MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
76, 4 (2024), 257–265
December 2024

research paper
оригинални научни рад
DOI: 10.57016/MV-o1hsah21

RIEMANNIAN SUBMERSIONS FROM RIEMANN SOLITONS

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Abstract. In the present paper, we study a Riemannian submersion π from a Riemann soliton (M_1, g, ξ, λ) onto a Riemannian manifold $(M_2, g^{'})$. We first calculate the sectional curvatures of any fibre of π and the base manifold M_2. Using them, we give some necessary and sufficient conditions for which the Riemann soliton (M_1, g, ξ, λ) is shrinking, steady or expanding. Also, we deal with the potential field ξ of such a Riemann soliton is conformal and obtain some characterizations about the extrinsic vertical and horizontal sectional curvatures of π.

1. Introduction

The term Riemann soliton was introduced by R. Hamilton in 1982 and corresponds to the self-similar solutions of the Riemann flow. A Riemann flow is given by

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)),$$

where $G := \frac{1}{2}(g \otimes g)$, for $\otimes$ the Kulkarni-Nomizu product, R is the Riemannian curvature tensor of g (for details see [10]).

A smooth vector field ξ on a smooth manifold (M_1, g) with Riemannian metric g defines a Riemann soliton if it satisfies

$$\frac{1}{2}((L_\xi g) \otimes g) + R = \lambda G, \tag{1}$$

where $G := \frac{1}{2}(g \otimes g)$, L_ξ is the Lie derivative in the direction of the vector field ξ, R is the Riemannian curvature tensor of g and λ is a constant. A Riemann soliton is denoted by (M, g, ξ, λ) and the vector field ξ is the potential field of the Riemann soliton.

A Riemann soliton (M_1, g, ξ, λ) is called shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively. Furthermore, (M_1, g, ξ, λ) is called trivial if M_1 is a space with constant sectional curvature. A Riemann soliton (M_1, g, ξ, λ) is a gradient

2020 Mathematics Subject Classification: 53C40, 32Q15
Keywords and phrases: Riemannian submersion; Riemann soliton; sectional curvature.

Riemann soliton if its potential field ξ is the gradient of a smooth function on M_1 and it is denoted by (M, g, f, λ). We recall that for $(0,2)-$ tensor fields T_1 and T_2 the Kulkarni-Nomizu product is defined as

$$(T_1 \otimes T_2)(X,Y,Z,H) := T_1(X,H)T_2(Y,Z) + T_1(Y,Z)T_2(X,H) \\ - T_1(X,Z)T_2(Y,H) - T_1(Y,H)T_2(X,Z),$$

for any vector fields X, Y, Z, H on M_1. Taking into account the Kulkarni-Nomizu product above, the equation (1) is equivalent to

$$2R(X,Y,Z,H) + g(X,H)(L_\xi g)(Y,Z) + g(Y,Z)(L_\xi g)(X,H) \\ - g(X,Z)(L_\xi g)(Y,H) - g(Y,H)(L_\xi g)(X,Z) = 2\lambda\Big(g(X,H)g(Y,Z) - g(X,Z)g(Y,H)\Big),$$

for any vector fields X, Y, Z, H on M_1. Note that the relations between Riemann soliton and Ricci soliton were studied in [4] and the authors showed that any gradient Riemann soliton with a potential vector field of constant length is steady and is a solenoidal vector field. Moreover, many authors studied the relations between the Ricci flow and the Riemann flow to determine some geometric properties of such flows on Riemannian manifolds [3–5, 11, 13, 14].

On the other hand, Riemannian submersions have many applications in theoretical physics, in particular in Kaluza-Klein theory, general relativity and modelling and control of certain types of redundant robotic chains. Therefore, the concept of Riemannian submersions between Riemannian manifolds has been intensively studied in the literature [1, 2, 8, 9, 12, 15].

In [7] the authors consider a Riemannian submersion whose entire manifold admits a Ricci soliton, and give some results on whether every fibre of such a submersion is a Ricci soliton or almost a Ricci soliton. Inspired by [7], in this paper we deal with a Riemannian submersion whose total space admits a Riemann soliton. Here we specify the sectional curvatures on an arbitrary fibre and the base manifold of π and thus obtain some necessary conditions for which the Riemann soliton (M_1, g, ξ, λ) is shrinking, steady or expanding. In the next section, we assume that the potential field ξ of the Riemann soliton is a conformal vector field, and in the case that ξ is horizontal or vertical, we calculate the extrinsic vertical and horizontal sectional curvature of π.

