MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 4 (2022), 289–300
December 2022

research paper
оригинални научни рад
DOI: 10.57016/MV-ockn6579

NAPOLEON'S THEOREM FROM THE VIEW POINT OF GRÖBNER BASES

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Abstract. In this article, we present a new proof of the Napoleon's theorem using algorithmic commutative algebra and algebraic geometry. We also show that, by using the same technique, several related theorems, with the same basic set of objects can be proved. Thus, from the new proof of Napoleon's theorem, we prove the Relative of Napoleon's theorem (result given by B. Grünbaum). Then, we present a new theorem related to Napoleon's theorem. In this theorem the existence of two more quadruplets of equilateral triangles associated with a given triangle was established.

1. Introduction

Napoleon's theorem from elementary geometry has a fascinating history. An earlier known formulation of this claim was published in 1825 in the journal *The Ladies' Diary*. W. Rutherford posed the problem, and according to all relevant data, neither the result nor the proof has anything to do with Napoleon. For a fuller treatment of the history of this theorem, we refer the reader to [6]. Nevertheless, the name Napoleon's theorem is generally accepted and will probably remain so.

Napoleon's theorem was proved by using various techniques. A proof of the theorem using elementary geometry can be found in [2, 9]. Numerous proofs use vectors and trigonometry, complex numbers, but also coordinate approach.

There are also various generalizations of this theorem. The most famous are undoubtedly the Napoleon-Barlotty theorem [7] and the Petr-Douglas-Neumann theorem [4].

The discovery of Gröbner bases and the development of algorithms for their calculation began in the 1980s. At the same time, algorithms for the automatic proof of geometric theorems were developed. Some of the automatic theorem proving algorithms use pseudodivision of polynomials, such as Wu's method [3]. Modern, sophisticated methods like the Gröbner Cover, make it possible to discover hypotheses

2020 Mathematics Subject Classification: 13P10, 68T15, 51M04.
Keywords and phrases: Napoleon's theorem; automatic theorem proving; elementary geometry; Gröbner bases.

and generalize theorems from geometry [8,10]. However, most of these algorithms are based on the calculation of Gröbner bases.

Note that not all theorems in geometry are suitable for proving using the method of commutative algebra. Even when possible, it is not an easy task.

The aim of our paper is to offer a new proof of Napoleon's theorem using algorithmic commutative algebra. We will also use the same method to prove the result discovered by Branko Grünbaum in his paper [5]. Grünbaum established two more quadruplets of equilateral triangles joining the basic construction from Napoleon's theorem and called this result a Relative of Napoleon's theorem. We will also present a new theorem that might be considered as a relative of Grünbaum's theorem. Starting from the construction from Napoleon's theorem, two quadruplets of equilateral triangles can be joined to a given triangle in different way. Of course, the new theorem can also be proved by the same technique as the previous two.

2. Gröbner bases and theorem proving

The theory of Gröbner bases is one of the algorithmic methods used in automatic geometric theorem proving for a long time. We consider the basic ideas underlying this method. For a more detailed presentation of the method, we refer the reader to [1].

We first introduce Cartesian coordinates in the Euclidean plane, which allows us to translate geometric statements to the language of algebra. We place the observed figure in the coordinate plane and assign coordinates to the vertices. We always presume that some coordinates depend upon our choices of other points. Coordinate of the points that are chosen arbitrarily are denoted by u_i. Such points are called free points. Coordinates of the points that depend upon our choices for free points are denoted by x_i.

Hypotheses and the conclusions of a large class of geometric theorems can be expressed as polynomial equations whose variables are coordinates of points specified in the statements. Such theorems are called admissible.

The process of hypotheses translation to the equations is not uniquely determined. If we add to this the fact that the choice of free points is not unique either, we can conclude that the same theorem can be translated into the language of equations in several different ways. Thus, we can obtain complex systems of equations determined by the theorem's assumptions or simple systems with a minimum number of variables.

The typical form of hypotheses of admissible geometric theorem translated to the language of polynomials in variables $u_1, \ldots, u_m, x_i, \ldots, x_n$ is the following:

$$\begin{aligned} h_1(u_1, \ldots, u_m, x_i, \ldots, x_n) &= 0 \\ h_2(u_1, \ldots, u_m, x_i, \ldots, x_n) &= 0 \\ &\vdots \\ h_l(u_1, \ldots, u_m, x_i, \ldots, x_n) &= 0. \end{aligned}$$

The conclusion of the theorem is expressed in the same form as polynomials in the $u_1, \ldots, u_m, x_i, \ldots, x_n$. It is sufficient to consider the case of one conclusion since we can treat one at a time if there are more. Hence we write the conclusion as $g(u_1, \ldots, u_m, x_i, \ldots, x_n) = 0$. If we want to show that g is satisfied if the assumptions $h_1, \ldots, h_l$ hold, then in the language of algebra, we want to show that g vanishes whenever $h_1, \ldots, h_l$ do. The hypotheses define a variety, which we denote by $V = \mathbf{V}(h_1, \ldots, h_l)$. This leads to the following definition.

