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ON WARPED-TWISTED PRODUCT MANIFOLDS WITH GRADIENT SOLITONS

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Abstract. In this paper, we give new characterizations for warped-twisted product manifolds. We study gradient Riemann, gradient Ricci, gradient Yamabe solitons and quasi-Einstein case on warped-twisted product manifolds and we investigate the effect of a gradient soliton and quasi-Einstein case on such manifolds to their factor manifolds. We also get some results when the factor manifolds of the warped-twisted product manifolds are compact, the twisting and the warping functions are harmonic.

1. Introduction

The notion of doubly twisted product [17] is a natural generalization of the notion of doubly warped product [11], warped-twisted product [18], twisted product [12], warped product [2] and direct product. The notion of warped product of pseudo-Riemannian manifolds was defined by Bishop and O'Neill in [2] in order to construct a large class of complete manifolds with negative curvature. Warped products are intensively studied in mathematical physics as well as in differential geometry, in particular in the theory of relativity. In fact, the standard spacetime models such as Robertson-Walker, Schwarschild, static and Kruskal are warped products. Even the simplest models of the neighborhoods of stars and black holes are warped products.

The theory of Ricci solitons is a rich area of research in differential geometry. Ricci solitons are closely related to Einstein and quasi-Einstein manifolds [6], and they are related to the warped product manifolds.

Recently, various geometric properties of Riemann [15], Ricci [14] and Yamabe solitons [7] have been studied. Moreover, gradient solitons such as gradient Riemann, gradient Ricci, gradient Yamabe were considered on twisted product and doubly warped product manifolds [4,12]. In addition, De et al. [10] studied Ricci solitons on warped product manifolds admitting either a conformal vector field or a concurrent vector

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field. Also Blaga [3] gave a way to construct a gradient η-Ricci soliton on a warped product manifold when the base manifold is oriented, compact and of constant scalar curvature.

In this paper, we review the basic background of warped-twisted product manifolds and the definitions of gradient solitons in Section 2. In Section 3, we investigate the effect of a gradient Riemann, gradient Ricci and gradient Yamabe soliton on a warped-twisted product manifold on the factor manifolds and we also study quasi-Einstein warped-twisted product manifolds. We establish a general inequality when the factor manifolds of the warped-twisted product manifolds are compact, the twisting and the warping functions are harmonic.

2. Preliminaries

2.1 Warped-twisted products

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $f_2 : M_2 \to (0, \infty)$ and $f_1 : M_1 \times M_2 \to (0, \infty)$ be smooth functions. Then, the *warped-twisted product* manifold ${}_{f_2}M_1 \times_{f_1} M_2$ [18] is the product manifold $M_1 \times M_2$ equipped with the metric tensor g defined by

$$g := (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + f_1^2 \pi_2^*(g_2), \tag{1}$$

where $\pi_i : M_1 \times M_2 \to M_i$ is the canonical projection, for $i = 1, 2$. The function f_2 is called a *warping function* and the function f_1 is called a *twisting function* of the warped-twisted product manifold.

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold with the Levi−Civita connection ∇, i.e., ∇ is calculated with respect to the metric g given in (1). Also denote by ∇^i the Levi−Civita connection of M_i, for $i \in \{1, 2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathcal{L}(M_i)$ and we use the same notation for a vector field and for its lift. On the other hand, each π_i is a positive homothety, so it preserves the Levi−Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas of the warped-twisted product manifold ${}_{f_2}M_1 \times_{f_1} M_2$ with the warping function f_2 and twisting function f_1 are given by

$$\nabla_X Y = \nabla^1_X Y - g(X, Y)\nabla \ln(f_2 \circ \pi_2), \tag{2}$$

$$\nabla_V X = \nabla_X V = V(\ln(f_2 \circ \pi_2))X + X(\ln f_1)V, \tag{3}$$

$$\nabla_U V = \nabla^2_U V + U(\ln f_1)V + V(\ln f_1)U - g(U, V)\nabla \ln f_1. \tag{4}$$

for any $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$.

Remark 2.1. From now on, we will put $k = \ln f_1$ (resp. $l = \ln f_2$) and use the same symbol for the function k (resp. l) and its pullback $k \circ \pi_1$ (resp. $l \circ \pi_2$).

