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MULTIPLICITY OF SOLUTIONS FOR ANISOTROPIC DISCRETE BOUNDARY VALUE PROBLEMS

Abdelrachid El Amrouss and Omar Hammouti

Abstract. In this paper, we study the existence and multiplicity of nontrivial solutions for an anisotropic discrete nonlinear problem with variable exponent. The analysis makes use of variational methods and critical point theory.

1. Introduction

Let $N \geq 2$ be an integer, $[1, N]_{\mathbb{Z}}$ be the discrete interval given by $\{1, 2, 3, \ldots, N\}$ and $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$. Define the forward difference operator Δ by $\Delta u(t) = u(t+1) - u(t)$, $t \in \mathbb{Z}$. This paper is concerned with the existence and multiplicity of nontrivial solutions for the following discrete anisotropic problem

$$\text{(P)} \qquad \begin{cases} -\Delta(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) = f(t, u(t)), & t \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

where $f\colon [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the second variable. For the function $p : [0, N]_{\mathbb{Z}} \longrightarrow [2, \infty[$ denote $p^+ = \max_{t\in[0,N]_{\mathbb{Z}}} p(t)$ and $p^- = \min_{t\in[0,N]_{\mathbb{Z}}} p(t)$. As usual, a solution of (P) is a function $u : [0, N+1]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (P).

In the case when $p(t) = p$ (a constant), the problem has been studied by A.R. El Amrouss and O. Hammouti in [7]. They obtained the existence of at least two nontrivial solutions, by using the critical point theory and Mountain Pass Theorem. Here, we will generalize this result.

We want to notice that problem (P) could be regarded as a discrete analogue of the variable exponent anisotropic problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(|\dfrac{\partial u}{\partial x_i}|^{p_i(x)-2}\dfrac{\partial u}{\partial x_i}\right) = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary, $f \in C\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ is a given function that satisfy certain properties, $p_i(x)$ are continuous functions on $\overline{\Omega}$, with $p_i(x) \geq 2$ for $(i, x) \in [1, N]_{\mathbb{Z}} \times \Omega$.

In the previous decades, the nonlinear difference equations have been intensively used for the mathematical modelling of various problems in different disciplines of science, such as mechanical engineering, statistics, computing, ecology, optimal control, neural network, electrical circuit analysis, population dynamics, economics, biology and other fields; (see, for example [2, 17]). In this context, anisotropic discrete nonlinear problems seem to have attracted a great deal of attention due to its usefulness of modelling some more complicated phenomenon such us fluid dynamics and nonlinear elasticity. We refer the reader to [1, 5, 6, 8, 11, 12] and references therein, where they can find the detailed background as well as many different approaches and techniques applied in the related area.

As is well known, critical point theory, variational methods and also monotonicity methods are powerful tools to investigate the existence and multiplicity of solutions of various problems, see the monographs [4, 9, 10, 15–17].

In this paper, we shall study the existence and multiplicity of nontrivial solutions of (P), via min-max methods and Mountain Pass Theorem.

Let $F(t, x) = \int_0^x f(t, s)\, ds$ for $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$. To state our main results, we consider the following conditions:

(H_1) There exists $\delta > \dfrac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}}$ such that $\lim\inf\limits_{|x|\to\infty} \dfrac{F(t,x)}{|x|^{p^+}} \geq \delta$, $\forall t \in [1, N]_{\mathbb{Z}}$.

(H_2) $\lim\limits_{|x|\to\infty} \left(F(t,x) - \dfrac{p^+}{(p^-)^2} \gamma_N |x|^{p^+} \right) = +\infty, \quad \forall t \in [1, N]_{\mathbb{Z}}$ where

$$\gamma_N = \sup \left\{ \frac{\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)}}{\sum\limits_{t=1}^{N} |u(t)|^{p^+}} \mid u \in E_N : \|u\| \geq 1 \right\}, \tag{1}$$

with
$$E_N = \{u : [0, N+1]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid u(0) = u(N+1) = 0\}, \tag{2}$$

and
$$\|u\| = \left(\sum_{t=1}^{N+1} |\Delta u(t-1)|^2 \right)^{\frac{1}{2}}.$$

It is easy to see that $\gamma_N > 0$ and we will see later that γ_N is finite.