2. Some notes on Riemannian submersions

Now we recall the following concepts of [8, 12].

Let $\pi : (M_1^m, g) \to (M_2^n, g^{'})$ be a submersion between two Riemannian manifolds and let $r = m - n$ denote the dimension of any closed fibre $\pi^{-1}(x)$, for any $x \in M_2$. For any $p \in M_1$, by setting $\mathcal{V}_p = ker\pi_{*p}$, we have an integrable distribution $\mathcal{V}$ corresponding to the foliation of M_1 determined by the fibres of π. We therefore have $\mathcal{V}_p = T_p\pi^{-1}(x)$ and $\mathcal{V}$ is called the vertical distribution. Let $\mathcal{H}$ be the horizontal distribution, which means that $\mathcal{H}$ is the orthogonal distribution of $\mathcal{V}$ with respect to

g, i.e. $T_p(M_1) = \mathcal{V}_p \oplus \mathcal{H}_p$, $p \in M_1$. We note that $\pi_* X$ is given by the basic vector field $X^{'}$ on M_2 which is $\pi-$related to X on M_1.

A mapping π between Riemannian manifolds M_1 and M_2 is called a Riemannian submersion if the following conditions hold:
(i) π has a maximal rank;

(ii) The differential π_{*p} preserves the length of the horizontal vector fields at each point of M_1.

Proposition 2.1. *Let $\pi : (M_1, g) \to (M_2, g^{'})$ be a Riemannian submersion between Riemannian manifolds. For the basic vector fields X, Y, $\pi-$related to $X^{'}, Y^{'}$, one has*
(i) $g(X, Y) = g^{'}(X^{'}, Y^{'}) \circ \pi$,

(ii) The basic vector field $h[X, Y]$ is π-related to $[X^{'}, Y^{'}]$,

(iii) The basic vector field $h(\nabla_X Y)$ is π-related to $\nabla^{'}_{X'} Y^{'}$,

(iv) $[X, V]$ is vertical, for any vertical vector field V,
where ∇ and $\nabla^{'}$ denote the Levi-Civita connections of M_1 and M_2, respectively (see [8]).

On the other hand, the tensor fields T and A are called the fundamental tensor fields on the total space M_1 of π, which are defined as

$$T(E, F) = T_E F = h(\nabla_{vE} vF) + v(\nabla_{vE} hF),$$
$$A(E, F) = A_E F = v(\nabla_{hE} hF) + h(\nabla_{hE} vF),$$

where vE and hE denote the vertical and horizontal components of E, respectively, for any $E, F \in \Gamma(TM_1)$.

The fundamental tensor fields T and A on M_1 satisfy the following relations:

$$g(T_E F, G) = -g(T_E G, F) \tag{2}$$
$$g(A_E F, G) = -g(A_E G, F) \tag{3}$$

for any $E, F, G \in \Gamma(TM_1)$. The following properties also apply to the fundamental tensor fields T and A

$$T_U V = T_V U, \tag{4}$$
$$A_X Y = -A_Y X, \tag{5}$$

for any vertical and horizontal vector fields U, V and X, Y respectively.

Note that the vanishing of the tensor field T or A has some geometric meanings. The tensor A vanishes if and only if the distribution $\mathcal{H}$ is integrable. The tensor T vanishes if and only if any fibre of π is a totally geodesic submanifold of M_1. Using the fundamental tensor fields T and A, we can see that

$$\nabla_U V = T_U V + \hat{\nabla}_U V, \qquad \nabla_V X = h(\nabla_V X) + T_V X, \tag{6}$$
$$\nabla_X V = A_X V + v(\nabla_X V), \qquad \nabla_X Y = h(\nabla_X Y) + A_X Y, \tag{7}$$

where ∇ and $\hat{\nabla}$ are the Levi-Civita connections of M_1 and any fibre of π, respectively, for any vertical vector fields U, V and horizontal vector fields X, Y.

On the other hand, the mean curvature vector field H on any fibre of Riemannian submersion π is given by $H = \frac{1}{r}\sum_{j=1}^{r} \mathcal{T}_{U_j}U_j$, where r denotes the dimension of any fibre of π and $\{U_1, U_2, \dots, U_r\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$.

