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A NOTE ON α-EQUIVALENT TOPOLOGIES

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Abstract. This paper responds to the question of when are two topologies α-equivalent by using some recently introduced classes of sets as well as the classes of regular open sets, nowhere dense sets and dense sets.

1. Introduction

In [10] Njåstad gave a characterization of α-equivalent topologies on a given set by means of semi-open sets. It is natural to ask whether α-equivalent topologies can be characterized by means of some other classes of subsets which are shared by α-equivalent topologies, that is by means of classes of regular open, preopen, semi-preopen, nowhere dense and dense sets. We answer that question in the affirmative and show that two topologies have the same collection of α-sets if and only if they share both the semi-regularisation topology and the γ-topology.

We first recall some definitions. Let A be a subset of a topological space (X,T). The closure of A and the interior of A with respect to T are denoted by $\operatorname{cl} A$ and $\operatorname{int} A$, respectively.

Definition. A subset A of (X,T) is called
(i) an *α-set* if $A \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$,
(ii) a *semi-open set* if $A \subset \operatorname{cl}(\operatorname{int} A)$,
(iii) a *preopen set* if $A \subset \operatorname{int}(\operatorname{cl} A)$,
(iv) a *semi-preopen set* if $A \subset \operatorname{cl}(\operatorname{int}(\operatorname{cl} A))$.

The first three notions were introduced by Njåstad [10], Levine [8] and Mashhour et al. [9], respectively. The fourth concept was introduced by Abd El-Monsef et al. [1] under the name of β-open set and it was called semi-preopen set in [3]. The classes of these sets in a space (X,T) are denoted by T_α, $\mathbf{SO}(T)$, $\mathbf{PO}(T)$ and $\mathbf{SPO}(T)$, respectively. All of these are larger than T and are closed under arbitrary unions. Njåstad [10] showed that T_α is a topology on X.

For a space (X,T) the family $\{\, A \subset X \mid A \cap B \in \mathbf{PO}(T) \text{ whenever } B \in \mathbf{PO}(T) \,\}$ will be denoted by T_γ. It was shown in [4] that T_γ is a topology on X larger than T_α. It will be called the *γ-topology* of T.

The classes of regular open sets, nowhere dense sets and dense sets in (X,T) will be denoted by $\mathbf{RO}(T)$, $\mathbf{N}(T)$ and $\mathbf{D}(T)$ respectively. The complement of a regular open set is called regular closed, and the complement of a dense set is called *codense*.

The following results will be needed in the sequel.

PROPOSITION 1. ([10], [2] and [3]) *Let (X,T) be a space. Then:*
(i) $T_\alpha = T_{\alpha\alpha}$, *(ii)* $\mathbf{SO}(T) = \mathbf{SO}(T_\alpha)$, *(iii)* $\mathbf{PO}(T) = \mathbf{PO}(T_\alpha)$, *(iv)* $\mathbf{SPO}(T) = \mathbf{SPO}(T_\alpha)$, *(v)* $\mathbf{RO}(T) = \mathbf{RO}(T_\alpha)$, *(vi)* $\mathbf{D}(T) = \mathbf{D}(T_\alpha)$, *(vii)* $\mathbf{N}(T) = \mathbf{N}(T_\alpha)$. ∎

PROPOSITION 2. ([3]) *A subset of a space (X,T) is semi-preopen if and only if* $\operatorname{cl} A$ *is regular closed.* ∎

PROPOSITION 3. ([3]) *In a space (X,T), the intersection of every open and each semi-preopen set is semi-preopen.* ∎

PROPOSITION 4. ([7]) *In a space (X,T), a subset A is semi-open if and only if it is semi-preopen and* $\operatorname{int}(\operatorname{cl} A) \subset \operatorname{cl}(\operatorname{int} A)$. ∎

PROPOSITION 5. ([7]) *If the topologies T and U on a set X have the same γ-topology, then they have the same class of nowhere dense sets.* ∎

PROPOSITION 6. ([7]) *Two topologies on a set X have the same class of semi-preopen sets if and only if their γ-topologies are the same.* ∎

2. α-equivalent topologies

DEFINITION. ([10]) Two topologies on a set X are called *α-equivalent* if their α-topologies are the same.

