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ON BETTER APPROXIMATION ORDER FOR THE NONLINEAR FAVARD-SZÁSZ-MIRAKJAN OPERATOR OF MAXIMUM PRODUCT KIND

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Abstract. Using maximum instead of sum, nonlinear Favard-Szász-Mirakjan operator of maximum product kind was introduced. The present paper deals with the approximation processes for this operator. Especially in a previous study, it was indicated that the order of approximation of this operator to the function f under the modulus is $\sqrt{x/n}$ and it could not be improved except for some subclasses of functions. Contrary to this claim, under some special conditions, we will show that a better order of approximation can be obtained with the help of classical and weighted modulus of continuities.

1. Introduction

For $f \in C[0,\infty)$, the classical Favard-Szász-Mirakjan operators are defined as

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}$$

were introduced in [16].

The construction logic of nonlinear operators of the maximum product type, which use the maximum instead of the sum, is based on the studies [8, 9, 14] (for details, see also, [6]).

There are some other notable articles such as [2–5, 7], which we will remind you in chronological order, that various nonlinear operators of the maximum product type have been introduced and their approximation and convergence properties have been studied.

Note that the paper [2] is the first one in which uniform convergence rates and shape-preserving properties are obtained with a remarkable approach, which is then used in the paper [7].

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In particular, in [5] the approximation properties, convergence rate and shape preserving properties of the maximum product type Favard-Szász-Mirakjan operator are investigated.

At this point, let us recall the following well-known concept of the classical module of continuity:

$$\omega(f,\delta) = \max\{|f(x) - f(y)|;\ x, y \in I, |x-y| \le \delta\}. \tag{1}$$

The approximation order for the Favard-Szász-Mirakjan operator of the maximum product type can be found in [5] using the continuity modulus as $\omega\left(f; \sqrt{x/n}\right)$.

The main goal of this paper is to obtain a better order for the Favard-Szász-Mirakjan operator of the maximum product type using the classical and the weighted continuity modulus.

2. The concept of nonlinear maximum product operators

Before the main results, we will recall basic definitions and theorems about nonlinear operators from [6–8].

Over the set of $\mathbb{R}_+$ we consider the operations $\vee$ (maximum) and "$\cdot$" product. Then $(\mathbb{R}_+, \vee, \cdot)$ has a semiring structure and is called a maximum product algebra.

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I) = \{f : I \to \mathbb{R}_+ : f \text{ continuous and bounded on } I\}.$$

Let us take the general form of $L_n : CB_+(I) \to CB_+(I)$, as

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) f(x_i) \quad \text{or} \quad L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) f(x_i),$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(., x_i) \in CB_+(I)$ and $x_i \in I$, for all i. These operators are non-linear, positive operators and also satisfy the following pseudo-linearity condition of the form

$$L_n(\alpha f \vee \beta g)(x) = \alpha L_n(f)(x) \vee \beta L_n(g)(x), \forall \alpha, \beta \in \mathbb{R}_+,\ f, g : CB_+(I).$$

In this section, we present some general results for this type of operator which we will use later.

Lemma 2.1 ([7]). *Let $I \subset \mathbb{R}$ be bounded or unbounded interval,*

$$CB_+(I) = \{f : I \to \mathbb{R}_+ : f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties:

(i) If $f, g \in CB_+(I)$ satisfy $f \le g$ then $L_n(f) \le L_n(g)$ for all $n \in \mathbb{N}$.

(ii) $L_n(f+g) \le L_n(f) + L_n(g)$ for $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have $|L_n(f)(x) - L_n(g)(x)| \le L_n(|f-g|)(x)$.

Remark 2.2 ([5]). 1) It is easy to see that the nonlinear Favard-Szász-Mirakjan maximum product operator satisfy the conditions (i) and (ii) of Lemma 2.1. In fact, instead of (i) t also satisfies the following stronger condition:

$$L_n (f \vee g)(x) = L_n (f)(x) \vee L_n (g)(x), \ f, g \in CB_+(I).$$

Indeed, taking into consideration of the equality above, for $f \leq g$, $f, g \in CB_+(I)$, *it easily follows* $L_n (f)(x) \leq L_n (g)(x)$.

