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SOME NEW OBSERVATIONS ON w-DISTANCE AND F-CONTRACTIONS

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Abstract. The aim of this paper is to present some new observations about w-distance (in the sense of O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica **44**, 2 (1996), 381–391) and F-contractions (in the sense of D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012**:94 (2012)). Both concepts have been examined separately a lot, but there have been few attempts to connect them. This article is a step in filling this gap. Besides, some comments and improvements of results in the existing literature are presented.

1. Introduction and preliminaries

1.1 w-distance

In the last thirty years, many mathematicians have introduced, in addition to the standard metric d, various other "distances" between points in an arbitrary non-empty set. One of them is the so-called w-distance, which was introduced by the Japanese mathematicians Osamu Kada, Tomonari Suzuki, and Wataru Takahashi in 1995 [14]. It served them to resolve some issues of non-convex minimization and to improve the famous results of Caristi, Ekeland and Takahashi. This version of distance is introduced as follows:

Definition 1.1. Let (X, d) be a metric space and let a mapping $p : X \times X \to [0, +\infty)$ satisfy:
(p1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(p2) for any $x \in X$, the function $p(x, \cdot) : X \to [0, +\infty)$ is d-lower semicontinuous;
(p3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $d(x, y) < \varepsilon$.

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Then, p is called a w-distance on X.

Recall that the property (p2) means that, for all $x, y \in X$, and any sequence $\{y_n\}$ which d-converges to y, $p(x,y) \leq \liminf p(x, y_n)$ holds.

In some later papers (e.g., [11, 16]), the authors used a stronger assumption, as follows:

Definition 1.2. A w-distance on a metric space (X, d) is called a w_0-distance if, in Definition 1.1, condition (p2) is replaced by
(p2') for all $x, y \in X$, the functions $p(x, \cdot) : X \to [0, +\infty)$ and $p(\cdot, y) : X \to [0, +\infty)$ are d-lower semicontinuous.

It is clear that a w-distance has some of the properties of a metric, but may lack some important ones. In particular, one should always have in mind the following facts when working with w-distances.

a) $p(\cdot, \cdot)$ is not necessarily a symmetric function.

b) $p(a, a) = 0$ does not necessarily hold.

The following easy example illustrates these two facts.

Example 1.3 ([14]). Let $X = \mathbb{R}$ be equipped with the standard metric $d(x, y) = |x - y|$ and define $p : X \times X \to [0, +\infty)$ by $p(x, y) = |y|$ for all $x, y \in X$. It is easy to check that p is a w-distance (see [14, Example 4]) which is not symmetric. Indeed, for $x = -1$, $y = 2$ we have $p(-1, 2) = |2| = 2$ while $p(2, -1) = |-1| = 1$, that is $p(-1, 2) \neq p(2, -1)$. Also, e.g., $p(1, 1) = 1 > 0$.

The following property was stated, e.g., in [20, Remark 2.2.2]. We provide an alternate proof.

c) If $x \neq y$, then $q(x, y) = \max\{p(x, y), p(y, x)\} > 0$.

Proof. It is sufficient to prove that $\max\{p(x, y), p(y, x)\} = 0$ implies $x = y$. Indeed, from $p(x, y) = p(y, x) = 0$ it follows by (p1) that also $p(x, x) = 0$. Now for any $\varepsilon > 0$, taking for δ any positive number, we get that, for the given x, y, $p(x, x) < \delta$ and $p(x, y) < \delta$, and so, according to (p3), $d(x, y) < \varepsilon$. Since ε is arbitrary, it follows that $d(x, y) = 0$, i.e., $x = y$. □

A part of the following property was stated without proof in [16, Remark 2.1].

d) If p is a w_0-distance, then $q(x, y) = \max\{p(x, y), p(y, x)\}$ is a symmetric w-distance on X.