(H_3) $\lim\limits_{|x|\to 0} \dfrac{F(t,x)}{|x|^{p^+}} = 0, \ \forall t \in [1, N]_{\mathbb{Z}}$.

Example 1.1. Let us consider a continuous function $f : [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by the formula

$$f(t,x) = \begin{cases} |\sin t|(1 + p^+ \ln|x|)|x|^{p^+-2}x, & |x| > 1, t \in [1, N]_{\mathbb{Z}}, \\ |\sin t||x|^{p^+-1}x, & |x| \leq 1, t \in [1, N]_{\mathbb{Z}}. \end{cases}$$

Clearly, we have

$$F(t,x)=\begin{cases}|\sin t|(|x|^{p^+}\ln|x|+\frac{1}{p^++1}), & |x|>1, t\in[1,N]_{\mathbb{Z}},\\ \frac{1}{p^++1}|\sin t||x|^{p^++1}, & |x|\leq 1, t\in[1,N]_{\mathbb{Z}}.\end{cases}$$

After a simple calculation, we get

$$\lim\inf_{|x|\to\infty}\frac{F(t,x)}{|x|^{p^+}}=+\infty,\ \lim_{|x|\to\infty}\left(F(t,x)-\frac{p^+}{(p^-)^2}\gamma_N|x|^{p^+}\right)=+\infty$$

and $$\lim_{|x|\to 0}\frac{F(t,x)}{|x|^{p^+}}=0, \text{ for any } t\in[1,N]_{\mathbb{Z}}.$$

Then F satisfies the conditions (H_1), (H_2) and (H_3), but f does not satisfy the conditions of the article [12].

Example 1.2. Put $f:[1,N]_{\mathbb{Z}}\times\mathbb{R}\longrightarrow\mathbb{R}$ by the formula

$$f(t,x)=\begin{cases}2^{p^++1}\dfrac{p^+}{p^-}(N+1)^{\frac{p^+}{2}}t|x|^{p^+-2}x, & |x|>1,\ t\in[1,N]_{\mathbb{Z}},\\ 2^{p^++1}\dfrac{p^+}{p^-}(N+1)^{\frac{p^+}{2}}t|x|^{p^+}x, & |x|\leq 1,\ t\in[1,N]_{\mathbb{Z}}.\end{cases}$$

By the expression of f we have

$$F(t,x)=\begin{cases}\dfrac{2^{p^++1}}{p^-}(N+1)^{\frac{p^+}{2}}t|x|^{p^+}-\dfrac{2^{p^++2}}{p^-(p^++2)}(N+1)^{\frac{p^+}{2}}t, & |x|>1,\ t\in[1,N]_{\mathbb{Z}},\\ 2^{p^++1}\dfrac{p^+}{p^-(p^++2)}(N+1)^{\frac{p^+}{2}}t|x|^{p^++2}, & |x|\leq 1,\ t\in[1,N]_{\mathbb{Z}}.\end{cases}$$

Direct calculations give $\lim\inf_{|x|\to\infty}\dfrac{F(t,x)}{|x|^{p^+}}=\dfrac{2^{p^++1}}{p^-}(N+1)^{\frac{p^+}{2}}t\geq\dfrac{2^{p^++1}}{p^-}(N+1)^{\frac{p^+}{2}}$ and $\lim_{|x|\to 0}\dfrac{F(t,x)}{|x|^{p^+}}=0$. Thus F satisfies the conditions (H_1) with $\delta=\dfrac{2^{p^++1}}{p^-}(N+1)^{\frac{p^+}{2}}$ and (H_3).

The main results in this paper are the following theorems.

Theorem 1.3. *Suppose that (H_1) and (H_3) hold. Then the problem* (P) *has at least two nontrivial solutions, in which one is non-negative and one is non-positive.*

Theorem 1.4. *Suppose that (H_2) and (H_3) hold. Then the problem* (P) *has at least two nontrivial solutions.*

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas. The main results will be proved in Section 3.

2. Preliminary lemmas

The vector space E_N defined in (2) is an N-dimensional Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{t=1}^{N} \Delta u(t-1)\Delta v(t-1), \quad \forall u, v \in E_N,$$

while the corresponding norm is given by

$$\|u\| = \left(\sum_{t=1}^{N+1} |\Delta u(t-1)|^2\right)^{\frac{1}{2}}.$$

We list also some inequalities that will be used later.

