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INCREASING SOLUTIONS OF $(r(x)y^{(n)})^{(n)} = yf(x)$

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Abstract. We study the existence of positive, monotonic, unbounded solutions of the equation $(r(x)y^{(n)})^{(n)} = yf(x)$. We obtain necessary and sufficient conditions for the existence of different classes of these solutions.

Previous investigations of the equation

$$(r(x)y^{(n)})^{(n)} = \pm y f(x,y) \tag{1.$\pm$1}$$

include that of Kusano and Naito [1], who studied $(1.-1)$ with $n=2$, and Kreith [2], who studied $(1.\pm 1)$ with $r(x)=1$. We consider the equation

$$(r(x)y^{(n)})^{(n)} = yf(x) \tag{2}$$

where $f(x)$ and $r(x)$ are positive and continuous on $[\tau,\infty)$ and $\int_\tau^\infty du/r(u) = \infty$.

Definition. Denote $E_k(x,y) = y^{(k)}$, $0 \leqslant k \leqslant n-1$, $E_k(x,y) = (r(x)y^{(n)})^{(k-n)}$ for $n \leqslant k \leqslant 2n$, and $E_k(x) = E_k(x,y(x))$. A solution $y(x)$ of the equation (2) is said to be of the type $2j$, $0 \leqslant j \leqslant n$ if $E_k(x) > 0$, $0 \leqslant k \leqslant 2j$ and $(-1)^k E_k(x) > 0$, for $2j \leqslant k \leqslant 2n$ and $\sigma \leqslant x < \infty$ for some $\sigma \geqslant \tau$. Denote also

$$R_k(x,t) = \begin{cases} \dfrac{(t-x)^k}{k!}, & \text{for } 0 \leqslant k \leqslant n-1, \\ \displaystyle\int_x^t \frac{(u-x)^{n-1}(t-u)^{k-n}}{(n-1)!\,(k-n)!\,r(u)}\,du, & \text{for } n \leqslant k \leqslant 2n-1, \end{cases} \qquad \tau \leqslant t, x < \infty.$$

We observe that $R_k(x,t) > 0$ for $\tau \leqslant t < s < \infty$, and $(-1)^k R_k(x,t) > 0$ for $x > t$.

The following facts are known:

1) A solution of (2) which is positive on $[\tau,\infty)$ must be of the type $2j$ for some j, $0 \leqslant j \leqslant n$.

2) Equation (2) has solutions of the type $2j$ for $j=0$ and $j=n$ [3].

Unlike earlier work on this subject, which considered solutions satisfying the asymptotic condition $0 < \lim_{x\to\infty} y(x) R_m^{-1}(x) < \infty$, we impose a stronger asymptotic condition

$$\lim_{x\to\infty} \left| y(x) - \sum_{k=0}^m A_k R_k(x) \right| = 0.$$

THEOREM. *$y(x)$ is a solution of the type $2j$, $0 < j < n$, of (2) if and only if $E_{2j}(x) \downarrow A_{2j} \geqslant 0$ as $x \to \infty$. In this case there exist positive constants α, β such that*

$$\alpha R_{2j-1}(\rho, x) < y(x) < \beta R_{2j}(\rho, \tau), \qquad \rho \leqslant x < \infty,$$

ρ sufficiently large. Further, $y(x) \approx R_{2j}(\tau, x)$ if and only if $A_{2j} > 0$; $y(x) \approx R_{2j-1}(\tau, x)$ if and only if $A_{2j} = 0$ and $E_{2j-1}(x) \uparrow A_{2j-1} > 0$, $x \to \infty$.

Proof. That $E_{2j} \downarrow A_{2j} \geqslant 0$ follows from the definition of the solutions of type $2j$, since E_{2j} is a positive decreasing function. It follows that for sufficiently large t, $0 \leqslant A_{2j} \leqslant E_{2j}(x) < A_{2j} + \varepsilon$. If $2j \leqslant n-1$, we can integrate these inequalities $2j$ times and obtain positive constants α, β such that

$$\alpha R_{2j-1}(\tau, x) < y(x) < \beta R_{2j}(\tau, x)$$

for sufficiently large t. If $2j \geqslant n$, we integrate $2j - n$ times obtaining $\alpha_1 R_{2j-n-1}(\tau, x) \leqslant r(x) y^{(n)}(x) \leqslant \beta_1 R_{2j-n}(\tau, x)$, then dividing by $r(x)$ and integrating n times we obtain

$$\alpha R_{2j-1}(\tau, x) < y(x) < \beta R_{2j}(\tau_1, x).$$

If A_{2j} is strictly positive, then the preceding argument in fact gives $\alpha R_{2j}(\tau, x) < y(x) < \beta R_{2j}(\tau, x)$ and $y(x)$ is in fact asymptotically equivalent to R_{2j}. If $A_{2j} = 0$ then E_{2j-1} is bounded and increasing shows that $y(x)$ is asymptotically equivalent to $R_{2j-1}(\tau_1, x)$.

We note that a solution of (2) can be written as

$$y(x) = E_0(x) = \sum_{k=0}^{2n-1} (-1)^k E_k(b) R_k(x, b) + \int_x^b R_{2n-1}(x, t) p(t) x(t)\, dt. \tag{3}$$

Formula (3) follows from the Taylor's theorem:

$$y(x) = \sum_{k=0}^{n-1} \frac{(-1)^k (b-x)^k y^{(k)}(b)}{k!} + (-1)^k \int_t^b \frac{(t-x)^{n-1} r(t) y^{(n)}(t)}{(n-1)!\, r(t)}\, dt \tag{a}$$

$$r(t) y^{(n)}(t) = \sum_{k=0}^{n-1} \frac{(-1)^k (b-t)^k (r(b) y^{(n)}(b))^{(k)}}{k!} + (-1)^n \int_x^b \frac{(t-n)^{n-1} p(u) y(u)}{(n-1)!}\, du; \tag{b}$$

if we substitute (b) into (a) and interchange the order of integration, (3) results.

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