Now, let ψ be a smooth function on a warped-twisted product ${}_{f_2}M_1 \times_{f_1} M_2$. Then, for any $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, by the definition of the Hessian tensor and by using (2) and (3), we have

$$h^{\psi}(X,U) = XU(\psi) - X(k)U(\psi) - X(\psi)U(l). \tag{5}$$

Next, we define $h_1^{\psi}(X,Y):=XY(\psi)-(\nabla^1_X Y)(\psi)$, for all $X,Y \in \mathfrak{L}(M_1)$ and $h_2^{\psi}(U,V):= UV(\psi) - (\nabla^2_U V)(\psi)$, for all $U,V \in \mathfrak{L}(M_2)$. By using (2) and (4), the Hessian tensor h^{ψ} of ψ satisfies

$$h^{\psi}(X,Y) = h_1^{\psi}(X,Y) + g(X,Y)g(\nabla l, \nabla\psi) \tag{6}$$

and
$$h^{\psi}(U,V) = h_2^{\psi}(U,V) - U(k)V(\psi) - V(k)U(\psi) + g(U,V)g(\nabla k, \nabla\psi). \tag{7}$$

Since $\nabla l \in \mathfrak{L}(M_2)$, from (6) and (7) we deduce that

$$h^{k}(X,Y) = h_1^{k}(X,Y) + g(X,Y)g(\nabla l, \nabla k), \tag{8}$$

$$h^{l}(X,Y) = g(X,Y)g(\nabla l, \nabla l), \tag{9}$$

$$h^{k}(U,V) = h_2^{k}(U,V) - U(k)V(k) - V(k)U(k) + g(U,V)g(\nabla k, \nabla k), \tag{10}$$

$$h^{l}(U,V) = h_2^{l}(U,V) - U(k)V(l) - V(k)U(l) + g(U,V)g(\nabla k, \nabla l). \tag{11}$$

Let ${}^1R$ and ${}^2R$ be the lifts of the Riemann curvature tensors of (M_1,g_1) and (M_2,g_2) respectively, and let R be the Riemann curvature tensor of the warped-twisted product ${}_{f_2}M_1\times_{f_1}M_2$. Then, by a direct computation and by using (2)–(4), we have the following relations.

Lemma 2.2. *Let $X,Y,Z \in \mathfrak{L}(M_1)$ and $U,V,W \in \mathfrak{L}(M_2)$. Then, we have*

$$R_{XY}Z = {}^1R_{XY}Z + H^l(Y)g(X,Z) - H^l(X)g(Y,Z), \tag{12}$$

$$R_{XY}U = U(l)\Big\{Y(k)X - X(k)Y\Big\}, \tag{13}$$

$$R_{UV}X = X(k)\Big\{V(l)U - U(l)V\Big\} + UX(k)V - VX(k)U, \tag{14}$$

$$R_{XU}Y = \Big\{h_1^k(X,Y) + X(k)Y(k)\Big\}U + Y(k)U(l)X + g(X,Y)\Big\{H^l(U) + U(l)\nabla l\Big\}, \tag{15}$$

$$\begin{aligned} R_{UX}V = {}& \Big\{h_2^l(U,V) + U(l)V(l) - U(l)V(k) - U(k)V(l)\Big\}X \\ &+ \Big\{V(l)X(k) - XV(k)\Big\}U + \Big\{X(k)\nabla k + H^k(X)\Big\}g(U,V), \end{aligned} \tag{16}$$

$$\begin{aligned} R_{UV}W = {}& {}^2R_{UV}W - \Big\{h_2^k(V,W) - W(k)V(k)\Big\}U + \Big\{h_2^k(U,W) - W(k)U(k)\Big\}V \\ &- \Big\{H^k(U) + U(k)\nabla k\Big\}g(V,W) + \Big\{H^k(V) + V(k)\nabla k\Big\}g(U,W), \end{aligned} \tag{17}$$

where H^f is the Hessian tensor of a smooth function f on ${}_{f_2}M_1\times_{f_1}M_2$, i.e., $H^f(E) := \nabla_E \nabla f$, for any vector field E on ${}_{f_2}M_1\times_{f_1}M_2$.

Let ${}^1\mathrm{Ric}$ and ${}^2\mathrm{Ric}$ be the lifts of the Ricci curvature tensors of (M_1,g_1) and (M_2,g_2) respectively, and let Ric be the Ricci curvature tensor of the warped-twisted product ${}_{f_2}M_1\times_{f_1}M_2$. Then, by a direct computation and by using (12)–(17) and (9)–(10), we have the following relations.

LEMMA 2.3. *Let $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. Then, we have*

$$\begin{aligned} \mathrm{Ric}(X,Y) &= {}^1\mathrm{Ric}(X,Y) + h^l(X,Y) - m_2\Big\{h_1^k(X,Y) + X(k)Y(k)\Big\} \\ &\quad - g(X,Y)\Big\{\Delta l + g(\nabla l, \nabla l)\Big\}, \end{aligned} \tag{18}$$

$$\mathrm{Ric}(X,U) = (1-m_2)XU(k) + (m_1+m_2-2)X(k)U(l), \tag{19}$$

$$\begin{aligned} \mathrm{Ric}(U,V) &= {}^2\mathrm{Ric}(U,V) + h^k(U,V) + (1-m_2)h_2^k(U,V) + m_2U(k)V(k) \\ &\quad - g(U,V)\Big\{\Delta k + g(\nabla k, \nabla k)\Big\} \\ &\quad - m_1\Big\{h_2^l(U,V) + U(l)V(l) - U(l)V(k) - U(k)V(l)\Big\}, \end{aligned} \tag{20}$$

where Δ is the Laplacian operator on ${}_{f_2}M_1 \times_{f_1} M_2$ and $m_i = \dim(M_i)$, for $i \in \{1,2\}$.