Recall that any fibre of π is a total umbilic if $\mathcal{T}_U V = g(U,V)H$, is satisfied. Here H is the mean curvature vector field of π in M_1, for any vertical vector fields U, V.

We recall some formulas dealing with the sectional curvatures $K(\alpha)$, where α denotes a $2-$plane in T_pM_1, $p \in M_1$. More precisely, if $\{U, V\}$ is an orthonormal basis of the vertical $2-$plane α, one has

$$K(\alpha) = \hat{K}(\alpha) + \|T_U V\|^2 - g(T_U U, T_V V), \tag{8}$$

$\hat{K}(\alpha)$ denotes the sectional curvature in the fibre through p. If $\{X, Y\}$ is an orthonormal basis of the horizontal $2-$plane α and $K^{'}(\alpha^{'})$ denotes the sectional curvature in $(M_2, g^{'})$ of the plane $\alpha^{'}$ spanned by $\{\pi_* X, \pi_* Y\}$, then

$$K(\alpha) = K^{'}(\alpha^{'}) - 3\|A_X Y\|^2. \tag{9}$$

On the other hand, we recall the following notion from [6].

Definition 2.2. Let (M, g) be an $n-$dimensional Riemannian manifold. A vector field ξ is called conformal vector field if

$$L_\xi g = 2fg, \tag{10}$$

where L_ξ is the Lie-derivative with respect to ξ and f is a smooth function on M_1. Also, using the Koszul formula, for a vector field ξ, one can see that

$$2g(\nabla_X \xi, Y) = (L_\xi g)(X, Y) + d\eta(X, Y), \tag{11}$$

where η denotes the 1-form dual to the vector field ξ, that is, $g(X, \xi) = 0$, for any vector fields X, Y. A skew-symmetric tensor field ϕ of type-$(1,1)$ on M_1 is defined by

$$d\eta(X, Y) = 2g(\phi X, Y), \tag{12}$$

for any vector fields X, Y on M_1. Then, from the equations (10)-(12), we obtain

$$\nabla_X \xi = fX + \phi X, \qquad X \in \Gamma(TM_1). \tag{13}$$

Here, the skew-symmetric tensor field ϕ in (13) is called the associate tensor field of the conformal vector field ξ.

3. Riemannian submersions whose total space admits an Riemann soliton

Here a Riemannian submersion from a Riemann soliton (M_1, g, ξ, λ) onto a Riemannian manifold $(M_2, g^{'})$ is considered and the sectional curvatures of an arbitrary fibre and a base manifold M_2 are calculated as follows.

Theorem 3.1. *Let (M_1, g, ξ, λ) be a Riemann soliton with horizontal potential field ξ and let $\pi : (M_1, g) \to (M_2, g^{'})$ be a Riemannian submersion with totally umbilical fibres. Then the sectional curvature $\hat{K}$ on any fibre is given by*

$$\hat{K}(U, V) = 2g(H, \xi) + \|H\|^2 + \lambda,$$

where H denotes the mean curvature vector field for any orthonormal vertical vector fields U, V.

Proof. Since the total space M_1 admits a Riemann soliton of (1), one has

$$2K(U,V) + ((L_\xi g) \otimes g)(U,V,V,U) - \lambda(g \otimes g)(U,V,V,U) = 0, \tag{14}$$

where K is the sectional curvature of M_1, for any orthonormal vertical vector fields U, V. Using equation (8), we also get

$$K(U,V) = \hat{K}(U,V) + g(T_U V, T_U V) - g(T_U U, T_V V). \tag{15}$$

Since any fibre of π is totally umbilical, the equation (15)

$$K(U,V) = \hat{K}(U,V) + g(U,V)^2\|H\|^2 - \|U\|^2\|V\|^2\|H\|^2,$$

which means

$$K(U,V) = \hat{K}(U,V) - \|H\|^2. \tag{16}$$

On the other hand, using the Nomizu-Kulkarni product in the equation (14), we get

$$((L_\xi g)\otimes g)(U,V,V,U) = g(U,U)(L_\xi g)(V,V) + g(V,V)(L_\xi g)(U,U) - 2g(U,V)(L_\xi g)(U,V)$$
$$= (L_\xi g)(V,V) + (L_\xi g)(U,U) = 2\big(g(\nabla_U \xi, U) + g(\nabla_V \xi, V)\big).$$