Definition 2.1. The conclusion g follows strictly from the hypotheses $h_1, \ldots h_l$ if $g \in \mathbf{I}(V) \subseteq \mathbb{R}[u_1, \ldots, u_m, x_1, \ldots, x_n]$, where $V = \mathbf{V}(h_1, \ldots, h_l)$.

Since we are working over $\mathbb{R}$, we do not have an effective method for determining $\mathbf{I}(V)$, but we do have the following useful criterion (see [1, Section 6.4.]).

Proposition 2.2. *If $g \in \sqrt{\langle h_1, h_2, \ldots h_l \rangle}$, then g follows strictly from $h_1, \ldots, h_l$.*

In the described way, we have reduced the geometric problem to the problem of belonging to a given ideal in a polynomial ring which is solved by applying the theory of the Gröbner bases. Namely, only when the Gröbner bases generate the observed ideal, the remainder of the division of a polynomial with ideal generators determines whether or not the polynomial belongs to a given ideal.

For most geometric theorems, Definition 2.1 is too strict. There are two main reasons for this. The first reason is that ideal $\mathbf{I}(V)$ is an ideal in a ring of polynomials with coefficients from a set of real numbers $\mathbb{R}$. We do not have effective methods for determining this ideal, but the different criteria can be used (see [1]). The second reason is that the variety of ideals is reducible. Such varieties contain subvarieties of degenerate cases in which the assertions of the theorem often do not hold or do not make sense. We overcome this problem by isolating the corresponding subvarieties on which the theorem is observed.

However, the theorems we are interested in, belong to the category of very rare geometric theorems to which the Definition 2.1 refers.

3. Napoleon's theorem and its relatives

We now use the described method of application of Gröbner bases to prove the famous Napoleon's theorem, Grünbaum's theorem, and an entirely new theorem related to Napoleon's theorem.

We have already noted that there are several different ways a theorem can be translated into the language of equations. We intend to obtain the most compact and elegant form of the ideal generated by the hypotheses of the theorem. That is why we need the following proposition.

Proposition 3.1. *Given an arbitrary triangle ABC:*
(i) Let the equilateral triangles BA_1C, CB_1A, AC_1B be constructed not overlapping ABC and let their centroids be denoted by T_1, T_2, T_3. Then centroids of the triangles ABC and $T_1T_2T_3$ coincide.

(ii) Let the equilateral triangles CA_2B, AB_2C, BC_2A be constructed so that each one overlaps ABC and let their centroids be denoted by S_1, S_2, S_3. Then centroids of the triangles ABC and $S_1S_2S_3$ coincide.

Proof. Denote the midpoints of CB, AC, and AB by A', B' and C', respectively. Let the equilateral triangles BA_1C, CB_1A and AC_1B be constructed on the sides of the triangle ABC externally and their centroids denoted by T_1, T_2, T_3. Denote by T the centroid of the triangle ABC (see Figure 1).

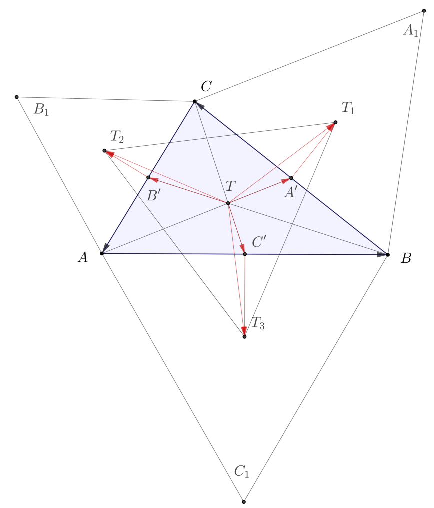

Figure 1: Proposition 2.2

We know that T is the centroid of the triangle ABC if and only if the following equation holds

$$\overrightarrow{TA} + \overrightarrow{TB} + \overrightarrow{TC} = \vec{0}. \tag{1}$$

We have

$$\begin{aligned}\overrightarrow{TT_1} + \overrightarrow{TT_2} + \overrightarrow{TT_3} = &\ \overrightarrow{TA'} + \overrightarrow{A'T_1} + \overrightarrow{TB'} + \overrightarrow{B'T_2} + \overrightarrow{TC'} + \overrightarrow{C'T_3} \\ = &-\frac{1}{2}\left(\overrightarrow{TA} + \overrightarrow{TB} + \overrightarrow{TC}\right) + \left(\overrightarrow{A'T_1} + \overrightarrow{B'T_2} + \overrightarrow{C'T_3}\right). \end{aligned} \tag{2}$$