REMARK 2.4. Let $\{e_1, \ldots, e_{m_1}, \omega_1, \ldots, \omega_{m_2}\}$ be an orthonormal basis of the warped-twisted product ${}_{f_2}M_1 \times_{f_1} M_2$, where $\{e_1, \ldots, e_{m_1}\}$ are tangent to M_1 and $\{\omega_1, \ldots, \omega_{m_2}\}$ are tangent to M_2. Then by (1), we see that $\{f_2e_1, \ldots, f_2e_{m_1}\}$ is an orthonormal basis of (M_1, g_1) and $\{f_1\omega_1, \ldots, f_1\omega_{m_2}\}$ is an orthonormal basis of (M_2, g_2).

Let ${}^1\tau$ and ${}^2\tau$ be the lifts of the scalar curvatures of (M_1, g_1) and (M_2, g_2) respectively, and let τ be the scalar curvature of the warped-twisted product ${}_{f_2}M_1 \times_{f_1} M_2$. Then, by Lemma 2.3 and Remark 2.4, we obtain

$$\begin{aligned} \tau &= \frac{\tau^1}{f_2^2} + \frac{\tau^2}{f_1^2} + \tilde{\Delta}_1 l + \tilde{\Delta}_2 k - \frac{m_2}{f_2^2}\Delta_1 k - \frac{m_1}{f_1^2}\Delta_2 l + \frac{(1-m_2)}{{f_1}^2}\Delta_2 k - m_2 g(P_1\nabla k, P_1\nabla k) - m_1\Delta l \\ &\quad -2m_1 g(\nabla l, \nabla l) - m_2\Big\{\Delta k + g(\nabla k, \nabla k)\Big\} + m_2 g(P_2\nabla k, P_2\nabla k) + 2m_1 g(P_2\nabla k, \nabla l), \end{aligned} \tag{21}$$

where Δ_i is the lift of the Laplacian operator on (M_i, g_i), $m_i = \dim(M_i)$, for $i \in \{1,2\}$:

$$\tilde{\Delta}_1(k) = \sum_{i=1}^{m_1} h^k(e_i, e_i), \quad \tilde{\Delta}_2(k) = \sum_{i=m_1+1}^{m_1+m_2} h^k(e_i, e_i), \quad \Delta k = \tilde{\Delta}_1(k) + \tilde{\Delta}_2(k)$$

and $\nabla k = P_1\nabla k + P_2\nabla k$, with $P_i : \mathcal{L}(M_1 \times M_2) \to \mathcal{L}(M_i)$.

2.2 Gradient Yamabe, gradient Ricci and gradient Riemann solitons

A semi-Riemannian manifold (M^m, g) is said to be a *gradient Yamabe soliton* [7] if there exists a smooth function ψ on M and a real constant λ satisfying

$$h^\psi = (\tau - \lambda)g, \tag{22}$$

where τ is the scalar curvature of (M, g). More generally, if

$$h^\psi = \gamma g \tag{23}$$

holds for some smooth function γ, then the triple (M, g, ψ) is called *conformal gradient soliton*.

A semi-Riemannian manifold (M^m, g) is said to be a *gradient Ricci soliton* [14] if there exists a smooth function ψ on M and a real constant λ satisfying

$$h^{\psi} + \mathrm{Ric} = \lambda g, \tag{24}$$

where Ric is the Ricci curvature of (M, g).

A semi-Riemannian manifold (M^m, g) is said to be a *gradient Riemann soliton* [15] if there exists a smooth function ψ on M and a real constant λ satisfying

$$h^{\psi} \wedge g + R = \lambda G, \tag{25}$$

where R is the Riemann curvature of (M, g), $G = \frac{1}{2}(g \wedge g)$ and $\wedge$ is the Kulkarni-Nomizu product. Then, for any vector field X, Y, Z, W on M, equation (25) is explicitly expressed as

$$\begin{aligned} R(X,Y,Z,W) &+ g(X,W)h^{\psi}(Y,Z) + g(Y,Z)h^{\psi}(X,W) - g(X,Z)h^{\psi}(Y,W) \\ &- g(Y,W)h^{\psi}(X,Z) = \lambda\big\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\big\}, \end{aligned}$$

which by contraction over X and W, gives

$$h^{\psi} + \frac{1}{m-2}\,\mathrm{Ric} = \frac{(m-1)\lambda - \Delta\psi}{m-2} g, \tag{26}$$

provided $m \geq 3$. If $m = 2$, then $\mathrm{Ric} = (\lambda - \Delta\psi)g$.