If the equations (6) and (2) are applied to the last equality, the result is

$$((L_\xi g) \otimes g)(U,V,V,U) = 2\big(g(T_U \xi, U) + g(T_V \xi, V)\big) = -2\big(g(T_U U, \xi) + g(T_V V, \xi)\big).$$

Since π has totally umbilical fibres, it follows

$$((L_\xi g) \otimes g)(U,V,V,U) = -4g(H,\xi). \tag{17}$$

We also get

$$\lambda(g \otimes g)(U,V,V,U) = \lambda\big(g(U,U)g(V,V) + g(V,V)g(U,U) - 2g(U,V)g(V,U)\big)$$

which is equivalent to

$$\lambda(g \otimes g)(U,V,V,U) = 2\lambda. \tag{18}$$

Putting the equations (16)-(18) into (14) we get $\hat{K}(U,V) = 2g(H,\xi) + \|H\|^2 + \lambda$. □

Particularly, if the Riemannian submersion π has the minimal fibres, by Theorem 3.1, we obtain the following result.

COROLLARY 3.2. *Let (M_1, g, ξ, λ) be a Riemann soliton with horizontal potential field ξ and let $\pi : (M_1, g) \to (M_2, g')$ be a Riemannian submersion with totally geodesic fibres. Then, the following hold:*

(i) Any fibre of π has positive sectional curvature if and only if the Riemann soliton (M_1, g, ξ, λ) is shrinking.

(ii) Any fibre of π is flat if and only if the Riemann soliton (M_1, g, ξ, λ) is steady.

(iii) Any fibre of π has negative sectional curvature if and only if the Riemann soliton (M_1, g, ξ, λ) is expanding.

If we choose the potential field ξ of the Riemann soliton as a concurrent on M_1, then we have the following.

Theorem 3.3. *Let (M_1, g, ξ, λ) be a Riemann soliton with vertical potential field ξ and $\pi : (M_1, g) \to (M_2, g^{'})$ a Riemannian submersion with totally umbilical fibres. If the vector field ξ is concurrent on M_1, then the sectional curvature $\hat{K}$ on any fibre is given by*

$$\hat{K}(U, V) = \|H\|^2 + \lambda - 2, \tag{19}$$

where H is the mean curvature vector field, for any vertical vector fields U, V.

Proof. Since the total space M_1 admits a Riemann soliton from (1), one has

$$2K(U, V) + ((L_\xi g) \otimes g)(U, V, V, U) - \lambda(g \otimes g)(U, V, V, U) = 0, \tag{20}$$

where K is the sectional curvature of M_1, for any orthonormal vertical vector fields U, V. Since any fibre of π is totally umbilical, the equation (8) gives

$$K(U, V) = \hat{K}(U, V) - \|H\|^2. \tag{21}$$

On the other hand, using the Nomizu-Kulkarni product in (20), we get

$$((L_\xi g)\otimes g)(U, V, V, U)=g(U, U)(L_\xi g)(V, V)+g(V, V)(L_\xi g)(U, U)-2g(U, V)(L_\xi g)(U, V)$$
$$=(L_\xi g)(V, V)+(L_\xi g)(U, U)=2\Big(g(\nabla_V \xi, V)+g(\nabla_U \xi, U)\Big).$$

Since the vector field ξ is concurrent on M_1, the last equality results in

$$((L_\xi g) \otimes g)(U, V, V, U) = 2\Big(g(V, V) + g(U, U)\Big) = 4. \tag{22}$$

If you use the Nomizu-Kulkarni product in (20), you also get

$$\lambda(g \otimes g)(U, V, V, U)=\lambda\big(g(U, U)g(V, V)+g(V, V)g(U, U)-2g(U, V)g(V, U)\big)=2\lambda. \tag{23}$$

So if you insert the equation (21)-(23) into (20), you get $2\Big(\hat{K}(U, V)-\|H\|^2\Big)+4-2\lambda = 0$, which gives (19). □

As a consequence of Theorem 3.3 we can give the next result.

Corollary 3.4. *Let (M_1, g, ξ, λ) be a Riemann soliton with vertical potential field ξ and $\pi : (M_1, g) \to (M_2, g^{'})$ a Riemannian submersion with totally geodesic fibres. If the vector field ξ is concurrent on M_1, then the following hold:*

(i) If any fibre has a positive sectional curvature, then (M_1, g, ξ, λ) is shrinking.