Proof. Let $x, y, z \in X$ be arbitrary. Then

$$\begin{aligned} q(x, z) &= \max\{p(x, z), p(z, x)\} \leq \max\{p(x, y) + p(y, z), p(z, y) + p(y, x)\} \\ &= \max\{p(x, y), p(y, x)\} + \max\{p(y, z), p(z, y)\} = q(x, y) + q(y, z), \end{aligned}$$

hence, (p1) holds for q. For $x, y \in X$ fixed, both functions $p(x, \cdot)$ and $p(\cdot, y)$ are d-lower semicontinuous, i.e.,

$$\liminf_{y_n \to y_0} p(x, y_n) \geq p(x, y_0), \quad \text{and} \quad \liminf_{x_n \to x_0} p(x_n, y) \geq p(x_0, y),$$

hold. Then

$$\begin{aligned}\liminf_{y_n\to y_0} q(x,y_n) &= \liminf_{y_n\to y_0} \max\{p(x,y_n),p(y_n,x)\}\\ &\geq \max\{\liminf_{y_n\to y_0} p(x,y_n), \liminf_{y_n\to y_0} p(y_n,x)\}\\ &\geq \max\{p(x,y_0),p(y_0,x)\} = q(x,y_0),\end{aligned}$$

which means that $q(x,y)$ is also d-semicontinuous (in both variables, since it is symmetric). Finally, for the given $\varepsilon>0$ choose $\delta>0$ such that $p(z,x)<\delta$ and $p(z,y)<\delta$ imply $d(x,y)<\varepsilon$. If $q(x,y)<\delta$ and $q(y,z)<\delta$ then $p(x,y)<\delta$ and $p(y,z)<\delta$, and hence $d(x,z)<\varepsilon$. □

REMARK 1.4. Recall that a mapping $\sigma : X\times X\to[0,+\infty)$ (where X is a nonempty set) is called metric-like [3] if, for all $x,y,z\in X$, the following hold:

(σ1) $\sigma(x,y)=0$ implies $x=y$;

(σ2) $\sigma(x,y)=\sigma(y,x)$;

(σ3) $\sigma(x,z)\leq\sigma(x,y)+\sigma(y,z)$.

According to properties (p1), **c)** and **d)**, the mapping q defined as in **d)** is a metric-like on the space X; see also [11, Remark 1.12]).

e) w-distance is not necessarily a d-continuous function (in two variables).

EXAMPLE 1.5 ([22]). Let $X=[0,1]\subset\mathbb{R}$ be equipped with the usual metric and let p be a w-distance on X given by

$$p(x,y)=\begin{cases}9, & \text{if } x=0, y\in X\\ y-x, & \text{if } 0<x\leq y,\\ 3x-3y, & \text{if } x>y.\end{cases}$$

Let $x_n=\dfrac{1}{n}$ and $y_n=\dfrac{1}{n}$. Then $x_n\to 0$, $y_n\to 0$ and $p(x_n,y_n)\to 0$ as $n\to+\infty$, but $p(0,0)=9\neq 0$. Hence, p is not a d-continuous function.

Note that the w-distance p considered in Example 1.3 is asymmetric and d-continuous.

For other standard examples of w-distances and their basic properties we refer to, e.g., [14, 20, 22].

We add the following two lemmas that will be, similarly as in the context of metric spaces (see, e.g., [9, 19]), useful in the proofs of our main results. Both these lemmas are used to prove the Cauchyness of the Picard sequence $x_n=Tx_{n-1}$, $n\in\mathbb{N}$, where $x_0\in X$ is a given point in a metric space X and $T:X\to X$.

LEMMA 1.6. *Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a Picard sequence in a metric space (X,d) with w-distance p such that*

$$p(x_{n+1},x_n)<p(x_n,x_{n-1}) \tag{1}$$

or

$$p(x_n,x_{n+1})<p(x_{n-1},x_n), \tag{2}$$

is satisfied for all $n \in \mathbb{N}$. *Then* $x_n \neq x_m$ *whenever* $n \neq m$.

Proof. Consider the first case (1). Suppose that $x_n = x_m$ for some $n < m$. Then $x_{n+1} = Tx_n = Tx_m = x_{m+1}$ and hence

$$p(x_{m+1}, x_m) = p(x_{n+1}, x_n) < p(x_n, x_{n-1}) < \cdots < p(x_{m+1}, x_m),$$

which is a contradiction. In the case (2) the proof is similar. □

Note that if one of the conditions (1), (2) is fulfilled then, for $n \neq m$, it is always $\max\{p(x_n, x_m), p(x_m, x_n)\} > 0$.