2) In addition, it is immediate that the nonlinear Favard-Szász-Mirakjan maximum product operator is positive homogenous, that is $L_n (\lambda f) = \lambda L_n (f)$ for all $\lambda \geq 0$.

After this point, we denote the monomials $e_r(t) := t^r$, $r \in \mathbb{N}_0$. The first three monomials are also known as Korovkin test functions.

Corollary 2.3 ([7]). *Let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the conditions (i) and (ii) in Lemma 2.1 and in addition being positive homogenous. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have*

$$|L_n (f)(x) - f(x)| \leq \left[\frac{1}{\delta} L_n (\varphi_x)(x) + L_n (e_0)(x)\right] \omega (f, \delta) + f(x) \, |L_n (e_0)(x) - 1|,$$

where $\omega (f, \delta)$ is the classical modulus of continuity defined by (1), $\delta > 0$, $e_0 (t) = 1$, $\varphi_x (t) = |t - x|$ *for all $t \in I$, $x \in I$, and if I is unbounded then we suppose that there exists $L_n (\varphi_x)(x) \in \mathbb{R}_+ \cup \{\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.*

A consequence of Corollary 2.3 is the following:

Corollary 2.4 ([7]). *Suppose that in addition to the conditions in Corollary 2.3, the sequence $(L_n)_n$ satisfies $L_n (e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_+ (I)$, $n \in \mathbb{N}$ and $x \in I$ we have*

$$|L_n (f)(x) - f(x)| \leq \left[1 + \frac{1}{\delta} L_n (\varphi_x)(x)\right] \omega (f, \delta)$$

where φ_x was introduced at Corollary 2.3 and $\omega (f, \delta)$ is the classical modulus of continuity defined by (1) *and* $\delta > 0$.

3. Nonlinear Favard-Szász-Mirakjan operator of maximum product kind

In the classical Favard-Szász-Mirakjan operator, the sum operator $\sum$ is replaced by the $\bigvee$ maximum operator and introduced by Bede et al. in [7]. The non-linear Favard-Szász-Mirakjan operator of the maximum product type is thus defined as

$$F_n^{(M)}(f)(x) = \frac{\bigvee\limits_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)}{\bigvee\limits_{k=0}^{\infty} \frac{(nx)^k}{k!}} \tag{2}$$

where $f \in C[0, \infty)$, $x \in [0, \infty)$, $n \in \mathbb{N}$.

In [5], the approximation and shape preserving properties of $F_n^{(M)}(f)(x)$ are investigated.

Remark 3.1 ([5]). It is clear that $F_n^{(M)}(f)(x)$ satisfies all conditions in Lemma 2.1, Corollary 2.3 and Corollary 2.4 for $I = [0, \infty)$.

4. Auxiliary results

From [5], we get $F_n^{(M)}(f)(0) - f(0) = 0$ for all n. In this part, we will also consider $x > 0$.

For each $k, j \in \{0, 1, 2, \ldots,\}$ and $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$, let us define

$$M_{k,n,j}(x) := \frac{s_{n,k}(x) \left|\frac{k}{n} - x\right|}{s_{n,j}(x)}, \; m_{k,n,j}(x) := \frac{s_{n,k}(x)}{s_{n,j}(x)}$$

similar to [5]. It is clear that if $k \geq j + 1$ then

$$M_{k,n,j}(x) = \frac{s_{n,k}(x) \left(\frac{k}{n} - x\right)}{s_{n,j}(x)}$$

and if $k \leq j - 1$ then

$$M_{k,n,j}(x) = \frac{s_{n,k}(x) \left(x - \frac{k}{n}\right)}{s_{n,j}(x)}$$

where $s_{n,k}(x) = \frac{(nx)^k}{k!}$.