(ii) If any fibre is flat, then (M_1, g, ξ, λ) is shrinking.

(iii) If (M_1, g, ξ, λ) is expanding, then any fibre has a negative sectional curvature.

(iv) If the Riemann soliton (M_1, g, ξ, λ) is steady, then any fibre has a negative sectional curvature.

The next theorem gives the relation between the characterization of the Riemann soliton (M_1, g, ξ, λ) and the sectional curvature on M_2.

Theorem 3.5. *Let (M_1, g, ξ, λ) be a Riemann soliton with vertical potential field ξ and $\pi : (M_1, g) \to (M_2, g^{'})$ a Riemannian submersion. Then, we have the following:*

(i) M_2 has negative sectional curvature or M_2 is flat if and only if (M_1, g, ξ, λ) is expanding.

(ii) If the Riemann soliton (M_1, g, ξ, λ) is steady or shrinking, then M_2 has positive sectional curvature.

Proof. Since (M_1, g, ξ, λ) is a Riemann soliton with a vertical potential field ξ, we obtain from (8)

$$2R(X,Y,Y,X) + ((L_\xi g) \otimes g)(X,Y,Y,X) - \lambda(g \otimes g)(X,Y,Y,X) = 0 \tag{24}$$

for any orthonormal horizontal vector fields X, Y. Using the Nomizu-Kulkarni product in (24) we get

$$\begin{aligned}((L_\xi g)\otimes g)(X,Y,Y,X) &= g(X,X)(L_\xi g)(Y,Y) \\ &\quad + (L_\xi g)(X,X)g(Y,Y) - 2g(X,Y)(L_\xi g)(X,Y) \\ &= (L_\xi g)(Y,Y) + (L_\xi g)(X,X) = 2g(\nabla_Y \xi, Y) + 2g(\nabla_X \xi, X).\end{aligned}$$

If you apply (7) and (5) to the last equality, you get

$$\begin{aligned}((L_\xi g) \otimes g)(X,Y,Y,X) &= 2g(\nabla_Y \xi, Y) + 2g(\nabla_X \xi, X) = 2g(A_Y \xi, Y) + 2g(A_X \xi, X) \\ &= -2g(A_Y Y, \xi) - 2g(A_X X, \xi) = 0. \end{aligned} \tag{25}$$

Furthermore, using the Nomizu-Kulkarni product, we get

$$\begin{aligned}\lambda(g \otimes g)(X,Y,Y,X) &= \lambda\big(g(X,X)g(Y,Y) + g(Y,Y)g(X,X) - 2g(X,Y)g(Y,X)\big) \\ &= 2\lambda(\|X\|^2\|Y\|^2 - g(X,Y)^2) = 2\lambda. \end{aligned} \tag{26}$$

If you insert the equations (9), (25), (26) into (24), the result is

$$K^{'}(\alpha^{'}) \circ \pi \ - 3\|A_X Y\|^2 - \lambda = 0, \tag{27}$$

where $X^{'}, Y^{'}$ is an orthonormal basis of the $2-$plane $\alpha^{'}$ on M_2. From the last equality we get the statement. □

Using (27), we have the following.

Corollary 3.6. *Let (M_1, g, ξ, λ) be a Riemann soliton with vertical potential field ξ and let $\pi : (M_1, g) \to (M_2, g^{'})$ be a Riemannian submersion. If the horizontal distribution $\mathcal{H}$ is integrable, then the sectional curvature of M_2 is λ.*

4. Conformal vector fields and Riemannian submersions

In this section we treat a Riemannian submersion $\pi : M_1 \to M_2$ between Riemannian manifolds such that the total manifold M_1 is endowed with a conformal vector field ξ. In the case that ξ is vertical or horizontal, we obtain some characterizations for a Riemannian submersion whose total manifold admits a Riemann soliton.