LEMMA 1.7. *Let* (X, d) *be a metric space with* w*-distance* p *and let* $\{x_n\}$ *be a sequence in* X *such that both* $p(x_{n+1}, x_n)$ *and* $p(x_n, x_{n+1})$ *tend to* 0 *as* $n \to +\infty$. *If* $\{x_n\}$ *is not a* d*-Cauchy sequence in* X, *then there exist* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers such that* $n_k > m_k > k$ *and the following sequences tend to* ε *from above, as* $k \to +\infty$*:*

$$\{p(x_{n_k}, x_{m_k})\}, \{p(x_{n_k}, x_{m_k-1})\}, \{p(x_{n_k+1}, x_{m_k})\},$$
$$\{p(x_{n_k+1}, x_{m_k-1})\}, \{p(x_{n_k+1}, x_{m_k+1})\}, \ldots$$

as well as

$$\{p(x_{m_k}, x_{n_k})\}, \{p(x_{m_k-1}, x_{n_k})\}, \{p(x_{m_k}, x_{n_{k+1}})\},$$
$$\{p(x_{m_k-1}, x_{n_k+1})\}, \{p(x_{m_k+1}, x_{n_k+1})\}, \ldots$$

Proof. Since $\{x_n\}$ is not a d-Cauchy sequence, it follows by [14, Lemma 1.(iii)] that $p(x_n, x_m)$ does not tend to 0 as $n, m \to +\infty$. This means that there exist $\varepsilon > 0$ and subsequences $\{n_k\}$ and $\{m_k\}$ such that $m_k > n_k > k$ and

$$p(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \text{and} \quad p(x_{n_k-1}, x_{m_k}) < \varepsilon.$$

Then, using (p1) and the fact that $p(x_{n+1}, x_n)$ and $p(x_n, x_{n+1})$ tend to 0 as $n \to +\infty$, it follows, in the same way as in metric spaces (see, e.g. [19])) that the given sequences tend to ε from above. □

A lot of fixed point and common fixed point results in metric spaces endowed with a w-distance have been obtained by various authors (see, for example, the articles [11, 16, 17, 22–24] and, in particular, the book [20] and the references therein).

This new kind of distance was also introduced in the context of b-metric spaces where it is usually called wt-distance. Results in this environment were obtained, e.g., in [10].

We note also that two variants of such distance (usually called c-distance and w-cone distance) were also introduced in cone metric spaces and tvs-cone metric spaces in the papers [6, 7] (for a survey on these kinds of spaces see [1]). Further on, several authors obtained fixed point results in this context. We are not going to consider here this approach.

1.2 F-contractions

In 2012, Polish mathematician Darius Wardowski published in [25] a very interesting generalization of Banach Contraction Principle (BCP). He replaced the Banach's condition $d(Tx, Ty) \leq \lambda d(x, y)$ with

$$d(Tx, Ty) > 0 \quad \text{implies} \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $\tau > 0$ and $F : (0, +\infty) \to \mathbb{R}$ is a strictly increasing function, satisfying some additional conditions. Later on, these conditions were modified by several authors, and finally shown to be redundant in the paper [18] (a bit shorter proof of this result is presented in [9]). We note just that it follows from the monotony of the function F that it has the left and right limit at each point $t > 0$. In the paper [25], Wardowski also showed, by giving an example, that his result is strictly stronger than BCP (although it reduces to it when $F(t) = \ln t$ is used as a "control" function).

A detailed survey on fixed point results using F-contractions in various environments (until 2021) can be found in [9].

To the best of our knowledge, the only papers where some results combining the use of w-distance and F-contractions are [11, 17].

In this paper, we present some new observations on w-distance and F-contractions, and their connection. Besides, some comments and improvements of results in the existing literature are presented.

2. Wardowski-type results in metric spaces using w-distance

2.1 A result using w_0-distance

In this subsection we consider Wardowski-type fixed point results for mappings in metric spaces endowed with a w_0-distance.

DEFINITION 2.1. Let (X, d) be a metric space equipped with a w_0-distance p and $q(x, y) = \max\{p(x, y), p(y, x)\}$ be the respective symmetric w-distance (see property **d)** in Section 1). Let $T : X \to X$ and let there exist $\tau > 0$ and a strictly increasing function $F : (0, +\infty) \to \mathbb{R}$ such that, for all $x, y \in X$,

$$q(Tx, Ty) > 0 \quad \text{implies} \quad \tau + F(q(Tx, Ty)) \leq F(q(x, y)) \tag{3}$$

holds. Then T is called an (F, τ, p)-contraction.

THEOREM 2.2. *Let T be an (F, τ, p)-contraction on a complete metric space (X, d) with a w_0-distance p and its respective symmetric w-distance q, and let*
(i) T be continuous, or

(ii) $\inf\{q(x, u) + q(x, Tx) : x \in X\} > 0$ whenever $u \in X$ and $u \neq Tu$.
Then T has a unique fixed point, say z. Moreover, $p(z, z) = 0$.