If we rotate the vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ by an angle of $90°$ clockwise, we get vectors that have the same direction as the vectors $\overrightarrow{A'T_1}$, $\overrightarrow{B'T_2}$ and $\overrightarrow{C'T_3}$, respectively. Since T_1, T_2 and T_3 are the centroids of equilateral triangles BA_1C, CB_1A and AC_1B it follows that

$$\left|\overrightarrow{A'T_1}\right| = \frac{\sqrt{3}}{6}\left|\overrightarrow{BC}\right|, \quad \left|\overrightarrow{B'T_2}\right| = \frac{\sqrt{3}}{6}\left|\overrightarrow{CA}\right|, \quad \left|\overrightarrow{C'T_3}\right| = \frac{\sqrt{3}}{6}\left|\overrightarrow{AB}\right|.$$

From this we obtain

$$\overrightarrow{A'T_1} + \overrightarrow{B'T_2} + \overrightarrow{C'T_3} = \vec{0}. \tag{3}$$

We conclude from (1), (2) and (3) that $\overrightarrow{TT_1} + \overrightarrow{TT_2} + \overrightarrow{TT_3} = \vec{0}$, hence T is centroid of the triangle $T_1T_2T_3$.

In the same manner we can see that (ii) holds. $\square$

THEOREM 3.2. *Given an arbitrary triangle ABC.*
(i) Let the equilateral triangles CA_1B, CB_1A, AC_1B be constructed not overlapping ABC and their centroids be denoted by T_1, T_2, T_3. Then $T_1T_2T_3$ is an equilateral triangle.

(ii) Let the equilateral triangles CA_2B, AB_2C, BC_2A be constructed so that each overlaps ABC and let their centroids be denoted by S_1, S_2, S_3. Then $S_1S_2S_3$ is an equilateral triangle.

(iii) If we denote by (ABC) the area of the triangle ABC, then the following equality holds:

$$(ABC) = (T_1T_2T_3) - (S_1S_2S_3). \tag{4}$$

Proof. Let us apply the technique described earlier to prove the theorem. First, we write the hypotheses and the conclusion in the form of equations.

The properties of triangles are unchanged under isometric transformations. Hence we can place the triangle so that the vertex A be at the origin and align the side AB with the horizontal coordinate axis. The unit of length can be chosen arbitrarily, so we assume that the coordinates of vertex B are given by $(1, 0)$. The vertex C of the triangle can be at any point (u_1, u_2), where u_1, u_2 are variables whose values are in $\mathbb{R}$. All other points are entirely determined by A, B and C.

Denote by A', B', C' midpoints of the BC, AC and AB (see Figure 2). Then we have $A'\left(\frac{u_1+1}{2}, \frac{u_2}{2}\right)$, $B'\left(\frac{u_1}{2}, \frac{u_2}{2}\right)$, $C'\left(\frac{1}{2}, 0\right)$.

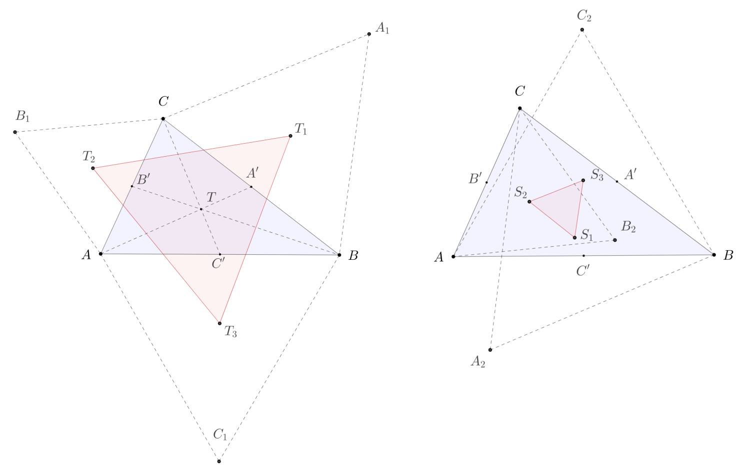

Figure 2: Illustration of parts (i) and (ii) of the Theorem 3.2

Denote by s_{AB}, s_{BC} and s_{AC} the bisectors of the sides of triangle ABC. Their equations are given by

$$\begin{aligned} s_{AB} &: x - \frac{1}{2} = 0 \\ s_{AC} &: 2u_1x + 2u_2y - u_1^2 - u_2^2 = 0 \\ s_{BC} &: 2u_2y - u_2^2 - 2(1-u_1)x + (1-u_1^2) = 0. \end{aligned}$$