Notice that, for all $k, j \in \{0, 1, 2, \ldots,\}$, $M_{k,n,j}(x)$ and $m_{k,n,j}(x)$ were defined in [5].

At this point, let us recall the following two lemmas.

Lemma 4.1 ([5]). *For all $k, j \in \{0, 1, 2, \ldots\}$ and $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ we have $m_{k,n,j}(x) \leq 1$.*

Lemma 4.2. *One has*

$$\bigvee_{k=0}^{\infty} s_{n,k}(x) = s_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n}, \frac{j+1}{n}\right], j = 0, 1, \ldots,$$

where $s_{n,k}(x) = \frac{(nx)^k}{k!}$.

We now give the following lemma, which is proved using a different proof technique from that given in [2, 5].

Lemma 4.3. *Let $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ and $\alpha = 2, 3, \ldots$.*

(i) If $k \in \{j + 1, j + 2, \ldots\}$ is such that $k - (k + 1)^{1/\alpha} \geq j$, then we have

$$M_{k,n,j}(x) \geq M_{k+1,n,j}(x).$$

(ii) If $k \in \{1, 2, \ldots, j - 1\}$ is such that $k + (k)^{1/\alpha} \leq j$, then we get

$$M_{k,n,j}(x) \geq M_{k-1,n,j}(x).$$

Proof. (i) From [5, Lemma 3.2, case (i)], we can write

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq \frac{k+1}{j+1} \frac{k-j-1}{k-j}.$$

After this point we will use a different proof technique from [5].

By using the induction method, let's show that the following inequality

$$\frac{k+1}{j+1}\frac{k-j-1}{k-j} \geq 1 \tag{3}$$

holds for $k-(k+1)^{1/\alpha} \geq j$.

For $\alpha = 2$, since the condition $k-(k+1)^{1/2} \geq j$ holds, then we have $(k-j)^2 \geq k+1$. So we get $(k+1)(k-j-1) \geq (j+1)(k-j)$. Therefore we obtain the inequality (3) for $\alpha = 2$.

Now, we assume that (3) is correct for $\alpha - 1$. Since the inequality (3) holds for $k-(k+1)^{1/(\alpha-1)} \geq j$, we have$(k-j)^\alpha \geq (k+1)(k-j)$.

On the other hand, since $k \geq j+1$, we can write $(k-j)^\alpha \geq (k+1)(k-j) \geq k+1$. So the desired inequality is provided for $k-(k+1)^{1/\alpha} \geq j$. Thus we obtain,

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq \frac{k+1}{j+1}\frac{k-j-1}{k-j} \geq 1$$

for $\alpha = 2, 3, \ldots$.

(ii) From [5, Lemma 3.2, case (ii)], we can write

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \frac{j}{k}\frac{j-k}{j-k+1}.$$

After this point we will use our proof technique again.

By using the induction method, let's show that the following inequality

$$\frac{j}{k}\frac{j-k}{j-k+1} \geq 1 \tag{4}$$

holds for $k+(k)^{1/\alpha} \leq j$.

For $\alpha = 2$, because of the condition $k+(k)^{1/2} \leq j$, we get $k \leq (j-k)^2$ and $j(j-k) \geq k(j-k+1)$. Thus we see that (4) is satisfied for $\alpha = 2$.

Now, we assume that (4) is correct for $\alpha - 1$. Since the inequality (4) holds for $k+(k)^{1/(\alpha-1)} \leq j$, we have$k(j-k) \leq (j-k)^\alpha$.