Proof. It follows directly from the contractive condition (3) that there cannot exist more than one fixed point of the mapping T. Moreover, if Tz=z then $p(z, z)$=0. Indeed, suppose that $p(z, z) > 0$. Then, putting $x = y = z$ in (3), we get $\tau + F(p(z, z)) \leq F(p(z, z))$, which is a contradiction with $\tau > 0$.

We will prove the existence of a fixed point.

Choose an arbitrary $x_0 \in X$ and define the Picard sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. If $x_n = x_{n-1}$ holds for some n, then x_{n-1} is a (unique) fixed point of T.

Suppose, therefore, that $x_n \neq x_{n-1}$ (and hence $q(x_n, x_{n-1}) > 0$, see property **c)** in Section 1) for all $n \in \mathbb{N}$.

Putting $x = x_{n-1}$, $y = x_n$ in (3), we obtain that

$$\tau + F(q(x_n, x_{n+1})) \leq F(q(x_{n-1}, x_n)). \tag{4}$$

Since the function F is strictly increasing, it follows that the sequence $\{q(x_n, x_{n+1})\}$ is strictly decreasing, and hence it has a limit $q^* \geq 0$. If $q^* > 0$ then, passing to the right limit as $n \to +\infty$ in (4), we get that $\tau + F(q^* + 0) \leq F(q^* + 0)$, which is a contradiction with $\tau > 0$. Thus, we get that the sequence $\{q(x_n, x_{n+1})\}$ tends to 0 as $n \to +\infty$ and, hence, both sequences $\{p(x_n, x_{n+1})\}$ and $\{p(x_{n+1}, x_n)\}$ tend to 0.

In order to prove that the Picard sequence $\{x_n\}$ is a Cauchy sequence, we assume the contrary and apply Lemma 1.7. Putting $x = x_{m_k}$, $y = x_{n_k}$ in (3), we get that

$$\tau + F(q(x_{m_k+1}, x_{n_k+1})) \leq F(q(x_{m_k}, x_{n_k})).$$

Passing here to the right limit as $k \to +\infty$, we obtain $\tau + F(\varepsilon + 0) \leq F(\varepsilon + 0)$, which is a contradiction with $\tau > 0$. Hence, $\{x_n\}$ must be a Cauchy sequence. Since the metric space (X, d) is complete, the sequence $\{x_n\}$ converges to a point $z \in X$.

If condition (i) is satisfied, then d-continuity of the mapping T implies that $Tz = z$.

Suppose that condition (ii) holds. Since $q(x_n, x_m) \to 0$ as $m, n \to +\infty$, using property (p2'), we get that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $q(x_{n_k}, z) \to 0$ as $k \to +\infty$. If $z \neq Tz$, we would obtain

$$0 < \inf\{q(x, z) + q(x, Tx) : x \in X\} \leq \inf\{q(x_{n_k}, z) + q(x_{n_k}, x_{n_k+1}) : k \in \mathbb{N}\} \to 0$$

as $k \to +\infty$, implying that $\inf\{q(x, z) + q(x, Tx) : x \in X\} = 0$, a contradiction. It follows that $Tz = z$. $\square$

Remark 2.3. A similar result was obtained using different methods in [11, Corollary 3.4].

Remark 2.4. We note that in the proof of the previous theorem, just the properties of mapping q mentioned in Remark 1.4 were used. This means that Theorem 2.2 can be treated as a result on fixed points under F-contractions for metric-like spaces.

Example 2.5. Let $X = [0, 1]$ be equipped with standard metric d and w-distance defined by $p(x, y) = x + y$ [14] (in this case, obviously, $q \equiv p$). Let $T : X \to X$ be given by

$$Tx = \begin{cases} x/2, & \text{if} \quad 0 \leq x < 1, \\ 1/4, & \text{if} \quad x = 1. \end{cases}$$

Then, T is a p-contraction. Indeed, for $0 \leq x, y < 1$ it is

$$p(Tx, Ty) = \frac{x}{2} + \frac{y}{2} = \frac{1}{2} p(x, y).$$

If $x = 1$ and $0 \leq y < 1$ it is

$$p(Tx, Ty) = \frac{1}{4} + \frac{y}{2} < \frac{1}{2}(1 + y) = \frac{1}{2} p(x, y).$$

(and similarly for $y = 1$, $0 \le x < 1$). If $x = y = 1$ it is

$$p(Tx, Ty) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1 = \frac{1}{2}p(x, y).$$

Hence, for $\lambda = \frac{1}{2}$, $p(Tx, Ty) \le \lambda p(x, y)$ holds in all cases. But then, T is also an (F, τ, p)-contraction (with $F(t) = \ln t$ and $\tau = \ln 2$).