Generalizing the notions of Yamabe and Ricci soliton, we talk about η-Yamabe and η-Ricci solitons.

A semi-Riemannian manifold (M^m, g) is said to be a *gradient η-Yamabe soliton* [5] if there exist a smooth function ψ on M and two real constants λ and μ satisfying $h^{\psi} = (\tau - \lambda)g + \mu\eta \otimes \eta$, where τ is the scalar curvature of (M, g) and η is a 1-form on M.

A semi-Riemannian manifold (M^m, g) is said to be a *gradient η-Ricci soliton* [9] if there exist a smooth function ψ on M and two real constants λ and μ satisfying $h^{\psi} + \mathrm{Ric} = \lambda g + \mu\eta \otimes \eta$, where Ric is the Ricci curvature of (M, g) and η is a 1-form on M.

In all of the above cases, if $\lambda > 0, \lambda < 0$ or $\lambda = 0$, then the soliton is called a shrinking, expanding or steady, respectively. If λ is allowed be a smooth function on M, then (M, g, ψ) is called a *gradient almost Yamabe*, a *gradient almost Ricci*, a *gradient almost Riemann*, a *gradient almost η-Yamabe* and a *gradient almost η-Ricci soliton*, respectively.

Another generalization for the notion of Ricci soliton was introduced in [13].

A semi-Riemannian manifold (M^m, g) is said to be a *gradient f-almost Ricci soliton* [13] if there exist smooth functions f, ψ and λ on M satisfying $fh^{\psi} + \mathrm{Ric} = \lambda g$.

We generalize this notion to gradient f-almost η-Ricci soliton as follows.

A semi-Riemannian manifold (M^m, g) is said to be a *gradient f-almost η-Ricci soliton* if there exist smooth functions f, ψ, λ and μ on M satisfying $fh^{\psi} + \mathrm{Ric} = \lambda g + \mu\eta \otimes \eta$, where Ric is the Ricci curvature of (M, g) and η is a 1-form on M.

Finally, a Riemannian manifold (M^m, g), $m \geq 2$, is said to be an *Einstein manifold* [1] if its Ricci tensor Ric satisfies the condition $\mathrm{Ric} = \frac{\tau}{m} g$, where τ denotes the scalar curvature of (M, g). A non-flat Riemannian manifold (M, g), $m \geq 2$, is said to be a *quasi-Einstein* [6] if the condition

$$\mathrm{Ric} = \alpha g + \beta A \otimes A \tag{27}$$

is fulfilled on M, where α and β are scalar functions on M with $\beta \neq 0$ and A is non-zero 1-form such that $g(X, \xi) = A(X)$, for every vector field X on M, ξ is a unit vector field which is called the generator of the manifold M.

3. Main results

We first give some characterizations for warped-twisted product manifolds.

Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold. Then, by using (8)–(11) and Remark 2.4, we have

$$\Delta k = \frac{1}{f_2^2}\Delta_1 k + \frac{1}{f_1^2}\Delta_2 k + m_1 g(\nabla l, \nabla k) + m_2 f_2^2 g_1(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k) \tag{28}$$

and

$$\Delta l = \frac{1}{f_1^2}\Delta_2 l + m_1 f_1^2 g_2(\nabla l, \nabla l) - 2g(P_2\nabla k, \nabla l) + m_2 g(\nabla k, \nabla l). \tag{29}$$

Now, we suppose that the first factor manifold M_1 is compact and the twisting function k is harmonic with respect to the Laplacian Δ. Then from (28) follows

$$\Delta_1 k = f_2^2\left\{ -\frac{1}{f_1^2}\Delta_2 k - m_1 g(\nabla l, \nabla k) - m_2 f_2^2 g_1(\nabla k, \nabla k) + 2f_2^2 g_1(P_2\nabla k, \nabla l)\right\}.$$

If $\Delta_1 k$ has constant sign, for example, if $\Delta_1 k \leq 0$ on M_1, then, by Hopf's lemma, we conclude that f_1 is a constant function (say $f_1 = c_1 = \mathit{constant}$) on M_1, since M_1 is compact. Hence, we can write $g = f_2^2 g_1 \oplus \tilde{g}_2$, where $\tilde{g}_2 = c_1^2 g_2$. Namely, ${}_{f_2}M_1 \times_{f_1} M_2$ can be expressed as a warped product ${}_{f_2}M_1 \times M_2$ with the warping function f_2, where the metric tensor of M_2 is $\tilde{g}_2$ given above. Therefore, we can state the following theorem.