On the other hand, since $k \leq j-1$, we can write $k \leq k(j-k) \leq (j-k)^\alpha$. So the desired inequality is provided for $k+(k)^{1/\alpha} \leq j$. Thus we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq \frac{j}{k}\frac{j-k}{j-k+1} \geq 1$$

for $\alpha = 2, 3, \ldots$ which gives the desired result. □

5. Pointwise rate of convergence

Let us take an x_0 on the interval $[0, \infty)$. The main goal of this section is to obtain a better order of the pointwise approximation for the operators $F_n^{(M)}(f)(x_0)$ to the function $f(x_0)$ using the continuity modulus. According to the following theorem, it can be said that the order of the pointwise approximation can be improved if α is large enough. Moreover, if $\alpha = 2$, these approximation results turn out to be the results in [5].

THEOREM 5.1. *Let $f:[0,\infty)\to\mathbb{R}_+$ is bounded and continuous. Then for any fixed point x_0 on the interval $[0,\infty)$, we have the following order of approximation for the operators (2) to the function f by means of the modulus of continuity:*

$$\left|F_n^{(M)}(f)(x_0)-f(x_0)\right|\leq\left(1+4x_0^{\frac{1}{\alpha}}\right)\omega\left(f;\frac{1}{n^{1-\frac{1}{\alpha}}}\right),$$

for all $n\in\mathbb{N}$, $x_0\in[0,\infty)$, where $\omega(f;\delta)$ is the classical modulus of continuity defined by (1) and $\alpha=2,3,\ldots$.

Proof. Since nonlinear max-product Favard-Szász-Mirakjan operator satisfies the conditions in Corollary 2.4, for any $x_0\in[0,\infty)$, using the properties of $\omega(f;\delta)$, we get

$$\left|F_n^{(M)}(f)(x_0)-f(x_0)\right|\leq\left[1+\frac{1}{\delta_n}F_n^{(M)}(\varphi_{x_0})(x_0)\right]\omega(f,\delta), \tag{5}$$

where $\varphi_{x_0}(t)=|t-x_0|$. At this point let us denote

$$E_n(x_0):=F_n^{(M)}(\varphi_{x_0})(x_0)=\frac{\bigvee_{k=0}^{\infty}\frac{(nx_0)^k}{k!}\left|\frac{k}{n}-x_0\right|}{\bigvee_{k=0}^{\infty}\frac{(nx_0)^k}{k!}},\ x_0\in[0,\infty).$$

Let $x_0\in\left[\frac{j}{n},\frac{j+1}{n}\right]$, where $j\in\{0,1,\ldots,\}$ is fixed, arbitrary. By Lemma 4.2 we easily obtain

$$E_n(x_0)=\max_{k=0,1,\ldots,}\{M_{k,n,j}(x_0)\},\ x_0\in\left[\frac{j}{n},\frac{j+1}{n}\right].$$

Firstly let's check for $j=0$, where $x_0\in\left[0,\frac{1}{n}\right]$ and $\alpha=2,3,\ldots$.

Since $s_{n,0}(x_0)=1$, then we obtain

$$M_{k,n,0}(x_0)=\frac{(nx_0)^k}{k!}\left|\frac{k}{n}-x_0\right|.$$

For $k=0$, we get $$M_{0,n,0}(x_0)=x_0=x_0^{\frac{1}{\alpha}}x_0^{1-\frac{1}{\alpha}}\leq\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}.$$

Now, for any $k\geq 1$, we have

$$M_{k,n,0}(x_0)\leq\frac{(nx_0)^k}{k!}\frac{k}{n}=\frac{n^{k-1}x_0^{\frac{1}{\alpha}}x_0^{k-\frac{1}{\alpha}}}{(k-1)!}\leq\frac{n^{k-1}x_0^{\frac{1}{\alpha}}}{n^{k-\frac{1}{\alpha}}(k-1)!}\leq\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}.$$

So, we find an upper estimate for any $k=0,1,\ldots$, $E_n(x_0)\leq\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}$ when $j=0$.

As a result it remains to find an upper estimate for each $M_{k,n,j}(x_0)$ when $j=1,2,\ldots$, is fixed, $x_0\in\left[\frac{j}{n},\frac{j+1}{n}\right]$, $k\in\{0,1,\ldots,\}$ and $\alpha=2,3,\ldots$.