LEMMA 2.1 ([12]). *For every $u \in E_N$, we have:*

(A_1) $\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)} \geq N^{\frac{p^+-2}{2}} \|u\|^{p^+}$, *with* $\|u\| \leq 1$.

(A_2) $\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)} \geq N^{\frac{2-p^-}{2}} ||u||^{p^-} - (N+1)$, *with* $\|u\| > 1$.

(A_3) $\sum\limits_{t=1}^{N} |u(t)|^m \leq N(N+1)^{m-1} \sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^m$, $\forall m > 1$.

(A_4) $\max\limits_{t\in[1,N]_{\mathbb{Z}}} |u(t)| < (N+1)^{\frac{1}{q}} \left(\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^p\right)^{\frac{1}{p}}$, $\forall p, q > 1$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(A_5) $\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^m \leq 2^m \sum\limits_{t=1}^{N} |u(t)|^m$, $\forall m \geq 2$.

(A_6) $\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)} \leq (N+1)||u||^{p^+} + (N+1)$.

(A_7) $\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^m \leq (N+1)||u||^m$, $\forall m \geq 1$.

(A_8) $\sum\limits_{t=1}^{N+1} |\Delta u(t-1)|^m \geq (N+1)^{\frac{2-m}{2}} ||u||^m$, $\forall m \geq 2$.

REMARK 2.2. From (A_6), it is easy to see that γ_N defined in (1) is finite.

Let $u \in E_N$, we consider the functional as follows

$$\Phi(u) = \sum_{t=1}^{N+1} \frac{1}{p(t-1)} |\Delta u(t-1)|^{p(t-1)} - \sum_{t=1}^{N} F(t, u(t)).$$

It is easy to see that $\Phi \in C^1(E_N, \mathbb{R})$ and its derivative $\Phi'(u)$ at $u \in E_N$ is given by

$$\Phi'(u).v = \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2} \Delta u(t-1)\Delta v(t-1) - \sum_{t=1}^{N} f(t, u(t))v(t),$$

for any $v \in E_N$. By the summation by parts formula, Φ' can be written as

$$\Phi'(u).v = \sum_{t=1}^{N} \Big[-\Delta(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) - f(t, u(t))\Big] v(t),$$

for any $v \in E_N$. Thus, if $u \in E_N$ is a critical point of Φ, then u is a solution of (P).

Now, we consider the truncated problem

$$(P_\pm) \qquad \begin{cases} -\Delta(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)) = f_\pm(t,u(t)), & t \in [1,N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases}$$

where

$$f_\pm(t,x) = \begin{cases} f(t,x), & \text{if } \pm x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For $u \in E_N$, we denote by $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$ the positive and negative parts of u. It is clear to see that $u^+ \geq 0$, $u^- \geq 0$, $u = u^+ - u^-$, $u^+ \cdot u^- = 0$, $u^\pm = \frac{1}{2}(|u| \pm u)$ and $u^\pm \leq |u|$.

Lemma 2.3. *All solutions of (P_+) (resp. (P_-)) are non-negative (resp. non positive) solutions of* (P).

Proof. Define $\Phi_\pm : E_N \longrightarrow \mathbb{R}$,

$$\begin{aligned} \Phi_\pm(u) &= \sum_{t=1}^{N+1} \frac{1}{p(t-1)}|\Delta u(t-1)|^{p(t-1)} - \sum_{t=1}^{N} F_\pm(t,u(t)) \\ &= \sum_{t=1}^{N+1} \frac{1}{p(t-1)}|\Delta u(t-1)|^{p(t-1)} - \sum_{t=1}^{N} F(t,u^\pm(t)), \end{aligned}$$

where $F_\pm(t,x) = \int_0^x f_\pm(t,s)\,ds$. It is easy to see that $\Delta u^+(t-1)\Delta u^-(t-1) \leq 0$, $\forall t \in [1,N+1]_{\mathbb{Z}}$. Now, we show that $|\Delta u^-(t-1)| \leq |\Delta u(t-1)|$, $\forall t \in [1,N+1]_{\mathbb{Z}}$. Indeed,

$$\begin{aligned} |\Delta u^-(t-1)| &= |u^-(t) - u^-(t-1)| = |\frac{1}{2}(|u(t)| - u(t)) - \frac{1}{2}(|u(t-1)| - u(t-1))| \\ &\leq \frac{1}{2}\left[||u(t)| - u(t-1)|| + |u(t) - u(t-1)|\right] \leq |\Delta u(t-1)|. \end{aligned}$$