Let (x_1, x_2) be the coordinates of the vertex of an equilateral triangle constructed over the side BC. This point belongs to the intersection of the bisector and the circle whose centre is at point B and diameter is equal to BC. Therefore x_1 and x_2 satisfy the system of equations:

$$s_1 \begin{cases} 2(1-u_1)x_1 - 2u_2x_2 + u_1^2 + u_2^2 - 1 = 0 \\ (x_1-1)^2 + x_2^2 - (u_1-1)^2 - u_2^2 = 0. \end{cases}$$

If (x_3, x_4) denote the coordinates of the centroid of this equilateral triangle, then they satisfy the following system

$$s_2 \begin{cases} x_1 + u_1 + 1 - 3x_3 = 0 \\ x_2 + u_2 - 3x_4 = 0. \end{cases}$$

Similarly, we conclude that the coordinates (x_5, x_6) of the vertex of an equilateral triangle constructed over the side AC satisfy the following system of equations:

$$s_3 \begin{cases} 2u_2x_6 + 2u_1x_5 - u_1^2 - u_2^2 = 0 \\ x_5^2 + x_6^2 - u_1^2 - u_2^2 = 0. \end{cases}$$

If we denote by (x_7, x_8) the coordinates of the centroid of that equilateral triangle, then we have:

$$s_4 \begin{cases} x_5 + u_1 - 3x_7 = 0 \\ x_6 + u_2 - 3x_8 = 0. \end{cases}$$

Note that each of the systems s_1 and s_3 defines two points. Namely, we can construct an equilateral triangle from the outside and the inside over each side of the triangle. Similarly, the systems s_2 and s_4 determine a pair of points that are the centroids of the inner and outer constructed triangle. So, if we only use these systems, we will not distinguish between the inner and outer equilateral triangle. Therefore, we will use Proposition 3.1.

Also, due to the choice of vertices A and B of the triangle ABC, the coordinates of equilateral triangles constructed over side AB are given by $C_1\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $C_2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Thus, the coordinates of their centroids are entirely determined, $T_3\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right)$, $S_3\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$. Note that C_1 is the vertex of the outer triangle, and C_2 is the vertex of the inner triangle constructed over AB if $u_2 > 0$. If $u_2 < 0$ the role of these vertices is replaced with each other.

It follows from Proposition 2.2 that the centroids of the triangles ABC, $T_1T_2T_3$

and $S_1S_2S_3$ coincide, hence we conclude that

$$s_5 \begin{cases} x_3 + x_7 - u_1 - \frac{1}{2} = 0 \\ x_4 + x_8 - u_2 - \frac{\sqrt{3}}{6} = 0, \end{cases}$$

$$s_5' \begin{cases} x_3 + x_7 - u_1 - \frac{1}{2} = 0 \\ x_4 + x_8 - u_2 + \frac{\sqrt{3}}{6} = 0. \end{cases}$$

Statements (i) and (ii) of Napoleon's theorems are identical, but the assumptions differ. To prove them, we do not have to know which case refers to the outer and the inner triangles. It suffices to prove that the assertions are valid in both cases.

We can now represent the hypotheses of the theorem by two collections of polynomial equations in variables $u_1, u_2, x_1, \ldots, x_8$ and form the appropriate ideals. The ideal of hypotheses relating to triangles constructed not to overlap with the triangle ABC (for $u_2 > 0$) contains the equations of systems s_1, s_2, s_3, s_4 and s_5. We will denote this ideal by I_1.

$$\begin{aligned} I_1 = \langle & 2(1-u_1)x_1 - 2u_2x_2 + u_1^2 + u_2^2 - 1, (x_1-1)^2 + x_2^2 - (u_1-1)^2 - u_2^2, \\ & x_1 + u_1 + 1 - 3x_3, x_2 + u_2 - 3x_4, 2u_2x_6 + 2u_1x_5 - u_1^2 - u_2^2, x_5^2 + x_6^2 - u_1^2 - u_2^2, \\ & x_5 + u_1 - 3x_7, x_6 + u_2 - 3x_8, x_3 + x_7 - u_1 - \frac{1}{2}, x_4 + x_8 - u_2 - \frac{\sqrt{3}}{6}\rangle. \end{aligned}$$

The ideal of hypotheses relating to triangles constructed to overlap with the triangle ABC (for $u_2 > 0$) contains the equations of systems s_1, s_2, s_3, s_4 and s_5'. We will denote this ideal by I_2.

