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HARNACK ESTIMATES FOR THE POROUS MEDIUM EQUATION WITH POTENTIAL UNDER GEOMETRIC FLOW

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Abstract. Let $(M, g(t))$, $t \in [0, T)$ be a closed Riemannian n-manifold whose Riemannian metric $g(t)$ evolves by the geometric flow $\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$, where $S_{ij}(t)$ is a symmetric two-tensor on $(M, g(t))$. We discuss differential Harnack estimates for positive solution to the porous medium equation with potential, $\frac{\partial u}{\partial t} = \Delta u^p + Su$, where $S = g^{ij} S_{ij}$ is the trace of S_{ij}, on time-dependent Riemannian metric evolving by the above geometric flow.

1. Introduction

There are many results about the Harnack estimates for parabolic equations. The study of differential Harnack estimates and applications for parabolic equation originated in the famous paper [11] of Li and Yau, in which they discovered the celebrated differential Harnack estimate for any positive solution to the heat equation with potential on Riemannian manifolds with a fixed Riemannian metric. Afterwards, this method played an important role in the study of geometric flows, for instance, Hamilton proved Harnack inequalities for the Ricci flow on Riemannian manifolds with weakly positive curvature operator [7] and mean curvature flow [8], see also [3, 5]. Also, many authors have obtained recently a differential Harnack estimate for solutions of the parabolic equation on Riemannian manifold along the geometric flow, for instance, Fang in [6] proved differential Harnack estimates for backward heat equation with potentials under an extended Ricci flow and Ishida in [10] studied differential Harnack estimates for heat equation with potentials along the geometric flow.

Let M be a closed Riemannian manifold with a one parameter family of Riemannian metric $g(t)$ evolving by the geometric flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2S_{ij}(x,t) \tag{1}$$

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where S_{ij} is a general time-dependent symmetric two-tensor on $(M, g(t))$. For example, (1) becomes Ricci flow whenever $S_{ij} = R_{ij}$ is the Ricci tensor, where it introduced by Hamilton [9].

In [4], Cao and Zhu obtained Aronson-Bénilan estimates for the porous medium equation (PME) with potential

$$\frac{\partial u}{\partial t} = \Delta u^p + Ru \tag{2}$$

along the Ricci flow, where R is the scalar curvature of M. Differential equation (2) is a nonlinear parabolic equation and has applications in mathematics and physics. For $p > 1$ differential equations PME describe physical processes of gas through porous medium, heat radiation in plasmas [15]. Motivated by the above works, in this paper, we consider equation of type (2) with a linear forcing term

$$\frac{\partial u}{\partial t} = \Delta u^p + Su \tag{3}$$

under the geometric flow (1), where $S = g^{ij} S_{ij}$, Δ is the Laplace operator with respect to the evolving metric $g(t)$ of the geometric flow (1) and prove differential Harnack estimates for positive solutions to (3). Notice also that for any smooth solution u of (3) we have

$$\frac{\partial}{\partial t}(\int_M u\, d\mu) = \int_M \frac{\partial u}{\partial t}\, d\mu + u\frac{\partial d\mu}{\partial t} = \int_M (\frac{\partial u}{\partial t} - Su)\, d\mu = \int_M \Delta u^p\, d\mu = 0.$$

For $p = 1$, (3) is simply the equation $\frac{\partial u}{\partial t} = \Delta u + Su$, for which differential Harnack estimates for positive solution have been studied in [10]. Suppose that u is a positive solution of (3) and $v = \frac{p}{p-1} u^{p-1}$. Then we can rewrite (3) as follows

$$\frac{\partial v}{\partial t} = (p-1)v\Delta v + |\nabla v|^2 + (p-1)Sv. \tag{4}$$

To state the main results of the current article, analogous to definition from Müller [13] we introduce evolving tensor quantities associated with the tensor S_{ij}.

Definition 1.1. Let $g(t)$ be a solution of the geometric flow (1) and let $X = X^i \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ be a vector field on $(M, g(t))$. We define

$$\mathcal{I}(S, X) = (R^{ij} - S^{ij})X_i X_j, \qquad \mathcal{H}(S, X) = \frac{\partial S}{\partial t} + \frac{S}{t} - 2\nabla_i S X^i + 2S^{ij} X_i X_j,$$

$$\mathcal{D}(S) = \frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2, \quad \mathcal{E}(S, X) = \mathcal{D}(S) + 2\mathcal{I}(S, X) + 2(2\nabla^i S_{ij} - \nabla_j S)X^j.$$

2. Main results

The main results of this paper are the following.

Theorem 2.1. *Let $g(t)$, $t \in [0, T)$ be a solution to the geometric flow (1) on a complete Riemannian n-manifold M satisfying*

$$\mathcal{E}(S, X) \geq 0, \quad \mathcal{H}(S, X) \geq 0, \quad Ric \geq -(n-1)k_1, \quad -k_2 g \leq S_{ij} \leq k_3 g, \quad S \geq 0,$$

for all vector fields X and each time $t \in [0, T)$. Suppose u is a smooth positive solution to equation (3) *with $p > 1$ and $v = \frac{p}{p-1} u^{p-1}$. Then for any $d \in [2, \infty)$, on the geodesic ball $\mathcal{Q}_{\rho,T}$, we have*

$$\frac{|\nabla v|^2}{v} - 2\frac{v_t}{v} - \frac{S}{v} - \frac{d}{t} \leq \frac{2n(p-1)}{1+n(p-1)}\Big(\frac{E_1 v_{\max}}{\rho^2} + E_2\Big), \tag{5}$$

where $E_1 = \big(p^2 n + \frac{1}{2}\sqrt{k_1}\rho + \frac{9}{4}\big)c_1(p-1)$, $E_2 = \sqrt{c_2}(k_2+k_3)^2+1$ and c_1, c_2 are absolute positive constants.