Indeed, we will demonstrate it

$$M_{k,n,j}(x_0)\leq 4\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}} \tag{6}$$

for all $x_0 \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$, $k = 0, 1, \ldots,$ which directly will implies that

$$E_n(x_0) \le 4\,\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}},\ \text{for all } x_0 \in [0,\infty),\ n \in \mathbb{N}$$

and taking $\delta_n = \frac{1}{n^{1-\frac{1}{\alpha}}}$ in (5) we have the estimate in the statement immediately.

So, in order to complete the proof of (6), we consider the following cases.
Case 1) If $k = j$ then since $x_0 \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ we get

$$M_{j,n,j}(x_0) = \left|\frac{j}{n} - x_0\right| \le \left|\frac{j}{n} - \frac{j+1}{n}\right| = \frac{1}{n}.$$

Now, since $j \ge 1$ we have $x_0 \ge \frac{1}{n}$, which implies $\frac{1}{n} = \frac{1}{n^{\frac{1}{\alpha}}}\frac{1}{n^{1-\frac{1}{\alpha}}} \le \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}$.

Case 2) Let $k \ge j+1$.

Subcase a) Firstly, let's assume that $k - (k+1)^{\frac{1}{\alpha}} < j$. From Lemma 4.1 we have

$$M_{k,n,j}(x_0) = m_{k,n,j}(x_0)\left(\frac{k}{n} - x_0\right) \le \frac{k}{n} - x_0 \le \frac{k}{n} - \frac{j}{n}$$
$$\le \frac{k}{n} - \frac{k-(k+1)^{\frac{1}{\alpha}}}{n} = \frac{(k+1)^{\frac{1}{\alpha}}}{n}.$$

But we certainly have $k \le 3j$. In fact, if we assume that $k > 3j$, since $g'(x_0) = 1 - \frac{1}{\alpha(x_0+1)^{1-\frac{1}{\alpha}}} > 0$, then the function $g(x_0) = x_0 - (x_0+1)^{\frac{1}{\alpha}}$ is nondecreasing. It follows that we have$j > k - (k+1)^{\frac{1}{\alpha}} \ge 3j - (3j+1)^{\frac{1}{\alpha}}$. So we get$j > 3j - (3j+1)^{\frac{1}{\alpha}}$ which is also a contradiction.

In conclusion, taking into consideration $\left(\frac{j}{n}\right)^{\frac{1}{\alpha}} \le x_0^{\frac{1}{\alpha}}$ and $j \ge 1$, we get

$$M_{k,n,j}(x_0) \le \frac{(k+1)^{\frac{1}{\alpha}}}{n} \le \frac{(3j+1)^{\frac{1}{\alpha}}}{n} \le \frac{(4j)^{\frac{1}{\alpha}}}{n}$$
$$= \frac{(4j)^{\frac{1}{\alpha}}}{n^{\frac{1}{\alpha}}\,n^{1-\frac{1}{\alpha}}} = \frac{4^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}\left(\frac{j}{n}\right)^{\frac{1}{\alpha}} \le 4^{\frac{1}{\alpha}}\,\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}.$$

Subcase b) Assume now that $k - (k+1)^{\frac{1}{\alpha}} \ge j$. For the function $g(x_0) := x_0 - (x_0+1)^{\frac{1}{\alpha}}$, we have $g'(x_0) = 1 - \left(1/\alpha(x_0+1)^{1-\frac{1}{\alpha}}\right) > 0$.