Let u be a solution of (P_+), or equivalently let u be a critical point of Φ_+. Taking $v = u^-$ in $\langle \Phi'_+(u), v\rangle = \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Delta v(t-1) - \sum_{t=1}^{N} f_+(t,u(t))v(t)$, we have

$$\begin{aligned} \langle \Phi'_+(u), u^-\rangle &= \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Delta u^-(t-1) \\ &= \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta(u^+(t-1) - u^-(t-1))\Delta u^-(t-1) \\ &= \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}\Delta u^+(t-1)\Delta u^-(t-1) - |\Delta u(t-1)|^{p(t-1)-2}(\Delta u^-(t-1))^2. \end{aligned}$$

Therefore, we deduce that

$$\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2} \left[-\Delta u^{+}(t-1)\Delta u^{-}(t-1)\right] + \sum_{t=1}^{N+1} |\Delta u(t-1)|^{p(t-1)-2}(\Delta u^{-}(t-1))^2 = 0.$$

Since,

$$-\Delta u^{+}(t-1)\Delta u^{-}(t-1) \geq 0, \quad \forall t \in [1, N+1]_{\mathbb{Z}},$$

then, we get $$|\Delta u(t-1)|^{p(t-1)-2}(\Delta u^{-}(t-1))^2 = 0, \quad \forall t \in [1, N+1]_{\mathbb{Z}}.$$

On the other hand

$$\begin{aligned} |\Delta u^{-}(t-1)|^{p(t-1)} &= |\Delta u^{-}(t-1)|^{p(t-1)-2} \left(\Delta u^{-}(t-1)\right)^2 \\ &\leq |\Delta u(t-1)|^{p(t-1)-2} \left(\Delta u^{-}(t-1)\right)^2 = 0, \end{aligned}$$

for any $t \in [1, N+1]_{\mathbb{Z}}$. So $u^{-} = 0$ and $u = u^{+}$ is also a critical point of Φ with critical value $\Phi(u) = \Phi_{+}(u)$.

Similarly, nontrivial critical points of Φ_{-} are non-positive solutions of (P). □

DEFINITION 2.4. Let H be a real Banach space and $\Phi : H \longrightarrow \mathbb{R}$ be a C^1-functional. We say that a functional Φ satisfies the Palais-Smale (PS) condition, if every sequence $(u_n) \subset H$ such that $(\Phi(u_n))$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$, contains a convergent subsequence. The sequence (u_n) is called a (PS) sequence.

Let B_ρ denote the open ball in H about 0 of radius ρ and let ∂B_ρ the denote its boundary.

LEMMA 2.5 ([3, Mountain Pass Lemma]). *Let Φ be a C^1-functional on a Banach space H that satisfies the (PS) condition and $\Phi(0) = 0$. Suppose that:*
(σ_1) there exist ρ, $\alpha > 0$ such that $\Phi(u) \geq \alpha$ for all $u \in H$ with $\|u\|_H = \rho$,

(σ_2) there exists $e \in H$, with $\|e\|_H > \rho$ such that $\Phi(e) \leq 0$.
Then, $c = \inf_{h\in\Gamma}\max_{s\in[0,1]}\Phi(h(s)) \geq \alpha$, where $\Gamma = \{h \in C([0,1], H) \mid h(0) = 0,\ h(1) = e\}$, is a critical value of Φ.

3. Proofs of the main results

3.1 Proof of Theorem 1.3

To apply the Mountain Pass Theorem, we will do separate studies of the (PS) condition compactness of $\Phi_{\pm}$ and its geometry.

LEMMA 3.1. *Assume that (H_1) holds; then the functional Φ_{+} satisfies the (PS) condition.*

Proof. Let $(u_n) \subset E_N$ be a (PS) sequence for the functional Φ_+, i.e., $|\Phi_+(u_n)| \leq C$ and $\Phi'_+(u_n) \longrightarrow 0$ as $n \to \infty$, where C is a constant. Let us show that (u_n) is bounded in E_N. Since $u_n = u_n^+ - u_n^-$, we will prove that (u_n^+) and (u_n^-) are bounded.