However, T is not an F-contraction (in the sense of Wardowski). Otherwise, we would have

$$\tau + F(d(Tx, Ty)) \le F(d(x, y))$$

whenever $d(Tx, Ty) > 0$. But then, for $x = 1$, $y = 7/8$, we would get

$$\tau + F(d(1/4, 7/16)) = \tau + F(3/16) \le F(1/8) = F(d(1, 7/8)),$$

which is impossible since $\tau > 0$ and F has to be strictly increasing.

Recall also the basic example from [25].

Example 2.6 ([25]). Consider the set $X = \{x_n \mid n \in \mathbb{N}\}$ where $x_n = \sum_{k=1}^{n} k$, equipped with the standard metric d and let $p = d$. Then, (X, d) is a complete metric spaces. Let $T : X \to X$ be defined by $T(x_1) = x_1$ and $T(x_n) = x_{n-1}$, $n > 1$. It is easy to see that T is not a d-contraction (hence, also not a p-contraction), but it is an (F, τ, d)-contraction (and (F, τ, p)-contraction) with $\tau = 1$ and $F(t) = t + \ln t$.

2.2 Some comments on the paper [17]

Consider Definitions 7 and 8 and subsequent Lemma 3 and Theorem 1 of the paper [17]. First of all, in Definition 7 (page 4), there is a logical error – the quantifiers are mixed: instead “for all $x, y \in X$ there exists F ...” there should stay “there exists F such that for all $x, y \in X$ ...”. Then, we notice that the condition $0 \le \lambda < 1$ in the relation

$$p(fx, fy) > 0 \quad \text{implies} \quad \tau + F(p(fx, fy)) \le F(\lambda \mathcal{M}_p(x, y))$$

of Definition 7 is not correct because for $\lambda = 0$ the right-hand side of the last inequality becomes $-\infty$, which is then a contradiction with $\tau > 0$. Moreover, since F is supposed to be a strictly increasing function, the right-hand side in the inequality in the relation

$$p(fx, fy) > 0 \quad \text{implies} \quad \tau + F(p(fx, fy)) \le F(\lambda \mathcal{M}_p^g(x, y))$$

of Definition 8 is strictly smaller than $F(\mathcal{M}_p^g(x, y))$, i.e., the right-hand side in the inequality of the relation

$$p(fx, fy) > 0 \quad \text{implies} \quad \tau + F(p(fx, fy)) \le F(\mathcal{M}_p^g(x, y))$$

of Theorem 2. This shows that the results stated in [17] up to Theorem 2 are redundant. In other words, Theorem 1 is a consequence of Theorem 2 for $0 < \lambda \le 1$.

Finally, in all the results on common fixed points, commutativity of mappings f and g is supposed to hold, which is a very strong condition and is rarely satisfied. It would be interesting to investigate whether some of the known compatibility-type conditions (see, e.g., [15]) could be used instead of the commutativity (see further comments at the end of Subsection 3.1 on a similar situation in b-metric spaces).

Moreover, in Theorem 4, the function $F(t) = \ln t$ is used, which is of no interest, since it is well known (already from the basic paper [25]), that Wardowski-type results in that case easily reduce to classic ones.

3. Results in b-metric spaces with wt-distance

3.1 Some improvements of the results of paper [10]

Treating b-metric spaces as an environment for obtaining fixed point and related results goes back to Bakhtin [4] and Czerwick [8]. There is a huge bibliography of such results. Recall the basic definition.

DEFINITION 3.1 ([8]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d_b : X \times X \to [0, +\infty)$ is a b-metric on X if for all $x, y, z \in X$, the following conditions hold:

(d_b1) $d_b(x, y) = 0$ iff $x = y$,

(d_b2) $d_b(x, y) = d_b(y, x)$,

(d_b3) $d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)]$.

In this case, the pair (X, d_b) is called a b-metric space with parameter s.