Thus we can say that the function $g(x_0)$ is nondecreasing on the interval $[0,\infty)$, it follows that there exists a maximum value $\bar{k} \in \{1, 2, \ldots,\}$ satisfying the inequality $\bar{k} - \left(\bar{k}+1\right)^{\frac{1}{\alpha}} < j$. Then, for $k_1 = \bar{k}+1$, we have $k_1 - (k_1+1)^{\frac{1}{\alpha}} \ge j$ and

$$M_{\bar{k}+1,n,j}(x_0) = m_{\bar{k}+1,n,j}(x_0)\left(\frac{\bar{k}+1}{n} - x_0\right) \le \frac{\bar{k}+1}{n} - x_0$$
$$\le \frac{\bar{k}+1}{n} - \frac{j}{n} \le \frac{\bar{k}+1}{n} - \frac{\bar{k} - \left(\bar{k}+1\right)^{\frac{1}{\alpha}}}{n}$$

$$= \frac{\left(\bar{k}+1\right)^{\frac{1}{\alpha}}+1}{n} \leq \frac{(3j+1)^{\frac{1}{\alpha}}+1}{n} \leq \frac{(5j)^{\frac{1}{\alpha}}}{n} \leq 5^{\frac{1}{\alpha}} \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}.$$

In the above inequality, taking into consideration that $\bar{k}-\left(\bar{k}+1\right)^{\frac{1}{\alpha}} < j$ certainly implies $\bar{k} \leq 3j$ (see the similar reasonings in the previously mentioned subcase (a)). Also, we have $k_1 \geq j+1$. In fact, this results from the fact that g is nondecreasing, and since $g(j) = j-(j+1)^{\frac{1}{\alpha}} < j$ is obvious. By Lemma 4.3 (i) it follows that $M_{\bar{k}+1,n,j}(x_0) \geq M_{\bar{k}+2,n,j}(x_0) \geq \ldots$. We thereby obtain $M_{k,n,j}(x_0) \leq 5^{\frac{1}{\alpha}} \frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}$ for any $k \in \left\{\bar{k}+1, \bar{k}+2, \ldots,\right\}$.

Case 3) Let $k \leq j-1$.

Subcase a) Assume first that $k+k^{\frac{1}{\alpha}} > j$. Then we obtain

$$M_{k,n,j}(x_0) = m_{k,n,j}(x_0)\left(x_0-\frac{k}{n}\right) \leq \frac{j+1}{n}-\frac{k}{n} \leq \frac{k+k^{\frac{1}{\alpha}}+1}{n}-\frac{k}{n}$$
$$= \frac{k^{\frac{1}{\alpha}}+1}{n} \leq \frac{(j-1)^{\frac{1}{\alpha}}+1}{n} \leq \frac{2\,j^{\frac{1}{\alpha}}}{n^{\frac{1}{\alpha}}\,n^{1-\frac{1}{\alpha}}} \leq 2\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}$$

taking into consideration that $(j-1)^{\frac{1}{\alpha}}+1 \leq j^{\frac{1}{\alpha}}+j^{\frac{1}{\alpha}}$ and $\left(\frac{j}{n}\right)^{\frac{1}{\alpha}} \leq x_0^{\frac{1}{\alpha}}$.

Subcase b) Assume now that $k+k^{\frac{1}{\alpha}} \leq j$. Let $\tilde{k} \in \{0,1,\ldots,\}$ be the minimum value such that $\tilde{k}+\left(\tilde{k}\right)^{\frac{1}{\alpha}} > j$. Then $k_2 = \tilde{k}-1$ satisfies $k_2+(k_2)^{\frac{1}{\alpha}} \leq j$ and

$$M_{\tilde{k}-1,n,j}(x_0) = m_{\tilde{k}-1,n,j}(x_0)\left(x_0-\frac{\tilde{k}-1}{n}\right) \leq \frac{j+1}{n}-\frac{\tilde{k}-1}{n} \leq \frac{\tilde{k}+\left(\tilde{k}\right)^{\frac{1}{\alpha}}+1}{n}-\frac{\tilde{k}-1}{n}$$
$$= \frac{\left(\tilde{k}\right)^{\frac{1}{\alpha}}+2}{n} \leq \frac{(j+1)^{\frac{1}{\alpha}}+2}{n} \leq \frac{4}{n^{1-\frac{1}{\alpha}}}\left(\frac{j}{n}\right)^{\frac{1}{\alpha}} \leq 4\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}.$$