Suppose that (u_n^-) is unbounded. Then there exists an integer $n_0 > 0$ such that

$$\|u_n^-\| \geq N+1 \quad \text{for } n \geq n_0. \tag{4}$$

Since $\Delta u_n^+(t-1)\Delta u_n^-(t-1) \leq 0$ and $|\Delta u_n^-(t-1)| \leq |\Delta u_n(t-1)|$, $\forall t \in [1, N+1]_{\mathbb{Z}}$. Then, we have

$$\langle \Phi'_+(u_n), u_n^- \rangle = \sum_{t=1}^{N+1} |\Delta u_n(t-1)|^{p(t-1)-2} \Delta u_n(t-1)\Delta u_n^-(t-1) - \sum_{t=1}^{N} f_+(t, u_n(t))u_n^-(t)$$

$$= \sum_{t=1}^{N+1} |\Delta u_n(t-1)|^{p(t-1)-2}\Delta u_n^+(t-1)\Delta u_n^-(t-1) - |\Delta u_n(t-1)|^{p(t-1)-2}(\Delta u_n^-(t-1))^2$$

$$\leq - \sum_{t=1}^{N+1} |\Delta u_n^-(t-1)|^{p(t-1)}.$$

Using the above inequality and (A_2), we obtain for any $n \geq n_0$

$$\langle \Phi'_+(u_n), u_n^- \rangle \leq - N^{\frac{2-p^-}{2}} \|u_n^-\|^{p^-} + (N+1).$$

This implies that

$$N^{\frac{2-p^-}{2}} \|u_n^-\|^{p^-} - (N+1) \leq \langle \Phi'_+(u_n), -u_n^- \rangle \leq \|\Phi'_+(u_n)\| \|u_n^-\|.$$

Therefore,

$$N^{\frac{2-p^-}{2}} \|u_n^-\|^{p^-} \leq \|\Phi'_+(u_n)\| \|u_n^-\| + N + 1,$$

and

$$N^{\frac{2-p^-}{2}} \|u_n^-\|^{p^- -1} \leq \|\Phi'_+(u_n)\| + 1. \tag{5}$$

Since $\Phi'_+(u_n) \longrightarrow 0$ as $n \to \infty$, then for any $\varepsilon > 0$, there exists an integer n_1 with $n_1 \geq n_0$ such that $\|\Phi'_+(u_n)\| < \varepsilon$, $\forall n \geq n_1$. Combining the preceding inequality and (5), we get $\|u_n^-\|^{p^- -1} \leq (\varepsilon + 1) N^{\frac{p^- -2}{2}}$ for any $n \geq n_1$. Which means that (u_n^-) is bounded. Thus we obtain a contradiction.

Now, we will prove that (u_n^+) is bounded. According to (H_1), there exists $R > 0$ such that

$$\frac{F(t,x)}{|x|^{p^+}} \geq \delta - \varepsilon, \ \forall (t, |x|) \in [1, N]_{\mathbb{Z}} \times]R, +\infty[\,,$$

where

$$0 < \varepsilon < \delta - \frac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}}. \tag{6}$$

On the other hand, by continuity of $x \mapsto F(t,x) - (\delta - \varepsilon)|x|^{p^+}$, there exists $d > 0$ such that $F(t,x) - (\delta - \varepsilon)|x|^{p^+} \geq -d$, $\forall (t, |x|) \in [1, N]_{\mathbb{Z}} \times [0, R]$. Thus, we deduce that

$$F(t,x) \geq (\delta - \varepsilon)|x|^{p^+} - d, \ \forall (t,x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}. \tag{7}$$

According to (A_5), (A_8) and (7), we obtain

$$\begin{aligned}\sum_{t=1}^{N} F(t,u_n^+(t)) &\geq (\delta-\varepsilon)\sum_{t=1}^{N}|u_n^+(t)|^{p^+}-dN\\ &\geq 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}(\delta-\varepsilon)\|u_n^+\|^{p^+}-dN. \qquad (8)\end{aligned}$$