Twenty five years after Kada et al. introduced the notion of w-distance, Hussain, Saadati and Agrawal gave the respective definition in the case of b-metric spaces:

DEFINITION 3.2 ([10, Definition 3.1]). Let (X, d_b) be a b-metric space with parameter s. A mapping $p_b : X \times X \to [0, +\infty)$ is called a wt-distance on X if the following conditions are satisfied:

(p_b1) $p_b(x, z) \leq s[p_b(x, y) + p_b(y, z)]$ for all $x, y, z \in X$;

(p_b2) for any $x \in X$, the function $p_b(x, \cdot) : X \to [0, +\infty)$ is s-lower semicontinuous on X, i.e., if, for each $y_0 \in X$ and each sequence $\{y_n\}$ converging to y_0, $\frac{1}{s} p_b(x, y_0) \leq \liminf p_b(x, y_n)$ holds;

(p_b3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p_b(z, x) < \delta$ and $p_b(z, y) < \delta$ imply $d_b(x, y) < \varepsilon$.

For details and examples see [10].

In this subsection we show that some improvements of results in the paper [10] can be obtained. (For the sake of simplicity, we omit the assumptions about the ordering of the space, which are not important for our comments.)

First of all, we note that in Theorems 4.2 and 4.3 of the mentioned paper, for the "parameter of contraction" r it is supposed that $rs < 1$, i.e., $r \in [0, \frac{1}{s})$. We will show that this assumption can be relaxed to $r \in [0, 1)$ in the case of the simplest Banach-type fixed point result.

Theorem 3.3. *Let (X, d_b) be a complete b-metric space with parameter $s > 1$ and let p_b be a wt-distance on X. Suppose that $T : X \to X$ and that there exists a real number $r \in [0, 1)$ such that*

$$p_b(Tx, Ty) \le r\, p_b(x, y) \tag{5}$$

holds for all $x, y \in X$. If
(i) T is d_b-continuous, or

(ii) $\inf\{p_b(x, u) + p_b(x, Tx) : x \in X\} > 0$ whenever $u \in X$ and $u \ne Tu$,
then there exists a unique fixed point, say z, of T and it satisfies $p_b(z, z) = 0$. Moreover, the mapping T has the so-called property (P), i.e., for each $n \in \mathbb{N}$, the mappings T and T^n have the same set of fixed points.

Proof. The uniqueness of a fixed point z and the fact $p_b(z, z) = 0$ follow easily in a standard way.

Let us prove that for the sets of fixed points, $Fix(T) = Fix(T^n)$ holds for each $n \in \mathbb{N}$. The inclusion $Fix(T) \subset Fix(T^n)$ is obvious. For the converse, suppose that there exists $z \in Fix(T^n) \setminus Fix(T)$, i.e., $T^n z = z$, $Tz \ne z$, and so $p_b(Tz, z) > 0$. It follows from (5) that

$$p_b(Tz, z) = p_b(T^{n+1}z, T^n z) \le r\, p_b(T^n z, T^{n-1}z) \le \cdots \le r^n p_b(Tz, z),$$

hence $1 \le r^n$, a contradiction.

In order to prove the existence of a fixed point, suppose first that $r \in [0, \frac{1}{s})$. Let $x_0 \in X$ be arbitrary and let $\{x_n\}$ be the respective Picard sequence, i.e., $x_n = T^n x_0$. Then, using (5), it follows in a standard way that $p_b(x_n, x_{n+1}) \le r^n p_b(x_0, x_1)$ for each $n \in \mathbb{N}$ and that, for $m > n$,

$$p_b(x_n, x_m) \le \frac{sr^n}{1 - sr}\, p_b(x_0, x_1) \to 0 \quad \text{as} \quad m, n \to +\infty,$$

implying, by [10, Lemma 3.5], that $\{x_n\}$ is a d_b-Cauchy sequence in X. Since (X, d_b) is complete, it follows that there exists $\lim_{n\to+\infty} x_n = z \in X$. We will prove that z is a fixed point of T.

If (i) holds, the conclusion is trivial. Suppose that (ii) holds and that $Tz \ne z$. Then

$$0 < \inf\{p_b(x, z) + p_b(x, Tx) : x \in X\} \le \inf\{p_b(x_n, z) + p_b(x_n, x_{n+1}) : n \in \mathbb{N}\} = 0,$$

a contradiction. Hence, $Tz = z$.