When $\rho \to \infty$, we can get the gradient estimates for the nonlinear parabolic equation (3).

Corollary 2.2. *Let $g(t)$, $t \in [0, T)$ be a solution to the geometric flow* (1) *on a complete Riemannian n-manifold M satisfying*

$$\mathcal{E}(S, X) \geq 0, \quad \mathcal{H}(S, X) \geq 0, \quad Ric \geq -(n-1)k_1, \quad -k_2 g \leq S_{ij} \leq k_3 g, \quad S \geq 0,$$

for all vector fields X and each time $t \in [0, T)$. Suppose u is a bounded smooth positive solution to equation (3) *with $p > 1$ and $v = \frac{p}{p-1} u^{p-1}$. Then for any $d \in [2, \infty)$, we have*

$$\frac{|\nabla v|^2}{v} - 2\frac{v_t}{v} - \frac{S}{v} - \frac{d}{t} \leq \frac{2n(p-1)}{1+n(p-1)} E_2, \tag{6}$$

where $E_2 = \sqrt{c_2}(k_2+k_3)^2 + 1$ and c_2 is absolute positive constant.

As an application, we get the following Harnack inequality for v.

Theorem 2.3. *With the same assumption as in Corollary 2.2, if $d \geq 2$, then for any points (x_1, t_1) and (x_2, t_2) on $M \times [0, T)$ with $0 < t_1 < t_2$ we have the following estimate*

$$v(x_1, t_1) \leq v(x_2, t_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{d}{2}} \exp\Big(\frac{\Gamma}{2v_{\min}} + \Big(\frac{n(p-1)}{1+n(p-1)}E_2\Big)(t_2 - t_1)\Big), \tag{7}$$

where E_2 is the constants in Corollary 2.2 and $\Gamma = \inf_\gamma \int_{t_1}^{t_2} (S + |\frac{d\gamma}{dt}|^2)\, dt$ with the infimum taking over all smooth curves $\gamma(t)$ in M, $t \in [t_1, t_2]$, so that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.

Our results in this article are similar to those of Cao and Zhu [4] in the case $S_{ij} = R_{ij}$.

3. Examples

3.1 Static Riemannian manifold

In this case we have $S_{ij} = 0$ and $S = 0$. Then $\mathcal{D} = 0$, $\mathcal{H}(S, X) = 0$ and $\mathcal{I}(S, X) = R^{ij} X_i X_j$. Thus the assumption in Theorems 2.1, 2.3 and Corollary 2.2 can be replaced by $R_{ij} \geq 0$.

3.2 The Ricci flow

The Ricci flow was defined for the first time by Hamilton as follows

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}.$$

In this case we get $S_{ij} = R_{ij}$ and $S = R$ the scalar curvature. Along the Ricci flow we have

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2, \quad 2\nabla^i R_{il} - \nabla_l R = 0.$$

Therefore, $\mathcal{I}(S,X) = 0$, $\mathcal{D}(S) = 0$, $\mathcal{E}(S,X) = 0$, $\mathcal{H}(S,X) = \frac{\partial R}{\partial t} + \frac{R}{t} - 2\nabla_i R X^i + 2R^{ij}X_iX_j$.

Notice that for any vector field $X = X^i\frac{\partial}{\partial x^i}$ on M, if $g(t)$ is a complete solution to the Ricci flow with bounded curvature and nonnegative curvature operator then from [7] we have $\mathcal{H}(S,X) \geq 0$, that is $g(t)$ has weakly positive curvature operator. Hence, the assumption in Theorems 2.1, 2.3 and Corollary 2.2 hold.

3.3 List's extended Ricci flow

Extended Ricci flow was defined by List in [12] as follows

$$\begin{cases} \frac{\partial}{\partial t}g_{ij} = -2R_{ij} + 4\nabla_i f \nabla_j f, \\ \frac{\partial f}{\partial t} = \Delta f, \quad (g(0), f(0)) = (g_0, f_0), \end{cases}$$

where $f : M \to \mathbb{R}$ is a smooth function. In this case, $S_{ij} = R_{ij} - 2\nabla_i f\nabla_j f$ and $S = R - 2|\nabla f|^2$. Along the extended Ricci flow we have

$$\frac{\partial S}{\partial t} = \Delta S + 2|Ric|^2 + 4|\Delta f|^2, \qquad 2\nabla^i S_{il} - \nabla_l S + 4\Delta f \nabla_l f = 0.$$

Therefore, we obtain $\mathcal{I}(S,X) = 2(\nabla_X f)^2 \geq 0$, $\mathcal{D}(S) = 4|\Delta f|^2$, $\mathcal{E}(S,X) = 4|\Delta f - \nabla_X f|^2 \geq 0$.