Theorem 4.1. *Let (M_1, g, ξ, λ) be a Riemann soliton and let $\pi : (M_1, g) \to (M_2, g^{'})$ be a Riemannian submersion. If the conformal vector field ξ is vertical, then the extrinsic vertical sectional curvature $K \mid_{\mathcal{V}}$ is given by $K \mid_{\mathcal{V}} = \lambda - 2f$, where f is a smooth function on M_1.*

Proof. Since the Riemannian manifold M_1 admits a Riemann soliton, for any orthonormal vertical vector fields U, V, we obtain with (1):

$$((L_\xi g)\otimes g)(U,V,V,U)+2K(U,V)=\lambda(g\otimes g)(U,V,V,U). \tag{28}$$

If we apply the Nomizu-Kulkarni product to (28), we get

$$\begin{aligned}((L_\xi g)\otimes g)(U,V,V,U)=&g(U,U)(L_\xi g)(V,V)\\ &+g(V,V)(L_\xi g)(U,U)-2g(U,V)(L_\xi g)(U,V)\\ =&(L_\xi g)(U,U)+(L_\xi g)(V,V)=2\big(g(\nabla_U\xi,U)+g(\nabla_V\xi,V)\big)\end{aligned} \tag{29}$$

If we use (13) in (29), we also have:

$$\begin{aligned}&((L_\xi g)\otimes g)(U,V,V,U)=2\big(g(\nabla_U\xi,U)+(\nabla_V\xi,V)\big)\\ &=2\big(g(fU+\phi U,U)+g(fV+\phi V,V)\big)=2\big(fg(U,U)+g(\phi U,U)+fg(V,V)+g(\phi V,V)\big).\end{aligned}$$

Since the associated tensor field ϕ of ξ is skew-symmetric, the last equation is equivalent to

$$((L_\xi g)\otimes g)(U,V,V,U)=4f. \tag{30}$$

If we apply the Nomizu-Kulkarni product to the right-hand side of the equation (28), we obtain

$$\lambda(g\otimes g)(U,V,V,U)=\lambda\big(g(U,U)g(V,V)+g(V,V)g(U,U)-2g(U,V)g(V,U)\big)=2\lambda, \tag{31}$$

for any orthonormal vertical vector fields U, V. Putting the equations (30)-(31) into (28), we get $K\mid_{\mathcal{V}}=\lambda-2f$. □

Lemma 4.2. *Let $\pi:(M_1,g)\to(M_2,g')$ be a Riemannian submersion such that the conformal vector field ξ is vertical on M_1. Then the conformal vector field ξ is trivial.*

Proof. If you put the equations (3) and (5) into Lie derivative, you get

$$\begin{aligned}(L_\xi g)(X,Y)&=g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)=g(A_X\xi,Y)+(A_Y\xi,X)\\ &=-g(A_XY,\xi)-g(A_YX,\xi)=-g(A_XY,\xi)+g(A_XY,\xi)=0,\end{aligned} \tag{32}$$

for any horizontal vector fields X, Y. On the other hand, using the condition of skew-symmetric of ϕ and the equation (13) in the Lie derivative results in

$$\begin{aligned}(L_\xi g)(X,Y)&=g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)\\ &=g(fX+\phi X,Y)+g(fY+\phi Y,X)=2fg(X,Y).\end{aligned} \tag{33}$$

If we compare the equations (32)-(33), we obtain that $f=0$, which means that the vertical conformal vector field ξ is trivial. □

Theorem 4.3. *Let (M_1,g,ξ,λ) be a Riemann soliton with vertical conformal vector field ξ and let $\pi:(M_1,g)\to(M_2,g')$ be a Riemannian submersion. Then the extrinsic horizontal sectional curvature $K\mid_{\mathcal{H}}$ is given by $K\mid_{\mathcal{H}}=\lambda$.*

Proof. Since (M_1,g,ξ,λ) is a Riemann soliton, we obtain from the equation (1)

$$((L_\xi g)\otimes g)(X,Y,Y,X)+2K(X,Y)=\lambda(g\otimes g)(X,Y,Y,X) \tag{34}$$

for any orthonormal horizontal vector fields X, Y on M_1. If we apply the Nomizu-Kulkarni product to the right-hand side of (34), we obtain

$$\lambda(g\otimes g)(X,Y,Y,X)=\lambda\big(g(X,X)g(Y,Y)+g(Y,Y)g(X,X)-2g(X,Y)g(Y,X)\big)=2\lambda.$$

Here we note that according to Lemma 4.2 the vertical conformal vector field ξ is trivial. If we then substitute the previous equation into (34), we obtain $K\mid_{\mathcal{H}}=\lambda$ and the proof is complete. □

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(received 07.06.2022; in revised form 27.07.2023; available online 14.08.2024)

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