Suppose now that $r \in [\frac{1}{s}, 1)$. Then, for some $n \in \mathbb{N}$, $r^n \in (0, \frac{1}{s})$ holds. It follows from (5) that, for arbitrary $x, y \in X$,

$$p_b(T^n x, T^n y) \le r\, p_b(T^{n-1}x, T^{n-1}y) \le \cdots \le r^n\, p_b(x, y)$$

holds, i.e., the mapping T^n satisfies condition (5) with parameter $r^n < \frac{1}{s}$. Hence, by the previous part of the proof, there exists $z \in Fix(T^n)$. Since $Fix(T^n) = Fix(T)$, it follows that also $z \in Fix(T)$, which finishes the proof. □

Example 3.4. Let $X = \mathbb{R}$ be equipped with the b-metric $d_b(x, y) = (x - y)^2$ with parameter $s = 2$ and the wt-distance $p_b(x, y) = y^2$ (see [10, Example 3.4]). Consider the mapping $T : X \to X$ given by $Tx = \frac{1}{\sqrt{2}}x$. Then $p_b(Tx, Ty) = \frac{1}{2}y^2$, $p_b(x, y) = y^2$

and in order that $p_b(Tx,Ty) \leq r\,p_b(x,y)$ holds for all $x,y \in X$, the condition $r \geq \frac{1}{2}$ should hold. Hence, the condition $rs = 2r < 1$ could not be satisfied.

However, the condition $r < 1$ is clearly satisfied with $r = \frac{1}{2}$.

Recall now that the simplest common fixed point results for mappings $S,T : X \to X$ can be obtained if S and T commute. Obviously, this condition is too strong, and so it was natural to seek for weaker assumptions. Hence, several authors have introduced various other conditions which can be used in order to prove common fixed point results—a review of these conditions can be found in [15]. We mention here two of them:

1° The self-mappings S and T on a space X (with some convergence structure) are *compatible* [12] if, for each sequence $\{x_n\}$ in X, $\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Tx_n$ implies that $\lim_{n\to+\infty} STx_n = \lim_{n\to+\infty} TSx_n$.

2° The self-mappings S and T on a nonempty set are *weakly compatible* [13] if, for all $x \in X$, $Sx = Tx$ implies that $STx = TSx$.

Common fixed point results in the paper [10] were obtained using the condition of commutativity (as well as in most other papers in the literature where these problems were treated using w-distance or wt-distance). However, it is easy to see that, e.g., the result of Theorem 5.4 of that paper holds true when commutativity is replaced by a weaker condition of compatibility.

Indeed, in the proof of the mentioned theorem, a sequence $\{x_n\}$ in X is formed satisfying $Sx_n = Tx_{n-1}$, and it is proved that the sequences $\{Sx_n\}$ and $\{Tx_n\}$ converge to, say, y. By the continuity of S and T, we have that $STx_n \to Ty$ and $TSx_n \to Sy$ as $n \to +\infty$, and, by the compatibility of S and T, these limits are equal, i.e., $Sy = Ty$. Now the (weak) compatibility of these mappings implies in a standard way that S and T have a common fixed point.

3.2 A Reich-type fixed point result using wt-distance

It is well known that Banach-type fixed point results can be modified in various ways (see, e.g., the classical paper by Rhoades [21]). As a sample, we are going to show in this subsection how Reich-type results can be deduced in the environments that we are treating here.

Note first that assertions similar to **a)– e)** (Section 1) hold in the case of wt-distance p_b. In particular, if a wt-distance p_b is s-lower semicontinuous in each of its variables, then by $q_b(x,y) = \max\{p_b(x,y), p_b(y,x)\}$ a symmetric wt-distance is defined.

Recall now (see [2, Definition 2.4]) that a mapping $\sigma_b : X \times X \to [0,+\infty)$ is called a b-metric-like on a nonempty set X if it satisfies, for some $s \geq 1$ and all $x,y,z \in X$ the following:

(σ_b1) $\sigma_b(x,y) = 0$ implies $x = y$;

(σ_b2) $\sigma_b(x,y) = \sigma_b(y,x)$;

(σ_b3) $\sigma_b(x,z) \leq s[\sigma_b(x,y) + \sigma_b(y,z)]$.

We see that, similarly as in Remark 2.4, $q_b(x,y)$ is a b-metric-like on X.