In the same paper Njåstad obtained the following result on α-equivalence in terms of semi-open sets.

PROPOSITION 7. *Two topologies on a set X are α-equivalent if and only if they have the same class of semi-open sets.* ∎

We consider the same question in relation to the other classes of subsets which are shared by (X,T) and (X,T_α) mentioned in Proposition 1, to obtain analogous characterizations of α-equivalence. We begin with a consequence of Propositions 7 and 1.

PROPOSITION 8. *Let T and U be two α-equivalent topologies. Then:*
(i) $\mathbf{PO}(T) = \mathbf{PO}(U)$, *(ii)* $\mathbf{SPO}(T) = \mathbf{SPO}(U)$, *(iii)* $\mathbf{RO}(T) = \mathbf{RO}(U)$, *(iv)* $\mathbf{D}(T) = \mathbf{D}(U)$, *(v)* $\mathbf{N}(T) = \mathbf{N}(U)$. ∎

It should be noted that none of these five conditions is equivalent to the condition $\mathbf{SO}(T) = \mathbf{SO}(U)$. One can easily find the examples. But it is a natural question whether there are two conditions among these five which together imply $\mathbf{SO}(T) = \mathbf{SO}(U)$. We prove that any pair of independent conditions does so. From Propositions 5 and 6 it follows immediately that for the statements in Proposition

8, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (v) hold in general. The following result shows that (iv) $\Longrightarrow$ (v) is also true. The closure and the interior of a set A in (X,U) will be denoted by $\mathrm{cl}_U\,A$ and $\mathrm{int}_U\,A$ respectively.

PROPOSITION 9. *Let T and U be topologies on X having the same class of dense sets. Then their classes of nowhere dense sets coincide.*

Proof. Suppose that $A \in \mathbf{N}(T) - \mathbf{N}(U)$. Then $G = \mathrm{int}(\mathrm{cl}_U\,A) \neq \emptyset$ because (X,T) and (X,U) have the same class of codense sets. Define $W = \mathrm{int}_U(G - \mathrm{cl}\,A)$. Since A is T-nowhere dense, $G - \mathrm{cl}\,A$ is T-open and non-empty. Hence $W \subset \mathrm{cl}_U\,A - A$ and $W \neq \emptyset$, a contradiction. Therefore $\mathbf{N}(T) \subset \mathbf{N}(U)$. The reverse inclusion is shown in an analogous way. ∎

In order to prove our main result we first establish two lemmas.

LEMMA 1. *If T and U are topologies on X sharing the classes of semi-preopen sets and regular open sets, then they have the same class of dense sets.*

Proof. Assume that $\mathrm{cl}\,A = X$. Then $A \in \mathbf{SPO}(T) = \mathbf{SPO}(U)$ and so $\mathrm{cl}_U\,A$ is U-regular closed by Proposition 2. Hence $\mathrm{cl}_U\,A$ is T-regular closed and thus $\mathrm{cl}_U\,A = X$, i.e. $A \in \mathbf{D}(U)$. The reverse inclusion is obtained analogously. ∎

LEMMA 2. *Let T and U be topologies on X having the same class of dense sets, and let A be a subset of X. Then* $\mathrm{int}(\mathrm{cl}\,A) \subset \mathrm{cl}(\mathrm{int}\,A)$ *if and only if* $\mathrm{int}_U\,\mathrm{cl}_U\,A \subset \mathrm{cl}_U\,\mathrm{int}_U\,A$.