3.4 Müller coupled with harmonic map flow

Let (N,h) be a fixed Riemannian manifold. The harmonic-Ricci flow on M was introduced by Müller in [14] as follows

$$\begin{cases} \frac{\partial}{\partial t}g_{ij} = -2R_{ij} + 2\alpha(t)\nabla_i f \nabla_j f, \\ \frac{\partial f}{\partial t} = \tau_g f, \quad (g(0), f(0)) = (g_0, f_0) \end{cases}$$

where $\tau_g f$ is the tension field of the map $f : M \to N$ with respect to the metric $g(t)$ and $\alpha(t)$ is a positive non-increasing real function with respect to t. In this case, $S_{ij} = R_{ij} - \alpha(t)\nabla_i f\nabla_j f$ and $S = R - \alpha(t)|\nabla f|^2$. Along this flow we have

$$\frac{\partial S}{\partial t} = \Delta S + 2|Ric|^2 + 2\alpha(t)|\tau_g f|^2 - (\frac{\partial \alpha(t)}{\partial t})|\nabla f|^2, \quad 2\nabla^i S_{il} - \nabla_l S + 2\alpha(t)\tau_g f \nabla_l f = 0.$$

Therefore, we obtain

$$\mathcal{I}(S,X) = \alpha(t)\nabla^i f \nabla^j f X_i X_j = \alpha(t)(\nabla_X f)^2 \geq 0,$$

$$\mathcal{D}(S) = 2\alpha(t)|\tau_g f|^2 - (\frac{\partial \alpha(t)}{\partial t})|\nabla f|^2,$$

and $$\mathcal{E}(S,X)=2\alpha(t)|\tau_g f-\nabla_X f|^2-(\frac{\partial\alpha(t)}{\partial t})|\nabla f|^2.$$

Thus, $\mathcal{E}(S,X)\geq 0$ holds if $\alpha(t)\geq 0$ ia a non-increasing function. Notice that, to the best our knowledge, it is still unknown whether $\mathcal{H}(S,X)\geq 0$ is preserved under harmonic-Ricci flow in particular case of extended Ricci flow under suitable assumptions.

4. Proofs of the results

In this section, we suppose that u is a smooth positive solution to the equation (3) and $v=\frac{p}{p-1}u^{p-1}$. In order to prove the main results, we need the following lemmas and proposition.

LEMMA 4.1. *Let $(M,g(t))$ be a complete solution to the geometric flow* (1) *in some time interval $[0,T]$. Suppose that v is a positive solution of* (4),

$$\mathcal{L}=\frac{\partial}{\partial t}-(p-1)v\Delta \tag{8}$$

and

$$\begin{aligned} F=&\frac{|\nabla v|^2}{v}-b\frac{v_t}{v}+(1-b)\frac{S}{v}-\frac{d}{t}\\ =&-b(p-1)\Delta v+(1-b)\frac{|\nabla v|^2}{v}-b(p-1)S+(1-b)\frac{S}{v}-\frac{d}{t}. \end{aligned}\tag{9}$$

Then for any constants b,d we have

$$\begin{aligned} \mathcal{L}(F)=&2p\nabla_iF\nabla_iv-[\frac{b-1}{v}+p-1]\big(\frac{\partial S}{\partial t}-2\nabla_iS\nabla^iv+2S^{ij}\nabla_iv\nabla_jv\big)\\ &-2(p-1)(R^{ij}-S^{ij})\nabla_iv\nabla_jv-2(p-1)|\nabla^2v+\frac{b}{2}S_{ij}|^2+\frac{(b-2)^2}{2}(p-1)|S_{ij}|^2\\ &+(p-1)(1-b)\mathcal{D}(S)-\frac{1}{b}F^2-[(p-1)S-\frac{2(1-b)}{b}\frac{S}{v}+\frac{2d}{bt}]F\\ &-\frac{(1-b)^2}{b}\frac{S^2}{v^2}+(1-b)\frac{|\nabla v|^2}{v^2}S+\frac{1-b}{b}\frac{|\nabla v|^4}{v^2}-\frac{d^2}{bt^2}-d(p-1)\frac{S}{t}\\ &+2\frac{1-b}{b}\frac{d}{t}\frac{S}{v}+\frac{d}{t^2}-b(p-1)(2\nabla^iS_{il}-\nabla_lS)\nabla^lv. \end{aligned}\tag{10}$$

Proof. First of all, we have the following evolution equations, under the flow (1),

$$\frac{\partial}{\partial t}(\Delta v)=2S^{ij}\nabla_i\nabla_jv+\Delta(v_t)-g^{ij}\frac{\partial}{\partial t}(\Gamma^k_{ij})\nabla_kv \tag{11}$$

$$\frac{\partial}{\partial t}|\nabla v|^2=2S^{ij}\nabla_iv\nabla_jv+2\nabla^iv_t\nabla_iv \tag{12}$$

$$g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij}=-g^{kl}(2\nabla^iS_{il}-\nabla_lS). \tag{13}$$

Then from (4), (11) and (13) we get

$$\frac{\partial}{\partial t}(\Delta v)=2S^{ij}\nabla_i\nabla_jv+(p-1)v\Delta^2v+(p-1)(\Delta v)^2+2(p-1)\nabla_i(\Delta v)\nabla^iv$$

$$+\Delta|\nabla v|^2+(p-1)\Delta(Sv)+(2\nabla^i S_{il}-\nabla_l S)\nabla^l v. \qquad (14)$$