THEOREM 3.1. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold with the first factor manifold M_1 compact. If the twisting function k is harmonic with respect to the Laplacian Δ, then the manifold is a warped product of the form ${}_{f_2}M_1 \times M_2$ if and only if*

$$\frac{1}{f_1^2}\Delta_2 k + m_1 g(\nabla l, \nabla k) + m_2 f_2^2 g_1(\nabla k, \nabla k) \geq 2g(P_2\nabla k, P_2\nabla k). \tag{30}$$

By using (29), we can prove the following result in a similar way.

THEOREM 3.2. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold with the second factor manifold M_2 compact. If the warping function l is harmonic with respect to the Laplacian Δ, then the manifold is a twisted product of the form $M_1 \times_{f_1} M_2$ if and only if*

$$m_1 {f_1}^2 g_2(\nabla l, \nabla l) + m_2 g(\nabla k, \nabla l) \geq 2g(P_2\nabla k, \nabla l). \tag{31}$$

From Theorem 3.1 and Theorem 3.2, we get the following result.

THEOREM 3.3. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a compact warped-twisted product manifold. If the twisting function k and warping function l are harmonic with respect to the Laplacian Δ, then the manifold is a usual product of the form $M_1 \times M_2$ if and only if* (30) *and* (31) *hold.*

Next, we characterize gradient Yamabe solitons on warped-twisted product.

THEOREM 3.4. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold. If it is a gradient Yamabe soliton whose potential function ψ depends only on the points of M_1, then the manifold is a twisted product of the form $M_1 \times_{f_1} M_2$.*

Proof. Under the given conditions in the hypothesis, for any $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, we have $XU(\psi) - X(k)U(\psi) - X(\psi)U(l) = 0$, from (5) and (22). We know $U(\psi) = 0$, since ψ only depends on the points of M_1, so, we obtain $X(\psi)U(l) = 0$. Hence, we get $U(l) = 0$, for all $U \in \mathfrak{L}(M_2)$. Then, we find $l = const$, so $f_2 = c_2$ for some constant c_2. Thus, we can write $g = \tilde{g}_1 \oplus f_1^2 g_2$, where $\tilde{g}_1 = c_2^2 g_1$, that is, ${}_{f_2}M_1 \times_{f_1} M_2$ can be expressed as a twisted product $M_1 \times_{f_1} M_2$ with the twisting function f_1, where the metric tensor of M_1 is $\tilde{g}_1$ given above. □

Now, we shall investigate the geometry of the factor manifolds of a gradient Yamabe soliton with warped-twisted product structure.

THEOREM 3.5. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold. Then it is a gradient Yamabe soliton with the potential function ψ if and only if the following statements hold:*

(a) (M_1, g_1, ψ_1) is a gradient almost Yamabe soliton;

(b) $h_2^{\psi_2}(U, V) = ({}^2\tau - \lambda_2)g_2(U, V) + dk(U)d\psi(V) + dk(V)d\psi(U)$ for $U, V \in \mathcal{L}(M_2)$;

(c) $XU(\psi) = X(k)U(\psi) + X(\psi)U(l)$ for $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, where $\psi_i = \psi|_{M_i}$, for $i \in \{1, 2\}$.

Proof. From (22) we have $h^\psi = (\tau - \lambda)g$. Hence, using (1), (6) and (21), we obtain $h_1^{\psi_1} = ({}^1\tau - \lambda_1)g_1$ on M_1, where

$$\begin{aligned}\lambda_1 = &- f_2^2\bigg\{\frac{{}^2\tau}{f_1^2} + \tilde{\Delta}_1 l + \tilde{\Delta}_2 k - \frac{m_2}{f_2^2}\Delta_1 k - \frac{m_1}{f_1^2}\Delta_2 l + \frac{(1-m_2)}{f_1^2}\Delta_2 k \\ &- m_2 g(P_1\nabla k, P_1\nabla k) - m_1\Delta l - 2m_1 g(\nabla l, \nabla l) - m_2\{\Delta k + g(\nabla k, \nabla k)\} \\ &+ m_2 g(P_2\nabla k, P_2\nabla k) + 2m_1 g(P_2\nabla k, \nabla l) + \lambda + g(\nabla l, \nabla\psi)\bigg\},\end{aligned}$$

which means that (M_1, g_1, ψ_1) is a gradient almost Yamabe soliton, as desired. On the other hand, since $h^\psi = (\tau - \lambda)g$, using (7), we find

$$h_2^{\psi_2}(U, V) = ({}^2\tau - \lambda_2)g_2(U, V) + U(k)V(\psi) + V(k)U(\psi)$$