Proof. Suppose that $\mathrm{int}(\mathrm{cl}\,A) \subset \mathrm{cl}(\mathrm{int}\,A)$ and let $W = \mathrm{int}_U(\mathrm{cl}_U\,A - \mathrm{int}_U\,A)$ be non-empty. Then $G = \mathrm{int}\,W \neq \emptyset$ because (X,T) and (X,U) share the class of codense sets. Put $G_1 = G - \mathrm{cl}\,A$ and $G_2 = G \cap \mathrm{int}\,A$ and let $W_1 = \mathrm{int}_U\,G_1$ and $W_2 = \mathrm{int}_U\,G_2$. We observe that $W_1 \subset \mathrm{cl}_U\,A - \mathrm{cl}\,A$ which implies $W_1 = \emptyset$ and so $\mathrm{int}\,G_1 = G_1 = \emptyset$. On the other hand, $W_2 \subset A$ and $W_2 \cap \mathrm{int}_U\,A \subset W \cap \mathrm{int}_U\,A = \emptyset$. Hence $W_2 = \emptyset$ and thus $\mathrm{int}\,G_2 = G_2 = \emptyset$. Hence both G_1 and G_2 are empty, $G \subset \mathrm{cl}\,A - \mathrm{int}\,A$ and so $G \subset \mathrm{int}(\mathrm{cl}\,A) - \mathrm{cl}(\mathrm{int}\,A) = \emptyset$, a contradiction. Therefore $W = \emptyset$, that is $\mathrm{int}_U\,\mathrm{cl}_U\,A \subset \mathrm{cl}_U\,\mathrm{int}_U\,A$. ∎

THEOREM 1. *Let T and U be topologies on a set X. Then the following are equivalent:*

(a) T and U are α-equivalent,
(b) $\mathbf{RO}(T) = \mathbf{RO}(U)$ *and* $\mathbf{PO}(T) = \mathbf{PO}(U)$.
(c) $\mathbf{RO}(T) = \mathbf{RO}(U)$ *and* $\mathbf{SPO}(T) = \mathbf{SPO}(U)$.
(d) $\mathbf{RO}(T) = \mathbf{RO}(U)$ *and* $\mathbf{N}(T) = \mathbf{N}(U)$.
(e) $\mathbf{RO}(T) = \mathbf{RO}(U)$ *and* $\mathbf{D}(T) = \mathbf{D}(U)$.
(f) $\mathbf{PO}(T) = \mathbf{PO}(U)$ *and* $\mathbf{D}(T) = \mathbf{D}(U)$.
(g) $\mathbf{SPO}(T) = \mathbf{SPO}(U)$ *and* $\mathbf{D}(T) = \mathbf{D}(U)$.

Proof. (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (d) follow from Propositions 8, 5 and 6. Also, (a) $\Longrightarrow$ (e) and (a) $\Longrightarrow$ (f) follow from Proposition 8, (e) $\Longrightarrow$ (d) from Proposition 9 and (f) $\Longrightarrow$ (g) from Proposition 6.

(d) $\Longrightarrow$ (c): Suppose $A \in \mathbf{SPO}(T)$ and put $B = A - \mathrm{cl}_U\,\mathrm{int}_U\,\mathrm{cl}_U\,A$. Then $B \in \mathbf{N}(U) = \mathbf{N}(T)$ by (d). On the other hand, $\mathrm{cl}_U\,\mathrm{int}_U\,\mathrm{cl}_U\,A$ is U-regular closed and so T-regular closed. Hence $B \in \mathbf{SPO}(T)$ by Proposition 3 and thus $B = \emptyset$, that is $A \in \mathbf{SPO}(U)$.