If $\|u_n^+\|=0$ for any integer $n\geq 0$, then (u_n^+) is bounded. Otherwise, by (A_6) we have

$$\begin{aligned}\Phi_+(u_n) &= \sum_{t=1}^{N+1}\frac{1}{p(t-1)}|\Delta u_n(t-1)|^{p(t-1)}-\sum_{t=1}^{N}F(t,u_n^+(t))\\ &\leq \frac{N+1}{p^-}\left[\|u_n^+-u_n^-\|^{p^+}+1\right]-2^{-p^+}(N+1)^{\frac{2-p^+}{2}}(\delta-\varepsilon)\|u_n^+\|^{p^+}+dN\\ &\leq \frac{N+1}{p^-}\left[\left(\|u_n^+\|+\|u_n^-\|\right)^{p^+}+1\right]-2^{-p^+}(N+1)^{\frac{2-p^+}{2}}(\delta-\varepsilon)\|u_n^+\|^{p^+}+dN\\ &\leq 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}\left[\frac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}}\left(1+\frac{\|u_n^-\|}{\|u_n^+\|}\right)^{p^+}-(\delta-\varepsilon)\right]\|u_n^+\|^{p^+}+\frac{N+1}{p^-}+dN.\end{aligned}$$

So, we deduce that

$$-C\leq 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}\left[\frac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}}\left(1+\frac{\|u_n^-\|}{\|u_n^+\|}\right)^{p^+}-(\delta-\varepsilon)\right]\|u_n^+\|^{p^+}+\frac{N+1}{p^-}+dN.$$

If (u_n^+) is unbounded, up to a subsequence we may assume that $\|u_n^+\|\to\infty$. Then in view of (6) and the fact that (u_n^-) is bounded, we get

$$2^{-p^+}(N+1)^{\frac{2-p^+}{2}}\left[\frac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}}\left(1+\frac{\|u_n^-\|}{\|u_n^+\|}\right)^{p^+}-(\delta-\varepsilon)\right]\|u_n^+\|^{p^+}+\frac{N+1}{p^-}+dN\longrightarrow-\infty$$

as $n\to\infty$, what is a contradiction, hence (u_n^+) is bounded. It follows that (u_n) is bounded. □

Lemma 3.2. *Assume that (H_3) holds; then there exist $r>0$ and $\alpha>0$ such that $\Phi_+(u)\geq\alpha$, for all $u\in E_N$ with $\|u\|=r$.*

Proof. Using the condition (H_3), for any $\varepsilon>0$ there exists $R>0$ such that $|F(t,x)|\leq\varepsilon|x|^{p^+}$, $\forall(t,|x|)\in[1,N]_{\mathbb{Z}}\times[0,R]$.

Let $u\in E_N$, $\|u\|\leq r$ with $r=\min\left\{\frac{R}{\sqrt{N+1}},1\right\}$. From (A_4) it follows $|u^+(t)|\leq|u(t)|\leq\max_{t\in[1,N]_{\mathbb{Z}}}|u(t)|\leq R$, $\forall t\in[1,N]_{\mathbb{Z}}$. Therefore, we deduce that $|F(t,u^+(t))|\leq\varepsilon|u^+(t)|^{p^+}\leq\varepsilon|u(t)|^{p^+}$, $\forall t\in[1,N]_{\mathbb{Z}}$. Using the preceding inequality and (A_1), (A_3), (A_7), we obtain

$$\Phi_+(u)=\sum_{t=1}^{N+1}\frac{1}{p(t-1)}|\Delta u(t-1)|^{p(t-1)}-\sum_{t=1}^{N}F(t,u^+(t))\geq\left[\frac{N^{\frac{p^+-2}{2}}}{p^+}-\varepsilon N(N+1)^{p^+}\right]\|u\|^{p^+}.$$

Let us choose $\varepsilon>0$ such that $\varepsilon<\frac{N^{\frac{p^+-4}{2}}(N+1)^{-p^+}}{p^+}$. It follows that there exist $r>0$

and $\alpha > 0$ such that $\Phi_+(u) \geq \alpha$, $\forall u \in E_N : \|u\| = r$. □