$$\begin{aligned} I_2 = \langle & 2(1-u_1)x_1 - 2u_2x_2 + u_1^2 + u_2^2 - 1, (x_1-1)^2 + x_2^2 - (u_1-1)^2 - u_2^2, \\ & x_1 + u_1 + 1 - 3x_3, x_2 + u_2 - 3x_4, 2u_2x_6 + 2u_1x_5 - u_1^2 - u_2^2, x_5^2 + x_6^2 - u_1^2 - u_2^2, \\ & x_5 + u_1 - 3x_7, x_6 + u_2 - 3x_8, x_3 + x_7 - u_1 - \frac{1}{2}, x_4 + x_8 - u_2 + \frac{\sqrt{3}}{6}\rangle. \end{aligned}$$

Using lex order with $x_1 > x_2 > \cdots > x_8 > u_1 > u_2$, the Gröbner bases for ideal I_1 is given by $GBI_1 = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$, where

$$\begin{aligned} f_1 &= -\sqrt{3}u_1 - 3u_2 + 6x_8, & f_5 &= -\sqrt{3} + \sqrt{3}u_1 - 3u_2 + 6x_4, \\ f_2 &= -3u_1 + \sqrt{3}u_2 + 6x_7, & f_6 &= -3 - 3u_1 - \sqrt{3}u_2 + 6x_3, \\ f_3 &= -\sqrt{3}u_1 - u_2 + 2x_6, & f_7 &= -\sqrt{3} + \sqrt{3}u_1 - u_2 + 2x_2, \\ f_4 &= -u_1 + \sqrt{3}u_2 + 2x_5, & f_8 &= -1 - u_1 - \sqrt{3}u_2 + 2x_1. \end{aligned}$$

Using the same order, we find the Gröbner basis $GBI_2 = \{f_1', f_2', f_3', f_4, f_5', f_6', f_7', f_8'\}$, for the ideal I_2, where

$$\begin{aligned} f_1' &= -u_1 + \sqrt{3}u_2 - 2\sqrt{3}x_8, & f_5' &= -1 + u_1 + \sqrt{3}u_2 - 2\sqrt{3}x_4, \\ f_2' &= 3u_1 + \sqrt{3}u_2 - 6x_7, & f_6' &= 3 + 3u_1 - \sqrt{3}u_2 - 6x_3, \\ f_3' &= 3u_1 - \sqrt{3}u_2 + 2\sqrt{3}x_6, & f_7' &= 3 - 3u_1 - \sqrt{3}u_2 + 2\sqrt{3}x_2, \\ f_4' &= -u_1 - \sqrt{3}u_2 + 2x_5, & f_8' &= -1 - u_1 + \sqrt{3}u_2 + 2x_1. \end{aligned}$$

The triangle $T_1T_2T_3$ is equilateral if and only if $T_1T_2 = T_1T_3$ and $T_1T_2 = T_2T_3$. Therefore, from statements (i) and (ii), the conclusion of the theorem translated into the language of algebra, can be expressed by

$$g_1 =(x_3 - x_7)^2 + (x_4 - x_8)^2 - \left(x_7 - \frac{1}{2}\right)^2 - \left(x_8 + \frac{\sqrt{3}}{6}\right)^2,$$

$$g_2 =(x_3 - x_7)^2 + (x_4 - x_8)^2 - \left(x_3 - \frac{1}{2}\right)^2 - \left(x_4 + \frac{\sqrt{3}}{6}\right)^2.$$

Let us examine whether the polynomials g_1 and g_2 belong to the ideal I_1. Therefore, we divide the polynomials g_1 and g_2 by the set $\{f_1, \dots, f_8\}$ which is the Gröbner basis of the ideal I_1. Since

$$\begin{aligned} g_1 =& -\frac{1}{3}\left(\frac{1}{2\sqrt{3}} - x_4\right) f_1 + \frac{1}{3}\left(\frac{1}{2} - x_3\right) f_2 \\ &+ \frac{1}{3}\left(\frac{1}{4\sqrt{3}} - \frac{\sqrt{3}u_1}{4} - \frac{u_2}{4} + \frac{x_4}{2}\right) f_5 + \frac{1}{3}\left(\frac{1}{4} - \frac{u_1}{4} + \frac{\sqrt{3}u_2}{4} + \frac{x_3}{2}\right) f_6, \\ g_2 =& \frac{1}{3}\left(\frac{u_1}{4\sqrt{3}} + \frac{u_2}{4} - x_4 + \frac{x_8}{2}\right) f_1 + \frac{1}{3}\left(\frac{u_1}{4} - \frac{u_2}{4\sqrt{3}} - x_3 + \frac{x_7}{2}\right) f_2 \\ &+ \frac{1}{3}\left(-\frac{1}{2\sqrt{3}} - \frac{u_1}{2\sqrt{3}} - \frac{u_2}{2}\right) f_5 + \frac{1}{3}\left(\frac{1}{2} - \frac{u_1}{2} + \frac{u_2}{2\sqrt{3}}\right) f_6, \end{aligned}$$

the remainder is zero, and we have $g_1, g_2 \in I_1$.