Using the Bochner-Weitzebnböck formula $\frac{1}{2}\Delta|\nabla v|^2=\nabla_i(\Delta v)\nabla^i v+|\nabla^2 v|^2+R^{ij}\nabla_i v\nabla_j v$, we obtain

$$\begin{aligned}\mathcal{L}(\Delta v)=&2p\nabla_i(\Delta v)\nabla^i v+2S^{ij}\nabla_i\nabla_j v+(p-1)(\Delta v)^2+2|\nabla^2 v|^2+2R^{ij}\nabla_i v\nabla_j v\\&+(p-1)v\Delta S+2(p-1)\nabla_i S\nabla^i v+(p-1)S\Delta v+(2\nabla^i S_{il}-\nabla_l S)\nabla^l v \qquad (15)\end{aligned}$$

On the other hand, again (4) and (12) imply that

$$\begin{aligned}\mathcal{L}(|\nabla v|^2)=&2S^{ij}\nabla_i v\nabla_j v+2(p-1)|\nabla v|^2\Delta v+2\nabla_i|\nabla v|^2\nabla^i v+2(p-1)v\nabla_i S\nabla^i v\\&+2(p-1)S|\nabla v|^2-2(p-1)v|\nabla^2 v|^2-2(p-1)vR^{ij}\nabla_i v\nabla_j v. \qquad (16)\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{L}(\frac{|\nabla v|^2}{v})=&\frac{1}{v}\mathcal{L}(|\nabla v|^2)-\frac{|\nabla v|^2}{v^2}\mathcal{L}(v)+2(p-1)\nabla_i(\frac{|\nabla v|^2}{v})\nabla^i v\\=&2p\nabla_i(\frac{|\nabla v|^2}{v})\nabla^i v+\frac{2}{v}S^{ij}\nabla_i v\nabla_j v+2(p-1)\frac{|\nabla v|^2}{v}\Delta v+\frac{|\nabla v|^4}{v^2}\\&+2(p-1)\nabla_i S\nabla^i v+(p-1)\frac{|\nabla v|^2}{v}S-2(p-1)|\nabla^2 v|^2\\&-2(p-1)R^{ij}\nabla_i v\nabla_j v. \qquad (17)\end{aligned}$$

Also, we obtain

$$\mathcal{L}(\frac{S}{v})=2p\nabla_i\frac{S}{v}\nabla^i v+\frac{|\nabla v|^2}{v^2}S-\frac{2}{v}\nabla_i S\nabla^i v+\frac{1}{v}\frac{\partial S}{\partial t}-(p-1)\frac{S^2}{v}-(p-1)\Delta S. \qquad (18)$$

From (9), (15), (17) and (18) we get

$$\begin{aligned}\mathcal{L}(F)=&(1-b)\mathcal{L}(\frac{|\nabla v|^2}{v})-b(p-1)\mathcal{L}(\Delta v)-b(p-1)\mathcal{L}(S)+(1-b)\mathcal{L}(\frac{S}{v})-\mathcal{L}(\frac{d}{t})\\=&2p\nabla_i F\nabla_i v+\frac{1-b}{v}\big(\frac{\partial S}{\partial t}-2\nabla_i S\nabla^i v+2S^{ij}\nabla_i v\nabla_j v\big)-2(p-1)|\nabla^2 v|^2\\&-(p-1)\big(b\frac{\partial S}{\partial t}+(1-b)\Delta S-2\nabla_i S\nabla^i v+2R^{ij}\nabla_i v\nabla_j v\big)\\&-2b(p-1)S^{ij}\nabla_i\nabla_j v-b(p-1)^2(\Delta v)^2+2(1-b)(p-1)\frac{|\nabla v|^2}{v}\Delta v\\&-b(p-1)^2 S\Delta v+(1-b)(p-1)\frac{|\nabla v|^2}{v}S+(1-b)\frac{|\nabla v|^4}{v^2}+(1-b)\frac{|\nabla v|^2}{v^2}S\\&-(1-b)(p-1)\frac{S^2}{v}-b(p-1)(2\nabla^i S_{il}-\nabla_l S)\nabla^l v+\frac{d}{t^2}.\end{aligned}$$

Since $\Delta S=\frac{\partial S}{\partial t}-2|S_{ij}|^2-\mathcal{D}(S)$ and

$$\begin{aligned}&-b(p-1)^2(\Delta v)^2+2(1-b)(p-1)\frac{|\nabla v|^2}{v}\Delta v-b(p-1)^2 S\Delta v\\&+(1-b)(p-1)\frac{|\nabla v|^2}{v}S+(1-b)\frac{|\nabla v|^4}{v^2}+(1-b)\frac{|\nabla v|^2}{v^2}S-(1-b)(p-1)\frac{S^2}{v}\\=&-\frac{1}{b}\left(-F+(1-b)\frac{|\nabla v|^2}{v}-b(p-1)S+(1-b)\frac{S}{v}-\frac{d}{t}\right)^2\end{aligned}$$