Proof (of Theorem 1.3). In order to apply the Mountain Pass Theorem, we must prove that $\Phi_+(su) \longrightarrow -\infty$ as $s \to \infty$, for certain $u \in E_N$. Let $u \in E_N$, $u > 0$ and $s > 1$. From (A_6) and (8), we have

$$\begin{aligned}\Phi_+(su) &\leq \frac{N+1}{p^-}\left[s^{p^+}\|u\|^{p^+}+1\right] - 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}(\delta-\varepsilon)s^{p^+}\|u\|^{p^+} + dN\\ &\leq 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}s^{p^+}\left(\frac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}} - (\delta-\varepsilon)\right)\|u\|^{p^+} + \frac{N+1}{p^-} + dN,\end{aligned}$$

where $0 < \varepsilon < \delta - \dfrac{2^{p^+}}{p^-}(N+1)^{\frac{p^+}{2}}$. Therefore $\Phi_+(su) \longrightarrow -\infty$ as $s \to \infty$. It follows that there exists $u^* \in E_N$ such that $\|u^*\| > r$ and $\Phi_+(u^*) < 0$.

According to the Mountain Pass Theorem, Φ_+ admits a critical value $c \geq \alpha$ which is characterized by $c = \inf_{g\in\Gamma}\max_{s\in[0,1]}\Phi_+(g(s))$, where $\Gamma = \{g \in C([0,1], E_N) \mid g(0) = 0, g(1) = u^*\}$. Then, the functional Φ_+ has a critical point u_+ with $\Phi_+(u_+) \geq \alpha$. But, $\Phi_+(0) = 0$, that is $u_+ \neq 0$. Therefore, the problem (P_+) has a nontrivial solution which by Lemma 2.3, is a non-negative solution of the problem (P).

Similarly, using Φ_-, we show that there furthermore exists a non-positive solution. □

3.2 Proof of Theorem 1.4

Proof. From the condition (H_3), for $\varepsilon = \dfrac{N^{\frac{p^+-2}{2}}}{2p^+N(N+1)^{p^+}}$ there exists $R > 0$ such that $|F(t,x)| \leq \varepsilon|x|^{p^+}$, $\forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R]$.

Let $u \in E_N$, $\|u\| \leq \rho$ with $\rho = \min\left\{\dfrac{R}{\sqrt{N+1}}, 1\right\}$. By (A_4) it follows that $|u(t)| \leq \max_{t\in[1,N]_{\mathbb{Z}}}|u(t)| \leq R$, $\forall t \in [1,N]_{\mathbb{Z}}$. So, we deduce that $|F(t,u(t))| \leq \varepsilon|u(t)|^{p^+}$, $\forall t \in [1,N]_{\mathbb{Z}}$. By (A_1), (A_3) and (A_7), we have

$$\begin{aligned}\Phi(u) &\geq \frac{N^{\frac{p^+-2}{2}}}{p^+}\|u\|^{p^+} - \varepsilon N(N+1)^{p^+}\|u\|^{p^+}\\ &\geq \left[\frac{N^{\frac{p^+-2}{2}}}{p^+} - \varepsilon N(N+1)^{p^+}\right]\|u\|^{p^+} \geq \frac{N^{\frac{p^+-2}{2}}}{2p^+}\|u\|^{p^+}.\end{aligned}$$

Take $\alpha = \dfrac{N^{\frac{p^+-2}{2}}}{2p^+}\rho^{p^+} > 0$. Then,

$$\Phi(u) \geq \alpha > 0,\ \forall u \in E_N \quad \text{with } \|u\| = \rho. \tag{9}$$

Now, by contradiction we prove that Φ is anti-coercive. Let $K \in \mathbb{R}$ and $(u_n) \subset E_N$ such that $\|u_n\| \longrightarrow \infty$ and $\Phi(u_n) \geq K$. Putting $v_n = \dfrac{u_n}{||u_n||}$, one has $\|v_n\| = 1$. Since $dimE_N < \infty$, there exists $v \in E_N$ such that $\|v_n - v\| \longrightarrow 0$, as $n \to \infty$ and $\|v\| = 1$. In particular $v \neq 0$, we pose $\Theta = \{t \in [1,N]_{\mathbb{Z}}/\ v(t) \neq 0\}$.