on M_2 for $U, V \in \mathfrak{L}(M_2)$, where

$$\begin{aligned}\lambda_2 = &- f_1^2\bigg\{\frac{{}^1\tau}{f_2^2} + \tilde{\Delta}_1 l + \tilde{\Delta}_2 k - \frac{m_2}{f_2^2}\Delta_1 k - \frac{m_1}{f_1^2}\Delta_2 l + \frac{(1-m_2)}{f_1^2}\Delta_2 k \\ &- m_2 g(P_1\nabla k, P_1\nabla k) - m_1\Delta l - 2m_1 g(\nabla l, \nabla l) - m_2\{\Delta k + g(\nabla k, \nabla k)\}\end{aligned}$$

$$+ m_2 g(P_2\nabla k, P_2\nabla k) + 2m_1 g(P_2\nabla k, \nabla l) + \lambda + g(\nabla k, \nabla\psi)\Big\}.$$

So, we obtain the assertion (b). For $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, using (5) and (22), we easily get the assertion (c). The converse is just a verification. □

Remark 3.6. The authors gave illustrative examples for gradient almost η-Yamabe soliton and gradient η-Yamabe soliton when the potential vector field is a torse-forming vector field of gradient type in [5, Example 3.16 and Example 3.15].

Theorem 3.7. *A non-trivial warped-twisted product manifold ${}_{f_2}M_1 \times_{f_1} M_2$ does not admit a conformal gradient soliton.*

Proof. Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a non-trivial warped-twisted product manifold. Assume that (g, ψ) is a conformal gradient soliton on ${}_{f_2}M_1 \times_{f_1} M_2$. Then, for any vector fields $\bar{X}, \bar{Y}$ on ${}_{f_2}M_1 \times_{f_1} M_2$, we have

$$h^{\psi}(\bar{X}, \bar{Y}) = \gamma g(\bar{X}, \bar{Y}), \tag{32}$$

from (23), where γ is a smooth function on M. By (32) and [8, Lemma 4.1], we deduce that $\nabla\psi$ is a concircular vector field on (M, g), in which case, it follows that the manifold is locally a warped product of the form $I \times_{\varphi} F$ from [8, Theorem 3.1], where φ is a nowhere vanishing smooth function on an open interval I of the real line and F is an $(m-1)-$dimensional Riemannian manifold, which is a contradiction. □

Theorem 3.8. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold. Then it is a gradient Ricci soliton with potential function ψ and constant λ if and only if the following statements hold:*
(a) $(\nabla^1\varphi_1, \lambda_1, \mu_1)$ defines a gradient almost η-Ricci soliton on (M_1, g_1), where $\varphi_1 = \psi_1 - m_2 k$, $\lambda_1 = f_2^2\{\lambda + \Delta l - g(\nabla l, \nabla\psi)\}$, $\psi_1 = \psi|_{M_1}$, $\eta = \hat{dk}$ is the pullback of dk to M_1 and $\mu_1 = m_2$;

(b) ${}^2\mathrm{Ric}(U,V) + h_2^{\varphi_2}(U,V) = \lambda_2 g_2 + (2-m_2)\tilde{dk}(U)\tilde{dk}(V) + m_1\{dl(U)dl(V) - dl(U)\tilde{dk}(V) - \tilde{dk}(U)dl(V)\} + \tilde{dk}(U)d\psi(V) + \tilde{dk}(V)d\psi(U)$, where $\varphi_2 = \psi_2 - m_1 l - (m_2-2)k$, $\lambda_2 = f_1^2\{\lambda + \Delta k - g(\nabla k, \nabla\psi\}$, $\psi_2 = \psi|_{M_2}$ and $\tilde{dk}$ is the pullback of dk to M_2;

(c) $XU(\psi) - X(\psi)U(l) - X(k)U(\psi) = -(1-m_2)XU(k) - (m_1+m_2-2)X(k)U(l)$, for $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$.

Proof. Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a gradient Ricci soliton with the potential function ψ and constant λ. Then, we have $\mathrm{Ric} + h^{\psi} = \lambda g$, from (24). Hence, using (6) and (18), we obtain ${}^1\mathrm{Ric} + h_1^{\psi_1} = \lambda_1 g_1 + m_2 h_1^k + m_2 dk \otimes dk$ on M_1, where $\lambda_1 = f_2^2\{\lambda + \Delta l - g(\nabla l, \nabla\psi)\}$ and $\psi_1 = \psi|_{M_1}$.