$$
\begin{aligned}
&-2\frac{1-b}{b}\frac{|\nabla v|^2}{v}\big(F-(1-b)\frac{|\nabla v|^2}{v}+b(p-1)S-(1-b)\frac{S}{v}+\frac{d}{t}\big)\\
&+(p-1)S\big(F-(1-b)\frac{|\nabla v|^2}{v}+b(p-1)S-(1-b)\frac{S}{v}+\frac{d}{t}\big)\\
&+(1-b)(p-1)\frac{|\nabla v|^2}{v}S+(1-b)\frac{|\nabla v|^4}{v^2}+(1-b)\frac{|\nabla v|^2}{v^2}S-(1-b)(p-1)\frac{S^2}{v}\\
=&-\frac{1}{b}F^2-[(p-1)S-\frac{2(1-b)}{b}\frac{S}{v}+\frac{2d}{bt}]F-\frac{(1-b)^2}{b}\frac{S^2}{v^2}+(1-b)\frac{|\nabla v|^2}{v^2}S\\
&+\frac{1-b}{b}\frac{|\nabla v|^4}{v^2}-\frac{d^2}{bt^2}-d(p-1)\frac{S}{t}+2\frac{1-b}{b}\frac{d}{t}\frac{S}{v},
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathcal{L}(F)=&2p\nabla_iF\nabla_iv+\frac{1-b}{v}\big(\frac{\partial S}{\partial t}-2\nabla_iS\nabla^iv+2S^{ij}\nabla_iv\nabla_jv\big)\\
&-(p-1)\big(\frac{\partial S}{\partial t}-2\nabla_iS\nabla^iv+2R^{ij}\nabla_iv\nabla_jv\big)+2(p-1)(1-b)|S_{ij}|^2\\
&+(p-1)(1-b)\mathcal{D}(S)-2(p-1)|\nabla^2v|^2-2b(p-1)S^{ij}\nabla_i\nabla_jv-\frac{1}{b}F^2\\
&-[(p-1)S-\frac{2(1-b)}{b}\frac{S}{v}+\frac{2d}{bt}]F-\frac{(1-b)^2}{b}\frac{S^2}{v^2}+(1-b)\frac{|\nabla v|^2}{v^2}S+\frac{1-b}{b}\frac{|\nabla v|^4}{v^2}\\
&-\frac{d^2}{bt^2}-d(p-1)\frac{S}{t}+2\frac{1-b}{b}\frac{d}{t}\frac{S}{v}+\frac{d}{t^2}-b(p-1)(2\nabla^iS_{il}-\nabla_lS)\nabla^lv.
\end{aligned}
$$

The equation (10) follows directly from here. □

Definition 4.2. Suppose that $g(t)$ evolves by (1). Let S be the trace of S_{ij} and $X=X^i\frac{\partial}{\partial x^i}$ be a vector field on M. We define

$$\mathcal{E}_b(S,X)=(b-1)\mathcal{D}(S)+2\mathcal{I}(S,X)+b(2\nabla^iS_{ij}-\nabla_jS)X^j$$

where b is a constant.

Proposition 4.3. *Let $g(t)$, $t\in[0,T)$ be a solution to the geometric flow (1) on a closed Riemannian n-manifold M satisfying*

$$\mathcal{E}_b(S,X)\geq 0,\quad \mathcal{H}(S,X)\geq 0,\quad Ric\geq-(n-1)k_1,\quad -k_2g\leq S_{ij}\leq k_3g,\quad S\geq 0,$$

for all vector fields X and each time $t\in[0,T)$. Suppose u is a smooth positive solution to the equation (3) with $p>1$ and $v=\frac{p}{p-1}u^{p-1}$. Then for any $b\in[2,\infty)$ and $d\geq b$, on the geodesic ball $\mathcal{Q}_{\rho,T}$, we have

$$\frac{|\nabla v|^2}{v}-b\frac{v_t}{v}-(b-1)\frac{S}{v}-\frac{d}{t}\leq b\alpha(\frac{E_4v_{\max}}{\rho^2}+E_5)+E_6, \tag{19}$$

where $\alpha=\frac{bn(p-1)}{2+bn(p-1)}$, $E_4=\big(\frac{b^2p^2n}{4(b-1)}+\frac{\sqrt{k_1}\rho}{2}+\frac{9}{4}\big)c_3(p-1)$, $E_5=\sqrt{c_4}(k_2+k_3)^2+\frac{2(b-2)}{b}(k_2+k_3)+1$ and $E_6=n(k_2+k_3)(b-2)\sqrt{\frac{b(p-1)\alpha}{2}}$.

Proof. Let $x,x_0\in M$ and $d(x,x_0,t)$ be the geodesic distance x from x_0 with respect to the metric $g(t)$. Choose a smooth cut-off function $\psi(s)$ defined on $[0,+\infty)$ with $\psi(s)=1$ for $0\leq s\leq\frac{1}{2}$, $\psi(s)=0$ for $1\leq s$ and $\psi(s)>0$ for $\frac{1}{2}<s<1$ such that