Similarly,

$$\begin{aligned} g_1 =& \frac{1}{3}\left(-\frac{1}{2} + \sqrt{3}x_4\right) f_1' + \frac{1}{3}\left(-\frac{1}{2} + x_3\right) f_2' \\ &+ \frac{1}{3}\left(\frac{1}{4} - \frac{3u_1}{4} + \frac{\sqrt{3}u_2}{4} - \frac{\sqrt{3}x_4}{2}\right) f_5' + \frac{1}{3}\left(-\frac{1}{4} + \frac{u_1}{4} + \frac{\sqrt{3}u_2}{4} - \frac{x_3}{2}\right) f_6', \\ g_2 =& \frac{1}{3}\left(\frac{u_1}{4} - \frac{\sqrt{3}u_2}{4} + \sqrt{3}x_4 + \frac{\sqrt{3}x_8}{2}\right) f_1' + \frac{1}{3}\left(-\frac{u_1}{4} - \frac{u_2}{4\sqrt{3}} + x_3 - \frac{x_7}{2}\right) f_2' \\ &+ \frac{1}{3}\left(-\frac{1}{2} - \frac{u_1}{2} + \frac{\sqrt{3}u_2}{2}\right) f_5' + \frac{1}{3}\left(-\frac{1}{2} + \frac{u_1}{2} + \frac{u_2}{2\sqrt{3}}\right) f_6', \end{aligned}$$

so we conclude that $g_1, g_2 \in I_2$.

Since $I \subseteq \sqrt{I}$ is satisfied for every ideal I, it follows from above that $g_1, g_2 \in \sqrt{I_1}$ and $g_1, g_2 \in \sqrt{I_2}$. From Proposition 2.2 we now have that the conclusions g_1 and g_2 follow directly from hypotheses of the theorem. We have thus proved (i) and (ii) of Theorem 3.2.

Let us prove that (iii) holds. As mentioned earlier, the role of triangles $T_1T_2T_3$ and $S_1S_2S_3$ changes depending on the value of u_2. Therefore, condition (4) can be replaced by the condition

$$(ABC)^2 = ((T_1T_2T_3) - (S_1S_2S_3))^2 . \tag{5}$$

The triangle $T_1T_2T_3$ is equilateral, and we can quickly compute its area if we know the coordinates of its vertices. On the other hand, observing the polynomials from the Gröbner bases GBI_1, we see that each contains only one of the variables $x_1, x_2, \ldots, x_8$. This allows us to simply determine the area of the triangle $T_1T_2T_3$ depending only on u_1 and u_2. It follows that

$$(T_1T_2T_3) = \frac{1}{4\sqrt{3}}\left(1 - u_1 + u_1^2 + \sqrt{3}u_2 + u_2^2\right). \tag{6}$$

In the same manner, from the Gröbner bases GBI_2 we have

$$(S_1S_2S_3) = \frac{1}{4\sqrt{3}}\left(1 - u_1 + u_1^2 - \sqrt{3}u_2 + u_2^2\right). \tag{7}$$

Substituting (6) and (7) into (5), after simplifying, we obtain $((T_1T_2T_3) - (S_1S_2S_3))^2 = \frac{u_2^2}{4}$, which is the square of area of the triangle ABC. □

REMARK 3.3. This theorem is one of the very rare theorems in elementary geometry where conclusions directly follow from the hypotheses. The variety of the ideal of hypotheses is irreducible. Although geometrically speaking, the theorem does not make sense when the vertex C is on the x-axis (triangle ABC then becomes a line segment) algebraically, the conclusions of the theorem are still valid. A similar situation occurs in the case when a given triangle ABC is equilateral. Then, the outer Napoleon's triangle degenerates into a point. However, algebraically, the conclusions of the theorem remain satisfied.

THEOREM 3.4. *Given an arbitrary triangle ABC.*
(i) Let the equilateral triangles AC_1B, BA_1C, CB_1A be constructed not overlapping ABC. Denote the midpoints of B_1C_1, C_1A_1 and A_1B_1 by $\overline{A}_1$, $\overline{B}_1$ and $\overline{C}_1$, respectively. Then, $\overline{C}_1A\overline{B}_1$, $\overline{A}_1B\overline{C}_1$ and $\overline{B}_1C\overline{A}_1$ are equilateral triangles, and the centroids T_1^*, T_2^* and T_3^* of these triangles are vertices of an equilateral triangle.

