \mathcal{B}$ and (H7) becomes $2\lambda \int_0^T \sigma(t)\,dt < \frac{12}{T}$.

Since $0 < \lambda < 1$, it follows from (15) that $\Phi_\lambda(w) \to -\infty$ as $\|w\|_E \to +\infty$, $w \in \widetilde{E}$. If $\mathcal{C} \neq \emptyset$, we deduce from (15) and (H6) that

$$\Phi_\lambda(w) \le \Big[ -\frac{1}{2} + \lambda \frac{T}{12} \int_0^T \sigma(t)\,dt + \frac{T}{24} \sum_{(i,j)\in\mathcal{C}} b_{ij} \Big] \|w'\|_{L^2}^2 + \lambda \int_0^T \beta(t)\,dt + (L - \frac{1}{2}\delta)\|w\|_{L^2}^2$$

$$+ \sum_{j=1}^{p} \sum_{i=1}^{N} a_{ij} \frac{\sqrt{3T}}{6} \|w'\|_{L^2} + \sum_{(i,j)\in(\mathcal{A}\times\mathcal{B})\setminus\mathcal{C}} \frac{b_{ij}}{\gamma_{ij}+1} \Big( \frac{\sqrt{3T}}{6} \|w'\|_{L^2} \Big)^{\gamma_{ij}+1},$$

which combining (H7) yields that $\Phi_\lambda(w) \to -\infty$ as $\|w\|_E \to +\infty$, $w \in \widetilde{E}$.

Step 6. For any $w \in \widetilde{E}$ with $\|w\|_E \le M$, we deduce from (9) and $\|w'\|_{L^2} \le \|w\|_E$ that

$$|g(-w)| = \Big| -\sum_{j=1}^{p}\sum_{i=1}^{N} \int_0^{-w_i(t_j)} I_{ij}(y)\,dy \Big| \le \sum_{j=1}^{p}\sum_{i=1}^{N} \Big| \int_0^{-w_i(t_j)} I_{ij}(y)\,dy \Big| \le pNC_3 |w_i(t_j)|, \quad (16)$$

where $C_3 = \max_{i\in\mathcal{A}, j\in\mathcal{B}, |y| \le \frac{\sqrt{3T}}{6}M} |I_{ij}(y)|$. From (9), (16) and $\|w'\|_{L^2} \le \|w\|_E$ it follows that $|g(-w)| \le pNC_3|w_i(t_j)| \le pNC_3 \frac{\sqrt{3T}}{6} \|w'\|_{L^2} \le pNC_3 \frac{\sqrt{3T}}{6} M$, for any $w \in \widetilde{E}$ with $\|w\|_E \le M$. From this inequality and by the convexity of $g$, we have that

$$g(x+w) \ge 2g(\frac{x}{2}) - g(-w) \ge 2g(\frac{x}{2}) - pNC_3 \frac{\sqrt{3T}}{6} M, \quad (17)$$

for all $x \in \mathbb{R}^N$ and $w \in \widetilde{E}$ with $\|w\|_E \le M$. It follows from (2) that, for any $w \in \widetilde{E}$ $\big| \int_0^T F(t, -w(t))\,dt \big| \le \int_0^T |F(t, -w(t))| dt \le \int_0^T a(|w(t)|) b(t)\,dt$, which combining (8) and $\|w\|_E \le M$ yields that

$$\Big| \int_0^T F(t, -w(t))\,dt \Big| \le \max_{0 \le s \le \frac{\sqrt{3T}}{6}M} a(s) \int_0^T b(t)\,dt. \quad (18)$$

We deduce from (H1) and (18) that

$$\int_0^T F(t, x+w(t))\,dt \ge 2\int_0^T F(t, \frac{x}{2})\,dt - \int_0^T F(t, -w(t))\,dt$$

$$\ge 2\int_0^T F(t, \frac{x}{2})\,dt - \max_{0 \le s \le \frac{\sqrt{3T}}{6}M} a(s) \int_0^T b(t)\,dt, \quad (19)$$

for all $x \in \mathbb{R}^N$ and $w \in \widetilde{E}$ with $\|w\|_E \le M$. Thus it follows from (4), (5), (10), (12), (17) and (19) that

$$\Phi_\lambda(x+w) \ge -\frac{1}{2}M^2 + 2\Big[ g(\frac{x}{2}) + \lambda \int_0^T F(t, \frac{x}{2})\,dt \Big] - \Big( LT + \frac{1}{2}TC_2 \Big) \Big( |x| + \frac{\sqrt{3T}}{6} M \Big)^2$$

$$- \lambda \max_{0\le s\le \frac{\sqrt{3T}}{6}M} a(s) \int_0^T b(t)\,dt - pNC_3 \frac{\sqrt{3T}}{6} M,$$

for all $x \in \mathbb{R}^N$ and $w \in \widetilde{E}$ with $\|w\|_E \le M$. Hence, (H4) implies that for every $M > 0$, $\Phi_\lambda(x+w) \to +\infty$ as $|x| \to +\infty$, $x \in \mathbb{R}^N$, uniformly for $w \in \widetilde{E}$ with $\|w\|_E \le M$. Finally, from Theorem 2.8, for every $0 < \lambda < 1$, the functional $\Phi_\lambda$ has at least one critical point in $E$ and Corollary 2.5 shows that the conclusion is reached. □

Now, we give an example illustrating Theorem 3.1.

Example 3.2. Take $N = 1$, $0 < \lambda < 1$ and consider the problem

$$\begin{cases} -u''(t) - u(t) = \lambda \nabla F(t, u(t)) + \nabla H(u(t)), & a.e. \quad t \in [0,1], \\ \triangle(u'(t_j)) = I_j(u(t_j)), & j = 1,2, \\ u(0) - u(1) = u'(0) - u'(1) = 0, \end{cases} \tag{20}$$

where $F(t,x) = tx^2$ for every $t \in [0,1]$ and $x \in \mathbb{R}$, $H(x) = \frac{1}{2}x^2$ for all $x \in \mathbb{R}$. Also, $t_1 = 1$, $t_2 = 2$, $I_j(y) = -y + 1$ for $j = 1,2$ and every $y \in \mathbb{R}$. Clearly, $F$ satisfies (2) by choosing $a(x) = x^2$ and $b(t) = 2|t|$. Also, the conditions (H1), (H2) and (H3) hold. Moreover, for $x \in \mathbb{R}$

$$-\sum_{j=1}^{2} \int_0^x (-y+1)\,dy + \lambda \int_0^1 (tx^2)\,dt = (1 + \frac{\lambda}{2})x^2 - 2x,$$

converges to $+\infty$ as $|x| \to +\infty$ and (H4) holds. Also, (H5) and (H6) hold by choosing $\sigma(t) = t$, $\beta(t) = 0$ and $a_j = b_j = \gamma_j = 1$ for $j = 1,2$. Since $0 < \lambda < 1$

$$\sum_{j\in\{1,2\}} b_j + 2\lambda \int_0^T \sigma(t)\,dt = \sum_{j=1}^{2} 1 + 2\lambda \int_0^1 t dt = 2 + \lambda < 3 < 12,$$

and (H7) holds. Therefore, applying Theorem 3.1, for each $\lambda \in (0,1)$, problem (20) has at least one weak solution in $E$.

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