$-c_1\psi^{\frac{1}{2}} \leq \psi'(s) \leq 0$ and $-c_2 \leq \psi''(s) \leq c_2$ for some absolute constants $c_1, c_2 > 0$. For any fixed point $x_0 \in M$ and any positive number $\rho > 0$, we define $\phi(x,t) = \psi(\frac{r(x,t)}{2\rho})$ on $\mathcal{Q}_{\rho,T} = B(x_0, 2\rho) \times [0,T) \subset M \times [0,+\infty)$, where $B(x_0,2\rho)$ is the ball of radius $2\rho > 0$ centered at x_0 and $r(x,t) = d(x,x_0,t)$. Using an argument of Calabi [2], since $\psi(s)$ is in general Lipschitz we can assume everywhere smoothness of $\phi(x,t)$ with support in $\mathcal{Q}_{\rho,T}$. By the Laplacian comparison theorem in [1], the Laplacian of the distance function satisfies $\Delta r(x,t) \leq (n-1)\sqrt{|k_1|}\coth(2\sqrt{|k_1|}\rho)$, for all $x \in M$, $d(x,x_0) \leq 2\rho$. From the definition of ϕ, a direct calculation shows that

$$\frac{|\nabla\phi|^2}{\phi} = \frac{|\psi'|^2|\nabla r|^2}{4\psi\rho^2} \leq \frac{c_1}{\rho^2}, \quad \text{and}$$

$$\Delta\phi = \frac{\psi'\Delta r}{2\rho} + \frac{\psi''|\nabla r|^2}{4\rho^2} \geq -\frac{c_1}{2\rho}(n-1)\sqrt{|k_1|}\coth(2\sqrt{|k_1|}\rho) - \frac{c_1}{4\rho^2} \geq -\frac{c_1\sqrt{|k_1|}}{2\rho} - \frac{c_1}{4\rho^2}.$$

On the other hand, along the geometric flow (1), for a fixed smooth path $\gamma : [a,b] \to M$ whose length at time t is given by $d(\gamma) = \int_a^b |\gamma'(s)|_{g(t)}\, ds$, where s is the arc length along the path, we have

$$\frac{\partial d(\gamma)}{\partial t} = -\int_a^b |\gamma'(s)|^{-1}_{g(t)} S_{ij}(X,X)\, ds,$$

where X is the unit tangent vector to the path γ. $-k_2 g \leq S_{ij} \leq k_3 g$ results that $-(k_2+k_3)g \leq S_{ij} \leq (k_2+k_3)g$, hence $\sup_M |S_{ij}|^2 \leq n(k_2+k_3)^2$. Now, we get

$$\frac{\partial\phi}{\partial t} = \frac{\psi'}{2\rho}\frac{\partial r}{\partial t} = \frac{\psi'}{2\rho}\int_\gamma S_{ij}(X,X)\, ds \leq \sqrt{c_2}(k_2+k_3)^2.$$

Suppose that $t\phi F$ achieves its positive maximum value at (v_0, t_0). Then at (x_0,t_0), we have $\nabla(t\phi F)(x_0,t_0) = 0$, $\frac{\partial}{\partial t}(t\phi F)(x_0,t_0) \geq 0$, $\mathcal{L}(t\phi F)(x_0,t_0) \geq 0$. Suppose that $y = \frac{|\nabla v|^2}{v} + \frac{S}{v}$, $\tilde{y} = t\phi y$, $z = \frac{v_t}{v} + \frac{S}{v} + \frac{d}{bt}$, $\tilde{z} = t\phi z$, then $F = y - bz$, $t\phi F = \tilde{y} - b\tilde{z}$ and from Lemma 4.1 we get

$$\begin{aligned}
\mathcal{L}(F) =& 2p\nabla_i F\nabla_i v - [\frac{b-1}{v} + p - 1]\big(\frac{\partial S}{\partial t} - 2\nabla_i S\nabla^i v + 2S^{ij}\nabla_i v\nabla_j v\big)\\
&- 2(p-1)(R^{ij} - S^{ij})\nabla_i v\nabla_j v - 2(p-1)|\nabla^2 v + \frac{b}{2}S_{ij}|^2\\
&+ \frac{(b-2)^2}{2}(p-1)|S_{ij}|^2 + (p-1)(1-b)\mathcal{D}(S) - \frac{1}{b}F^2\\
&- [(p-1)S - \frac{2(1-b)}{b}\frac{S}{v} + \frac{2d}{bt}]F - \frac{b-1}{b}y^2 - \frac{(b-1)(b-2)}{b}y\frac{S}{v}\\
&- \frac{d^2}{bt^2} - d(p-1)\frac{S}{t} + 2\frac{1-b}{b}\frac{d}{t}\frac{S}{v} + \frac{d}{t^2} - b(p-1)(2\nabla^i S_{il} - \nabla_l S)\nabla^l v.
\end{aligned}$$

Therefore,

$$\begin{aligned}
t\phi\mathcal{L}(t\phi F) =& t\phi^2 F + t^2\phi\phi_t F + t^2\phi^2\mathcal{L}(F) - (p-1)t^2\phi v F\Delta\phi - 2t^2(p-1)\phi v\nabla_i\phi\nabla^i F\\
=& \phi(\tilde{y} - b\tilde{z}) + t\phi_t(\tilde{y} - b\tilde{z}) - (p-1)tv\Delta\phi(\tilde{y} - b\tilde{z}) - 2t^2(p-1)\phi v\nabla_i\phi\nabla^i F\\
&+ 2pt^2\phi^2\nabla_i F\nabla_i v - t^2\phi^2[\frac{b-1}{v} + p - 1]\mathcal{H}(S,\nabla v) - 2(p-1)t^2\phi^2|\nabla^2 v
\end{aligned}$$

$$+\frac{b}{2}S_{ij}|^2+\frac{(b-2)^2}{2}(p-1)t^2\phi^2|S_{ij}|^2-\frac{1}{b}(\tilde{y}-b\tilde{z})^2$$
$$-[(p-1)S-\frac{2(1-b)}{b}\frac{S}{v}+\frac{2d}{bt}]t^2\phi^2F-\frac{b-1}{b}\tilde{y}^2-\frac{(b-1)(b-2)}{b}t\phi\frac{S}{v}\tilde{y}$$
$$-d(p-1)t\phi^2S+2\frac{1-b}{b}dt\phi^2\frac{S}{v}-t^2\phi^2(p-1)\mathcal{E}_b(S,\nabla v)$$
$$-\frac{d}{b}(d-b)\phi^2+t\phi^2(\frac{b-1}{v}+p-1)S.$$