(ii) Let the equilateral triangles AC_2B, BA_2C, CB_2A be constructed so that each overlaps ABC. Denote the midpoints of B_2C_2, C_2A_2 and A_2B_2 by $\overline{A}_2$, $\overline{B}_2$ and $\overline{C}_2$, respectively. Then, $\overline{B}_2A\overline{C}_2$, $\overline{C}_2B\overline{A}_2$ and $\overline{A}_2C\overline{B}_2$ are equilateral triangles, and the centroids S_1^*, S_2^* and S_3^* of these triangles are vertices of an equilateral triangle.

(iii) If we denote by (ABC) the area of the triangle ABC, then the following equality holds: $(ABC) = 4\left((T_1^*T_2^*T_3^*) - (S_1^*S_2^*S_3^*)\right)$.

Proof. Note that in this theorem, we have the same basic construction as in Napoleon's theorem. Therefore, we can use the same ideals I_1 and I_2 generated by the hypotheses of Theorem 3.2. We will keep the same notation for the coordinates of the vertices and the essential points of the basic structure.

Denote by $\overline{A}_1$, $\overline{B}_1$, $\overline{C}_1$ the midpoints of the sides of triangle $A_1B_1C_1$, and by T_1^*, T_2^*, T_3^*, the centroids of triangles $\overline{C}_1A\overline{B}_1$, $\overline{A}_1B\overline{C}_1$ and $\overline{B}_1C\overline{A}_1$, respectively. The coordinates of these points can be expressed in terms of the coordinates of the points of the base structure (see Figure 3).

$$\overline{A}_1\left(\frac{2x_5+1}{4}, \frac{2x_6-\sqrt{3}}{4}\right),\ \overline{B}_1\left(\frac{2x_1+1}{4}, \frac{2x_2-\sqrt{3}}{4}\right),\ \overline{C}_1\left(\frac{x_1+x_5}{2}, \frac{x_2+x_6}{2}\right),$$

$$T_1^*\left(\frac{4x_1+2x_5+1}{12},\frac{4x_2+2x_6-\sqrt{3}}{12}\right),\ T_2^*\left(\frac{4x_5+2x_1+5}{12},\frac{4x_6+2x_2-\sqrt{3}}{12}\right),$$
$$T_3^*\left(\frac{2x_5+2x_1+4u_1+2}{12},\frac{2x_6+2x_2-2\sqrt{3}+4u_2}{12}\right).$$

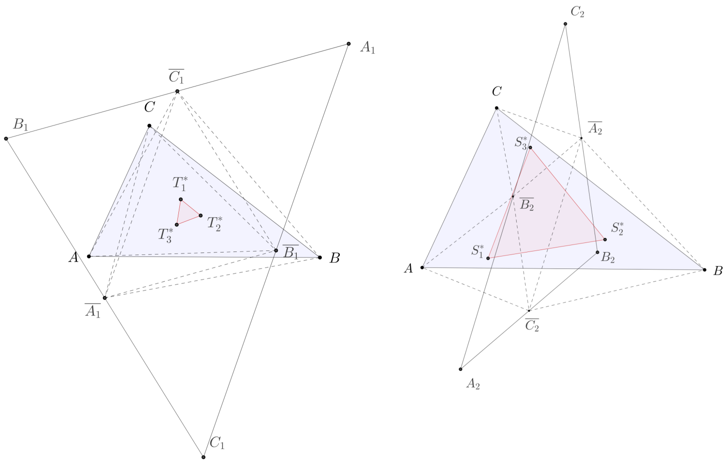

Figure 3: Illustration of parts (i) and (ii) of the Theorem 3.4

The conclusion that the triangle $\overline{C}_1A\overline{B}_1$ is equilateral is expressed by the following equations

$$c_1=(2x_1+1)^2+(2x_2-\sqrt{3})^2-(2x_5-1)^2-(2x_6+\sqrt{3})^2$$
$$c_2=4\left((x_1+x_5)^2+(x_2+x_6)^2\right)-(2x_5+1)^2-(2x_6-\sqrt{3})^2.$$

As in the proof of Theorem 3.2, after dividing by the polynomials that form the Gröbner bases of the ideal I_1, we find

$$\begin{aligned}c_1=&(-2\sqrt{3}-\sqrt{3}u_1-u_2-2x_6)f_3+(2-u_1+\sqrt{3}u_2-2x_5)f_4\\&+(-\sqrt{3}-\sqrt{3}u_1+u_2+2x_2)f_7+(3+u_1+\sqrt{3}u_2+2x_1)f_8,\\c_2=&(\sqrt{3}-2+4x_2)f_3+(2+4x_1)f_4\\&+(\sqrt{3}+\sqrt{3}u_1+3u_2+2x_2)f_7+(1+3u_1-\sqrt{3}u_2+2x_1)f_8.\end{aligned}$$

It follows that $c_1,c_2\in I_1\subseteq\sqrt{I_1}$. Thus, the conclusion that the triangle $\overline{C}_1A\overline{B}_1$ is equilateral follows directly from the hypotheses of the theorem.