By direct computations, we get ${}^1\mathrm{Ric} + h_1^{\varphi_1} = \lambda_1 g_1 + m_2 dk \otimes dk$, where $\varphi_1 = \psi_1 - m_2 k$. Thus, $(\nabla^1\varphi_1, \lambda_1, \mu_1)$ defines a gradient almost η-Ricci soliton on (M_1, g_1) as desired, with $\eta = \hat{dk}$ which is the pullback of dk to M_1. On the other hand, using (7) and (20), we find

$$\begin{aligned}{}^2\mathrm{Ric}(U,V) + h_2^{\psi_2}(U,V) &= \lambda_2 g_2(U,V) + m_1 h_2^l(U,V) + (m_2-2)h_2^k(U,V) \\ &+ (2-m_2)dk(U)dk(V) + m_1\{dl(U)dl(V) - dl(U)dk(V) - dk(U)dl(V)\}\end{aligned}$$

$$+ dk(U)d\psi(V) + dk(V)d\psi(U)$$

on M_2, where $\lambda_2 = f_1^2\{\lambda+\Delta k-g(\nabla k,\nabla\psi)\}$ and $\psi_2 = \psi|_{M_2}$. By direct computations, we get

$$^2\mathrm{Ric}(U,V) + h_2^{\varphi_2}(U,V) = \lambda_2 g_2 + (2-m_2)dk(U)dk(V)$$
$$+ m_1\{dl(U)dl(V) - dl(U)dk(V) - dk(U)dl(V)\} + dk(U)d\psi(V) + dk(V)d\psi(U),$$

where $\varphi_2 = \psi_2 - m_1 l - (m_2-2)k$. So, we obtain assertion (b).

Finally, for $X \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, we know that $g(X,U) = 0$. Thus, the assertion (c) follows immediately from (5) and (19). The converse is trivial. □

Remark 3.9. The authors gave illustrative examples for gradient η-Ricci soliton, gradient almost η-Ricci soliton in [5, Example 3.10 and Example 3.11]. Also Blaga obtained that the generalized cylinder $M \times S^3$ has a gradient η-Ricci soliton structure as a product manifold in [3, Example 4.9].

Theorem 3.10. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product manifold. If it is a gradient Riemann soliton, then the following statements hold:*
(a) $(\nabla^1\varphi_1, \lambda_1, \mu_1)$ *defines a gradient almost η-Ricci soliton on (M_1, g_1), where* $\varphi_1 = (m-2)\psi_1 - m_2 k$, $\lambda_1 = f_2^2\{(m-1)\lambda + \Delta l - \Delta\psi - (m-2)g(\nabla l,\nabla\psi)\}$, $\psi_1 = \psi|_{M_1}$, $\eta = \hat{dk}$ *is the pullback of dk to M_1 and $\mu_1 = m_2$;*

(b) $^2\mathrm{Ric}(U,V)+(m-2)h_2^{\varphi_2}(U,V) = \lambda_2 g_2(U,V)+(2-m_2)\tilde{dk}(U)\tilde{dk}(V)+m_1\{dl(U)dl(V) -dl(U)\tilde{dk}(V) - \tilde{dk}(U)dl(V)\} + (m-2)\{\tilde{dk}(U)d\psi(V) + \tilde{dk}(V)d\psi(U)\}$, *where* $\varphi_2 = (m-2)\psi_2 - m_1 l + (2-m_2)k$, $\lambda_2 = f_1^2\{(m-1)\lambda + \Delta k - \Delta\psi - (m-2)g(\nabla k,\nabla\psi)\}$, $\psi_2 = \psi|_{M_2}$ *and $\tilde{dk}$ is the pullback of dk to M_2.*

Proof. Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a gradient Riemann soliton with the potential function ψ. Then, we have $\mathrm{Ric} + (m-2)h^\psi = \{(m-1)\lambda - \Delta\psi\}g$, from (26). Hence, using (6) and (18), we obtain $^1\mathrm{Ric} + (m-2)h_1^{\psi_1} = \lambda_1 g_1 + m_2 h_1^k + m_2 dk \otimes dk$ on M_1, where $\lambda_1 = f_2^2\{(m-1)\lambda + \Delta l - \Delta\psi - (m-2)g(\nabla l,\nabla\psi)\}$ and $\psi_1 = \psi|_{M_1}$. By direct computations, we get $^1\mathrm{Ric} + h_1^{\varphi_1} = \lambda_1 g_1 + m_2 dk \otimes dk$, where $\varphi_1 = (m-2)\psi_1 - m_2 k$. Thus, $(\nabla^1\varphi_1, \lambda_1, \mu_1)$ defines a gradient almost η-Ricci soliton on (M_1, g_1) as desired, with $\eta = dk$. On the other hand, using (7) and (20), we find

$$^2\mathrm{Ric}(U,V) + (m-2)h_2^{\psi_2}(U,V) = \lambda_2 g_2(U,V) - (2-m_2)h_2^k(U,V)$$
$$+ m_1 h_2^l(U,V)(2-m_2)dk(U)dk(V)$$
$$+ m_1\{dl(U)dl(V) - dl(U)dk(V) - dk(U)dl(V)\}(m-2)\{dk(U)d\psi(V) + dk(V)d\psi(U)\}$$