(c) $\Longrightarrow$ (a): Suppose that $A \in \mathbf{SO}(U)$. According to Lemma 1, (X,T) and (X,U) share the class of codense sets and so $\operatorname{int}_U(A - \operatorname{cl}(\operatorname{int} A)) = \emptyset$, that is $\operatorname{int}_U A \subset \operatorname{cl}_U \operatorname{cl}(\operatorname{int} A)$. Therefore $A \subset \operatorname{cl}_U \operatorname{int}_U A \subset \operatorname{cl}_U \operatorname{cl}(\operatorname{int} A)$. On the other hand, $\operatorname{cl}(\operatorname{int} A)$ is T-regular closed and so U-regular closed by (c). Hence $A \subset \operatorname{cl}(\operatorname{int} A)$, that is $A \in \mathbf{SO}(T)$. The reverse inclusion is obtained analogously.

(g) $\Longrightarrow$ (a): Suppose that $A \in \mathbf{SO}(T)$. Then $A \in \mathbf{SPO}(T)$ and $\operatorname{int}(\operatorname{cl} A) \subset \operatorname{cl}(\operatorname{int} A)$ by Proposition 4. According to Lemma 2 and (g) we have that $A \in \mathbf{SPO}(U)$ and $\operatorname{int}_U \operatorname{cl}_U A \subset \operatorname{cl}_U \operatorname{int}_U A$ and thus $A \in \mathbf{SO}(U)$ again by Proposition 4. The reverse inclusion is obtained analogously. ∎

For a space (X,T) the topology T_s on X which has as a base the class $\mathbf{RO}(T)$ is called the semiregularisation topology of (X,T). Recall that $T_{ss} = T_s$ and $\mathbf{RO}(T) = \mathbf{RO}(U)$ if and only if $T_s = U_s$ [6]. Theorem 1 and Proposition 6 give the following characterization.

Theorem 2. *Let T and U be topologies on a set X. Then $T_\alpha = U_\alpha$ if and only if $T_\gamma = U_\gamma$ and $T_s = U_s$.* ∎

Corollary 1. *Let (X,T) be a space. Then $T_{s\alpha} = T_\alpha$ if and only if $T_{s\gamma} = T_\gamma$.* ∎

It was shown in [5] that the topologies T and T_γ on a set X are α-equivalent if and only if they share the class of dense sets. Our Theorem 3 will be a slight improvement of this result. For convenience we first establish two simple lemmas.

Lemma 3. ([4]) *Let (X,T) be a space. Then:*
(i) $\operatorname{cl}_{T_\gamma} G = \operatorname{cl} G$ *for any T-open set G,*
(ii) $\operatorname{int}_{T_\gamma} F = \operatorname{int} F$ *for any T-closed set F.* ∎

Lemma 4. *Let T and U be topologies on X such that $U \subset T_\gamma$. Then* $\operatorname{cl}(\operatorname{int} A) \subset \operatorname{cl}_U \operatorname{int} A$ *for any subset A.*

Proof. By the assumption and Lemma 3 we have that $\operatorname{cl}(\operatorname{int} A) = \operatorname{cl}_{T_\gamma} \operatorname{int} A \subset \operatorname{cl}_U \operatorname{int} A$. ∎

Theorem 3. *Let T and U be topologies on a set X satisfying $T \subset U_\gamma$ and $U \subset T_\gamma$. Then T and U are α-equivalent if and only if they have the same class of dense sets.*

Proof. Assume $\mathbf{D}(T) = \mathbf{D}(U)$ and let $A \in \mathbf{SO}(T)$. According to Proposition 4 and Lemma 2 it suffices to show that $A \in \mathbf{SPO}(U)$. By Lemma 4 we have $\operatorname{cl}_U A \subset \operatorname{cl}_U \operatorname{cl}(\operatorname{int} A) \subset \operatorname{cl}_U \operatorname{int} A$ and so $\operatorname{cl}_U A = \operatorname{cl}_U \operatorname{int} A$. Since $\operatorname{int} A \in T \subset U_\gamma \subset \mathbf{PO}(U)$ we have by Proposition 2 that $\operatorname{cl}_U A$ is U-regular closed and hence $A \in \mathbf{SPO}(U)$. ∎

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