For $t \in \Theta$, $|u_n(t)| \longrightarrow \infty$. Using (1), we have

$$K \leq \frac{1}{p^-}\gamma_N \sum_{t=1}^{N} |u_n(t)|^{p^+} - \sum_{t=1}^{N} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right] - \frac{p^+}{(p^-)^2}\gamma_N \sum_{t=1}^{N} |u_n(t)|^{p^+}$$
$$\leq \frac{1}{p^-}\left(1 - \frac{p^+}{p^-}\right)\gamma_N \sum_{t=1}^{N} |u_n(t)|^{p^+} - \sum_{t=1}^{N} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right]$$
$$\leq -\sum_{t\in\Theta} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right] - \sum_{t\in[1,N]_{\mathbb{Z}}\setminus\Theta} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right].$$

From the condition (H_2), we deduce that

$$-\sum_{t\in\Theta} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right] \longrightarrow -\infty, \quad \text{as} \quad n \to \infty.$$

The sequence $(u_n(t))$ is bounded for any $t \in [1,N]_{\mathbb{Z}}\setminus\Theta$ and F is continuous. Hence, there exists a constant $M \in \mathbb{R}$ such that

$$-\sum_{t\in[1,N]_{\mathbb{Z}}\setminus\Theta} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right] \leq M.$$

Therefore, we get

$$K \leq -\sum_{t\in\Theta} \left[F(t,u_n(t)) - \frac{p^+}{(p^-)^2}\gamma_N |u_n(t)|^{p^+}\right] + M \longrightarrow -\infty, \quad \text{as} \quad n \to \infty.$$

This a contradiction. Hence Φ is anti-coercive on E_N. So, we can choose e large enough to ensure that $\Phi(e) < 0$, and that any (PS) sequence (u_n) is bounded. In view of the fact that the dimension of E_N is finite, we see that Φ satisfies the (PS) condition. Since $\Phi(0) = 0$, then all the conditions of Lemma 2.5 are satisfied. Thus Φ possesses a critical value $c \geq \alpha = \dfrac{1}{2p^+} N^{\frac{p^+-2}{2}} \rho^{p^+} > 0$, where $c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \Phi(h(s))$, and $\Gamma = \{h \in C([0,1], E_N)/\, h(0) = 0, h(1) = e\}$. Let $u_1 \in E_N$ such that $\Phi(u_1) = c$. Clearly, u_1 is a nontrivial solution of the problem (P).

On the other hand, since Φ is bounded from above and anti-coercive, then there is a maximum point of Φ at some $u_2 \in E_N$ i.e., $\Phi(u_2) = \sup_{u\in E_N} \Phi(u)$. Using (9), we obtain $\Phi(u_2) = \sup_{u\in E_N} \Phi(u) \geq \sup_{u\in\partial B_\rho} \Phi(u) > 0$. Hence u_2 is a nontrivial solution of the problem (P).

If $u_1 \neq u_2$, then we have two nontrivial solutions u_1 and u_2. Otherwise, similarly to the proof of [7, Theorem 1.3], since $u_1 = u_2$, we deduce that $\Phi(u_1) = \max_{s\in[0,1]} \Phi(g(s)) = \Phi(u_2)$, $\forall g \in \Gamma$.

By the continuity of $\Phi(g(s))$ with respect to s, $\Phi(0) = 0$ and $\Phi(\overline{u}) < 0$ imply that there exists $s_1 \in]0,1[$ such that $\Phi(u_1) = \Phi(g(s_1))$. Choose $g_2, g_3 \in \Gamma$ such that $\{g_2(s) \mid\ s \in]0,1[\} \cap \{g_3(s) \mid; s \in [0,1]\} = \varnothing$; then there exists $s_2, s_3 \in]0,1[$ such that $\Phi(g_2(s_2)) = \Phi(g_3(s_3)) = \Phi(u_1) = \max_{s\in[0,1]} \Phi(g(s))$. We get two different critical points of Φ on E_N.

Consequently, the problem (P) has at least two nontrivial solutions. $\square$

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Laboratory of Applied Mathematics and Information Systems, Department of Mathematics and Computer, Faculty of Sciences, Mohammed I University, Morocco

E-mail: elamrouss@hotmail.com

ORCID iD: https://orcid.org/0000-0003-3536-398X

Laboratory of Applied Mathematics and Information Systems, Department of Mathematics and Computer, Faculty of Sciences, Mohammed I University, Morocco

E-mail: omar.hammouti.83@gmail.com

ORCID iD: https://orcid.org/0000-0002-6065-1361