On the other hand, we have

$$t_0^2\phi^2\nabla_iF\nabla^iv=-t_0^2\phi F\nabla_i\phi\nabla^iv\le t_0^2\phi F|\nabla_i\phi||\nabla^iv|\le\frac{\sqrt{c_1}}{\rho}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}(\tilde{y}-b\tilde{z}),$$
$$-(p-1)t_0v\Delta\phi(\tilde{y}-b\tilde{z})\le(p-1)t_0v(\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2})(\tilde{y}-b\tilde{z}),$$
$$-2(p-1)t_0^2v\phi\nabla_i\phi\nabla^iF=2(p-1)t_0^2v|\nabla\phi|^2F\le2(p-1)t_0v\frac{c_1}{\rho^2}(\tilde{y}-b\tilde{z}),$$

and
$$-2(p-1)t_0^2\phi^2|\nabla^2v+\frac{b}{2}S_{ij}|^2\le-\frac{2(p-1)t_0^2\phi^2}{n}(\Delta v+\frac{b}{2}S)^2$$
$$=-\frac{2(p-1)t_0^2\phi^2}{n}\left(-\frac{F}{b(p-1)}-\frac{b-1}{b(p-1)}\frac{|\nabla v|^2}{v}+\frac{b-2}{2}S-\frac{b-1}{b(p-1)}\frac{S}{v}\right)^2$$
$$=-\frac{2}{b^2n(p-1)}\left(\tilde{y}-b\tilde{z}+(b-1)\tilde{y}-\frac{b(b-2)}{2}(p-1)t_0\phi S\right)^2.$$

Thus
$$0\le t_0\phi\mathcal{L}(t_0\phi F)$$
$$\le(\tilde{y}-b\tilde{z})+t_0\sqrt{c_2}(k_2+k_3)^2(\tilde{y}-b\tilde{z})+(p-1)t_0v(\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2})(\tilde{y}-b\tilde{z})$$
$$+2(p-1)t_0v\frac{c_1}{\rho^2}(\tilde{y}-b\tilde{z})+2p\frac{\sqrt{c_1}}{\rho}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}(\tilde{y}-b\tilde{z})$$
$$-\frac{2}{b^2n(p-1)}\left(\tilde{y}-b\tilde{z}+(b-1)\tilde{y}-\frac{b(b-2)}{2}(p-1)t_0\phi S\right)^2$$
$$+\frac{(b-2)^2}{2}(p-1)t_0^2\phi^2S^2-\frac{1}{b}(\tilde{y}-b\tilde{z})^2.$$

Notice that $(r+s)^2\ge r^2+2rs$ results in

$$-\frac{2}{b^2n(p-1)}\left(\tilde{y}-b\tilde{z}+(b-1)\tilde{y}-\frac{b(b-2)}{2}(p-1)t_0\phi S\right)^2$$
$$\le-\frac{2}{b^2n(p-1)}(\tilde{y}-b\tilde{z})^2-\frac{4(b-1)}{b^2n(p-1)}\tilde{y}(\tilde{y}-b\tilde{z})+\frac{2(b-2)}{bn}t_0\phi S(\tilde{y}-b\tilde{z}).$$

Hence

$$0\le(\tilde{y}-b\tilde{z})+t_0\sqrt{c_2}(k_2+k_3)^2(\tilde{y}-b\tilde{z})+(p-1)t_0v(\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2})(\tilde{y}-b\tilde{z})$$

$$
\begin{aligned}
&+2(p-1)t_0v\frac{c_1}{\rho^2}(\tilde{y}-b\tilde{z})+2p\frac{\sqrt{c_1}}{\rho}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}(\tilde{y}-b\tilde{z})-\frac{1}{b\alpha}(\tilde{y}-b\tilde{z})^2\\
&-\frac{4(b-1)}{b^2n(p-1)}\tilde{y}(\tilde{y}-b\tilde{z})+\frac{2(b-2)}{bn}t_0\phi S(\tilde{y}-b\tilde{z})+\frac{(b-2)^2}{2}(p-1)t_0^2\phi^2S^2\\
\le&(\tilde{y}-b\tilde{z})\left[-\frac{4(b-1)}{b^2n(p-1)}\tilde{y}+2p\frac{\sqrt{c_1}}{\rho}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}+(\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2}+2\frac{c_1}{\rho^2})(p-1)t_0v\right]\\
&+(\tilde{y}-b\tilde{z})\left[t_0\sqrt{c_2}(k_2+k_3)^2+\frac{2(b-2)}{b}t_0(k_2+k_3)+1\right]\\
&+\frac{(b-2)^2}{2}(p-1)t_0^2n^2(k_2+k_3)^2-\frac{1}{b\alpha}(\tilde{y}-b\tilde{z})^2,
\end{aligned}
$$

where $\alpha=\frac{bn(p-1)}{2+bn(p-1)}$. For $a>0$, the inequality $-ax^2+bx\le\frac{b^2}{4a}$ implies that

$$
-\frac{4(b-1)}{b^2n(p-1)}\tilde{y}+2p\frac{\sqrt{c_1}}{\rho}\tilde{y}^{\frac{1}{2}}(t_0v)^{\frac{1}{2}}\le\frac{b^2p^2nc_1}{4(b-1)\rho^2}(p-1)t_0v.
$$