Theorem 3.5. *Let (X, σ_b) be a complete b-metric-like space with parameter $s > 1$ and let $T : X \to X$ be a mapping satisfying*

$$\sigma_b(Tx, Ty) \leq a\,\sigma_b(x, y) + b\,\sigma_b(x, Tx) + c\,\sigma_b(y, Ty) \tag{6}$$

for some $a, b, c \geq 0$ with $s(a + b) + c < 1$, and for all $x, y \in X$. If
(i) T is continuous, or

(ii) $\inf\{\sigma_b(x, u) + \sigma_b(x, Tx) : x \in X\} > 0$ whenever $u \in X$ and $u \neq Tu$,
then T has a unique fixed point, say z. Moreover, $\sigma_b(z, z) = 0$.

Proof. Let us prove first that T cannot have two distinct fixed points. Indeed, if z_1 and z_2 are such points, then $\sigma_b(z_1, z_2) > 0$ and (6) implies that $\sigma_b(z_1, z_2) \leq a\,\sigma_b(z_1, z_2)$. It follows that $a \geq 1$, a contradiction.

Denote $r = \frac{a+b}{1-c}$ (so that $r < \frac{1}{s}$). Let $x_0 \in X$ be arbitrary and let $\{x_n\}$ be the respective Picard sequence, i.e., $x_n = T^n x_0$. Then it follows from (6) that

$$\sigma_b(x_n, x_{n+1}) \leq r\,\sigma_b(x_{n-1}, x_n)$$

for $n \in \mathbb{N}$. Applying [2, Lemma 2.14], we conclude that $\lim_{m,n\to+\infty} \sigma_b(x_n, x_m) = 0$, i.e., $\{x_n\}$ is a Cauchy sequence in (X, σ_b). Since this space is complete, the Picard sequence converges to some (unique – see [2, Proposition 2.10]) $z \in X$ and

$$\lim_{n,m\to+\infty} \sigma_b(x_n, x_m) = \lim_{n\to+\infty} \sigma_b(x_n, z) = \sigma_b(z, z) = 0$$

holds.

Suppose that the mapping T is continuous in (X, σ_b). It follows that $\lim_{n\to+\infty} \sigma_b(Tx_n, Tz) = \sigma_b(Tz, Tz)$. On the other hand $\sigma_b(T_n, Tz) = \sigma_b(x_{n+1}, Tz)$, so $\sigma_b(x_{n+1}, Tz)$ tends to $\sigma_b(Tz, Tz)$ as $n \to +\infty$. Since the sequence $\{x_n\}$ has a unique limit, it follows that $Tz = z$.

If the condition (ii) is satisfied, equality $Tz = z$ can be proved in the same way as in the proof of Theorem 2.2. □

As a consequence, we deduce the following

Corollary 3.6. *Let (X, d_b) be a complete b-metric space with parameter s, let p_b be a wt-distance on X that is s-lower semicontinuous with respect to both variables and denote $q_b(x, y) = \max\{p_b(x, y), p_b(y, x)\}$. Let $T : X \to X$ be a mapping satisfying*

$$q_b(Tx, Ty) \leq a\,q_b(x, y) + b\,q_b(x, Tx) + c\,q_b(y, Ty)$$

for some $a, b, c \geq 0$ with $s(a + b) + c < 1$, and for all $x, y \in X$. If
(i) T is continuous, or

(ii) $\inf\{q_b(x, u) + q_b(x, Tx) : x \in X\} > 0$ whenever $u \in X$ and $u \neq Tu$,
then T has a unique fixed point, say z. Moreover, $p_b(z, z) = 0$.

3.3 F-contractions in b-metric spaces with a wt-distance

Taking into account known results in b-metric-like spaces, in particular [5, Corollary 2.14], we come to the following b-metric version of Theorem 2.2.

Corollary 3.7. *Let (X, d_b) be a complete b-metric space with parameter s, let p_b be a wt-distance on X that is s-lower semicontinuous with respect to both variables and* *denote* $q_b(x,y)=\max\{p_b(x,y),p_b(y,x)\}$. *Let* $T:X\to X$ *be continuous and such that there exist a real number* $\tau>0$ *and a strictly increasing function* $F:(0,+\infty)\to\mathbb{R}$ *such that, for all* $x,y\in X$,

$$q_b(Tx,Ty)>0 \quad \textit{implies} \quad \tau+F(q_b(Tx,Ty))\le F(q_b(x,y))$$

holds. Then T *has a unique fixed point, say* z. *Moreover,* $p_b(z,z)=0$.

4. Some suggestions for further work

There are several other classical fixed point results which have not yet been discussed in some of the environments that we have treated in this paper, or using Wardowski's method. Such are, for example, the results of Boyd and Wong, of Nemitzky and Edelstein, and many others.

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