on M_2, where $\lambda_2 = f_1^2\big((m-1)\lambda + \Delta k - \Delta\psi - (m-2)g(\nabla k,\nabla\psi)\big)$ and $\psi_2 = \psi|_{M_2}$. After some computations, we get

$$^2\mathrm{Ric}(U,V) + (m-2)h_2^{\varphi_2}(U,V) = \lambda_2 g_2(U,V) + (2-m_2)dk(U)dk(V)$$
$$+ m_1\{dl(U)dl(V) - dl(U)dk(V) - dk(U)dl(V)\}(m-2)\{dk(U)d\psi(V) + dk(V)d\psi(U)\},$$

where $\varphi_2 = (m-2)\psi_2 - m_1 l + (2-m_2)k$. Thus, the assertion (b) holds. □

Theorem 3.11. *Let ${}_{f_2}M_1 \times_{f_1} M_2$ be a warped-twisted product quasi-Einstein manifold with associated scalar functions α_0 and β_0. Then* *(a)* $^1\mathrm{Ric}(X,Y) - m_2 h_1^k(X,Y) = \lambda_1 g_1(X,Y) + \beta_0 \tilde{A}_1(X)\tilde{A}_1(Y) + m_2 \hat{dk}(X)\hat{dk}(Y)$, *where* $\lambda_1 = f_2^2(\alpha_0 + \Delta l)$, $\tilde{A}_1 = A|_{M_1}$, $\hat{dk}$ *is the pullback of* dk *to* M_1*;*

(b) $^2\mathrm{Ric}(U,V) + h_2^{\varphi_2}(U,V) = \lambda_2 g_2(U,V) + \beta_0 \tilde{A}_2(U)\tilde{A}_2(V) + (2-m_2)\tilde{dk}(U)\tilde{dk}(V) + m_1\left\{dl(U)dl(V) - dl(U)\tilde{dk}(V) - \tilde{dk}(U)dl(V)\right\}$, *where* $\varphi_2 = (2-m_2)k - m_1 l$, $\lambda_2 = f_1^2(\alpha_0 + \Delta k)$ *and* $\tilde{A}_2 = A|_{M_2}$, $\tilde{dk}$ *is the pullback of* dk *to* M_2.

Proof. For any $X, Y \in \mathfrak{L}(M_1)$, using (18), we have

$$\alpha_0 g(X,Y) + \beta_0 A(X)A(Y) = {}^1\mathrm{Ric}(X,Y) + h^l(X,Y) - m_2\left\{h_1^k(X,Y) + X(k)Y(k)\right\} - g(X,Y)\left\{\Delta l + g(\nabla l, \nabla l)\right\}$$

from (27). By using (9), we obtain

$$^1\mathrm{Ric}(X,Y) - m_2 h_1^k(X,Y) = \lambda_1 g_1(X,Y) + \beta_0 \tilde{A}_1(X)\tilde{A}_1(Y) + m_2 dk(X)dk(Y),$$

where, $\lambda_1 = f_2^2(\alpha_0 + \Delta l)$. Thus, the assertion (a) holds. On the other hand, for any $U, V \in \mathfrak{L}(M_2)$, using (20), we have

$$\alpha_0 g(U,V) + \beta_0 A(U)A(V) = {}^2\mathrm{Ric}(U,V) + h^k(U,V) + (1-m_2)h_2^k(U,V) + m_2 U(k)V(k) - g(U,V)\left\{\Delta k + g(\nabla k, \nabla k)\right\} - m_1\left\{h_2^l(U,V) + U(l)V(l) - U(l)V(k) - U(k)V(l)\right\},$$

from (27). By using (10), we obtain

$$^2\mathrm{Ric}(U,V) + (2-m_2)h_2^k(U,V) - m_1 h_2^l(U,V) = \alpha_0 g(U,V) + \beta_0 A(U)A(V) + (2-m_2)U(k)V(k) + g(U,V)\Delta k + m_1\left\{U(l)V(l) - U(l)V(k) - U(k)V(l)\right\}.$$

After some computations, we get

$$^2\mathrm{Ric}(U,V) + h_2^{\varphi_2}(U,V) = \lambda_2 g_2(U,V) + \beta_0 \tilde{A}_2(U)\tilde{A}_2(V) + (2-m_2)U(k)V(k) + m_1\left\{U(l)V(l) - U(l)V(k) - U(k)V(l)\right\},$$

where $\varphi_2 = (2-m_2)k - m_1 l$ and $\lambda_2 = f_1^2(\alpha_0 + \Delta k)$. Thus, assertion (b) holds. □

Remark 3.12. The authors established the examples of warped product on mixed generalized quasi-Einstein manifold in [16, Example 5.1 and Example 5.2].

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