For this final inequality, we used the self-evident relationship $\tilde{k}-1 = k_2 \leq k_2+(k_2)^{\frac{1}{\alpha}} \leq j$, which implies $\tilde{k} \leq j+1$ and $\left(\tilde{k}\right)^{\frac{1}{\alpha}}+2 \leq (j+1)^{\frac{1}{\alpha}}+2 \leq 4j^{\frac{1}{\alpha}}$. And, since $j \geq 1$, it is clear that $k_2 \leq j-1$.

By Lemma 4.3 (ii) it follows that $M_{\tilde{k}-1,n,j}(x_0) \geq M_{\tilde{k}-2,n,j}(x_0) \geq \ldots \geq M_{0,n,j}(x_0)$. Thus we obtain $M_{k,n,j}(x_0) \leq 4\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}$ for any $k \leq j-1$ and $x_0 \in \left[\frac{j}{n},\frac{j+1}{n}\right]$.

If we use

$$\max\left\{\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, 4^{\frac{1}{\alpha}}\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, 2\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, 5^{\frac{1}{\alpha}}\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, 4\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}\right\} = 4\frac{x_0^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}},$$

then we obtain desired result. □

6. Rate of weighted approximation

We see that the previous results work for a fixed point x_0 or finite intervals. However, if we want to achieve a uniform approximation on infinite intervals, we should use weighted modules of continuity.

Before giving useful properties of this type of continuity moduli, let us recall the following spaces and norms (see [12]):

$B_\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} |$ a constant M_f depending on f exists such that $|f| \leq M_f \rho\}$,

$C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}) |$ f continuous on $\mathbb{R}\}$,

endowed with the norm $\|f\|_\rho = \sup_{0\leq x} \frac{|f(x)|}{\rho(x)}$.

To obtain the rate of weighted approximation of positive linear operators defined on infinite intervals, various weighted modules of continuity are introduced. Some of them contain the term h in the denominator of the supremum expression. In chronological order, we refer to some related papers ss [1, 10, 11, 13, 15].

In [13] the authors have introduced the following weighted module of continuity:

$$\Omega(f;\delta) = \sup_{0\leq x, |h|\leq \delta} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}. \tag{7}$$

And in [10] the following weighted modulus of continuity were defined:

$$\omega_\rho(f;\delta) = \sup_{0\leq x, |h|\leq \delta} \frac{|f(x+h) - f(x)|}{\rho(x+h)} \tag{8}$$

where $\rho(x) \geq \max(1, x)$. In the same paper the author has developed a generalization of the Gadjiev-Ibragimov operators, which includes many well-known operators, and obtains their weighted convergence rate using $\omega_\rho(f;\delta)$, defined in (8).

In [15], Moreno introduced a different type of modulus of continuity in (8) as follows

$$\overline{\Omega}_\alpha(f;\delta) = \sup_{0\leq x, |h|\leq \delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^\alpha}.$$

It is obvious that by choosing $\alpha = 2$, in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we get $\overline{\Omega}_2(f;\delta) = \omega_{\rho_0}(f;\delta)$ for $\rho_0(x) = 1+x^2$.

Moreover, let $C^0_\rho(\mathbb{R})$ be the subspace of all functions in $C_\rho(\mathbb{R})$ such that $\lim_{|x|\to\infty} \frac{f(x)}{\rho(x)}$ exists finitely.

In the light of these definitions, we can give the following theorem.