Therefore,

$$
\begin{aligned}
0\le&(\tilde{y}-b\tilde{z})\left[\left(\frac{b^2p^2nc_1}{4(b-1)\rho^2}+\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2}+2\frac{c_1}{\rho^2}\right)(p-1)t_0v+t_0\sqrt{c_2}(k_2+k_3)^2\right.\\
&\left.+\frac{2(b-2)}{b}t_0(k_2+k_3)+1\right]+\frac{(b-2)^2}{2}(p-1)t_0^2n^2(k_2+k_3)^2-\frac{1}{b\alpha}(\tilde{y}-b\tilde{z})^2.
\end{aligned}
$$

If $0\le-ax^2+bx+c$ for $a,b,c>0$, then $x\le\frac{b}{a}+\sqrt{\frac{c}{a}}$. Hence

$$
\begin{aligned}
\tilde{y}-b\tilde{z}\le&b\alpha\left[\left(\frac{b^2p^2nc_1}{4(b-1)\rho^2}+\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2}+2\frac{c_1}{\rho^2}\right)(p-1)t_0v+t_0\sqrt{c_2}(k_2+k_3)^2\right.\\
&\left.+\frac{2(b-2)}{b}t_0(k_2+k_3)+1\right]+t_0n(k_2+k_3)(b-2)\sqrt{\frac{b(p-1)\alpha}{2}}.
\end{aligned}
$$

If $d(x,x_0,\tau)<\rho$, then $\phi(x,\tau)=1$. Since (x_0,t_0) is the maximum point for $t\phi F$ in $\mathcal{Q}_{\rho,T}$, we have $\tau F(x,\tau)=(\tau\phi F)(x,\tau)\le(t_0\phi F)(x_0,t_0)$, for all $x\in M$, such that $d(x,x_0,\tau)<\rho$ and $\tau\in[0,T]$ is arbitrary. Then we have

$$
\begin{aligned}
F\le&b\alpha\left[\left(\frac{b^2p^2nc_1}{4(b-1)\rho^2}+\frac{c_1\sqrt{k_1}}{2\rho}+\frac{c_1}{4\rho^2}+2\frac{c_1}{\rho^2}\right)(p-1)v+\sqrt{c_2}(k_2+k_3)^2\right.\\
&\left.+\frac{2(b-2)}{b}(k_2+k_3)+1\right]+n(k_2+k_3)(b-2)\sqrt{\frac{b(p-1)\alpha}{2}}.
\end{aligned}
$$

□

Proof of Theorem 2.1. In Proposition 4.3, suppose that $b=2$. Then inequality (5) follows from (19).

Proof of Corollary 2.2. If u is bounded on $M\times[0,T]$, then assume that $\rho\to\infty$. Then the inequality of Theorem 2.1 yields (6).

Proof of Theorem 2.3. For any curve $\gamma(t)$, $t\in[t_1,t_2]$, from $\gamma(t_1)=x_1$ to $\gamma(t_2)=x_2$, we have

$$
\log\frac{v(x_2,t_2)}{v(x_1,t_1)}=\int_{t_1}^{t_2}\frac{d}{dt}\log v(\gamma(t),t)\,dt=\int_{t_1}^{t_2}\frac{v_t}{v}+\frac{\nabla v}{v}\frac{d\gamma}{dt}\,dt.
$$

From the inequality $xy \geq -\frac{x^2}{2} - \frac{y^2}{2}$ for any x, y, it follows $\nabla v . \frac{d\gamma}{dt} \geq -\frac{|\nabla v|^2}{2} - \frac{1}{2}|\frac{d\gamma}{dt}|^2$. Hence,

$$\log \frac{v(x_2,t_2)}{v(x_1,t_1)} \geq \int_{t_1}^{t_2} (\frac{v_t}{v} - \frac{|\nabla v|^2}{2v} - \frac{1}{2v}|\frac{d\gamma}{dt}|^2)\, dt.$$

Corollary 2.2 implies that

$$\begin{aligned}\log \frac{v(x_2,t_2)}{v(x_1,t_1)} &\geq \int_{t_1}^{t_2} \big(-\frac{n(p-1)}{1+n(p-1)}E_2 - \frac{S}{2v_{\min}} - \frac{d}{2t} - \frac{1}{2v_{\min}}|\frac{d\gamma}{dt}|^2\big)\, dt \\ &= -\frac{n(p-1)}{1+n(p-1)}E_2(t_2-t_1) - (\frac{t_2}{t_1})^{\frac{d}{2}} - \frac{1}{2v_{\min}} \int_{t_1}^{t_2} \big(S + |\frac{d\gamma}{dt}|^2\big)\, dt.\end{aligned}$$

By taking exponents, we arrive at (7).

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