The rest of the proof runs as before. □

The following theorem provides the existence of two more quadruplets of equilateral triangles associated with a given triangle ABC.

The basic construction of points in this theorem remains the same as in the previous two (see Figure 4). The conclusions of the theorem can be expressed in polynomial form with the same variables as before. We can use ideals I_1 and I_2 as ideals of hypotheses, and their Gröbner bases GBI_1 and GBI_2 that are already calculated.

So, the proof of the following theorem is omitted because it can be obtained in the same way as the previous one.

THEOREM 3.5. *Given an arbitrary triangle ABC, denote midpoints of BC, CA and AB by A', B' and C', respectively.*

(i) Let the equilateral triangles AC_1B, BA_1C, CB_1A be constructed not overlapping ABC. Denote the midpoints of B_1C_1, C_1A_1 and A_1B_1 by $\overline{A_1}$, $\overline{B_1}$ and $\overline{C_1}$, respectively. Then, $\overline{A_1}B'C'$, $\overline{B_1}A'C'$ and $\overline{C_1}B'A'$ are equilateral triangles, and the centroids T_1', T_2' and T_3' of these triangles are vertices of an equilateral triangle.

(ii) Let the equilateral triangles AC_2B, BA_2C, CB_2A are constructed so that each overlaps ABC. The midpoints of B_2C_2, C_2A_2 and A_2B_2 are denoted by $\overline{A_2}$, $\overline{B_2}$ and $\overline{C_2}$ respectively. Then, $\overline{A_2}B'C'$, $\overline{B_2}C'A'$ and $\overline{C_2}A'B'$ are equilateral triangles, and the centroids S_1', S_2' and S_3' of these triangles are vertices of an equilateral triangle.

(iii) If we denote by (ABC) the area of the triangle ABC, then the following equality holds: $(ABC) = 4\left((T_1'T_2'T_3') - (S_1'S_2'S_3')\right)$.

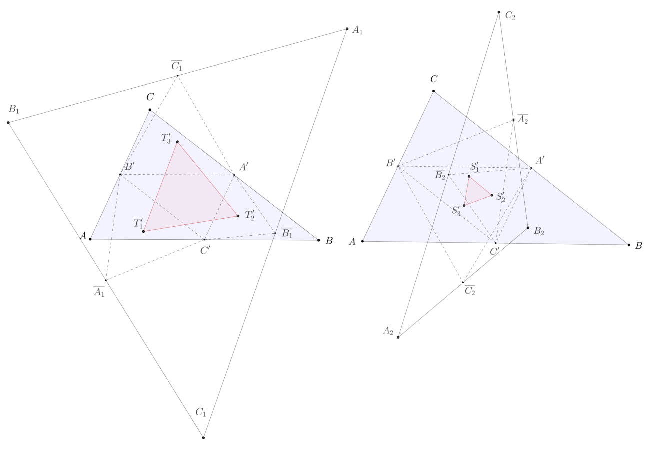

Figure 4: Illustration of parts (i) and (ii) of the Theorem 3.5

REMARK 3.6. The triangles $T_1^*T_2^*T_3^*$ and $S_1'S_2'S_3'$ are obtained by completely different constructions. Nevertheless, comparing the coordinates of their vertices we conclude that they coincide. The same applies to triangles $S_1^*S_2^*S_3^*$ and $T_1'T_2'T_3'$.

4. Conclusion

The main advantage of the proof of Napoleon's theorem, which we presented in this paper, is that it can be used quickly and efficiently to prove theorems relating to the same basic construction of points. We can join several objects such as Fermat's point, Napoleon's hexagon, Fermat's hexagon and others to the basic construction of the outer or inner Napoleon's triangles. The proof of the theorem via algorithmic commutative algebra can be adapted to prove the existence and properties of the above objects.

Napoleon's theorem has many generalizations. It would certainly be interesting to see how Grünbaum's and Theorem 3.5 can be generalized to regular n-gones or in some other direction.

It is easily shown algebraically that the pairs of triangles, whose vertices are the centroids of the corresponding triangles, from Theorems 3.4 and 3.5, coincide. It would be interesting, as an exercise, to give a purely geometric proof of this fact.

Acknowledgement. The symbolic algebra computations in the paper were performed with the aid of Wolfram Mathematica 12.3.

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(received 16.10.2021; in revised form 10.01.2022; available online 27.07.2022)

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