THEOREM 6.1. *Let $f : [0,\infty) \to \mathbb{R}_+$ be continuous. Then for all $x \in [0,\infty)$, we have the following rate of weighted approximation for the operators* (2) *to the function f by means of the weighted modulus of continuity defined in* (8)*. Then for each $f \in C^0_{\rho_0}(\mathbb{R}_+)$, we have*

$$\frac{\left|F_n^{(M)}(f)(x) - f(x)\right|}{(\rho_0(x))^2} \leq \frac{(1+9x^2)(1+4x^{\frac{1}{\alpha}})}{(1+x^2)^2}\, \omega_{\rho_0}\left(f; \frac{1}{n^{1-\frac{1}{\alpha}}}\right), \tag{9}$$

for all $n \in \mathbb{N}$, *where* $\omega_{\rho_0}(f;\delta)$ *is the weighted modulus of continuity defined by* (8), $\delta > 0$, $\rho_0(x) = 1 + x^2$ *and* $\alpha = 2, 3, \dots$.

Proof. By using the properties of $\omega_{\rho_0}(f;\delta)$,(see [15]), we can write

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \le \left(1 + (2x + F_n^{(M)}(e_1)(x))^2\right)\left(\frac{1}{\delta}F_n^{(M)}(\varphi_x)(x) + 1\right)\omega_{\rho_0}(f;\delta) \quad (10)$$

where φ_x was introduced at Corollary 2.3 and $\omega_{\rho_0}(f;\delta)$ is the weighted modulus of continuity defined by (8) and $\delta > 0$.

From the proof of Theorem 5.1, we have

$$E_n(x) \le 4\frac{x^{\frac{1}{\alpha}}}{n^{1-\frac{1}{\alpha}}}, \text{ for all } n \in \mathbb{N}. \quad (11)$$

On the other hand, after simple calculations, we have

$$F_n^{(M)}(e_1)(x) = \frac{x \bigvee_{k=1}^{\infty} x^{k-1}\frac{n^{k-1}}{(k-1)!}}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} = \frac{x \bigvee_{k=0}^{\infty} x^k \frac{n^k}{k!}}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} = x. \quad (12)$$

So, using the inequalities (11) and (12) in (10) and by choosing $\delta = \frac{1}{n^{1-\frac{1}{\alpha}}}$, the proof is completed. □

This theorem allows us to express the following weighted approximation result.

Theorem 6.2. *Let* $f : [0,\infty) \to \mathbb{R}_+$ *be continuous. Then for all* $x \in [0,\infty)$, *we have the following rate of weighted approximation for the operators* (2) *to the function* f *by means of the weighted modulus of continuity defined in* (8). *Then for each* $f \in C^0_{\rho_0}(\mathbb{R}_+)$, *we have*

$$\left\|F_n^{(M)}(f)(x) - f(x)\right\|_{\rho_0^2(x)} \le 50\ \omega_{\rho_0}\left(f;\frac{1}{n^{1-\frac{1}{\alpha}}}\right),$$

for all $n \in \mathbb{N}$, *where* $\omega_{\rho_0}(f;\delta)$ *is the weighted modulus of continuity defined by (8),* $\delta > 0$, $\rho_0(x) = 1 + x^2$ *and* $\alpha = 2, 3, \dots$.

Proof. By using the inequalities $\frac{1}{1+x^2} \le 1$, $\frac{x^2}{1+x^2} \le 1$ and $\frac{x^{\frac{1}{\alpha}}}{1+x^2} \le 1$, we have

$$\frac{(1+9x^2)(1+4x^{\frac{1}{\alpha}})}{(1+x^2)^2} \le 50. \quad (13)$$

If we use (13) in (9), we obtain desired result. □

Remark 6.3. Theorem 5.1, Theorem 6.1 and Theorem 6.2 thus show that the orders of the pointwise approximation, the weighted approximation andthe weighted uniform approximation are $1/n^{1-\frac{1}{\alpha}}$. For sufficiently large α , $1/n^{1-\frac{1}{\alpha}}$ inclines to $1/n$. As a result, since $1 - \frac{1}{\alpha} \ge \frac{1}{2}$ for $\alpha = 2, 3, \dots$, this choice of α improves the order of approximation. Thus, these results, including the classical and weighted modulus, of continuities show that a better order of approximation can be obtained.

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