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SOME INEQUALITIES FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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Abstract. In this paper we give simple proofs of some inequalities for entire functions of exponential type.

1. Introduction

Thera are many L^p-inequalities, as well as inequalities in the uniform norm, concerning entire functions of exponential type and their derivatives. One of the most known inequalities is Bernstein's inequality

$$\sup_{x\in\mathbf{R}} |f'(x)| \leqslant \sigma \sup_{x\in\mathbf{R}} |f(x)|, \tag{1}$$

$$\int_{-\infty}^{\infty} |f'(x)|^p \, dx \leqslant \sigma^p \int_{-\infty}^{\infty} |f(x)|^p \, dx, \tag{2}$$

which holds for entire functions of type $\leqslant \sigma$. In the case $p \geqslant 1$ the inequalities (1) and (2) were proved in a more general form in [1]. An extension to the case $0 < p < 1$ was done in [4], where it was proved that for arbitrary numbers A, B with $\operatorname{Im}(A/B) \geqslant 0$ and an arbitrary entire function of exponential type $\leqslant \sigma$

$$\int_{-\infty}^{\infty} |Af(x) + Bf'(x)|^p \, dx \leqslant |A + i\sigma B|^p \int_{-\infty}^{\infty} |f(x)|^p \, dx. \tag{3}$$

In this paper we give simple proofs of some inequalities that are interesting by themselves and may be of some interest in other investigations.

2. Results

THEOREM 1. *Let f be an entire function of exponential type σ and P a polynomial. Let $P = P_+P_-$, where P_+ and P_- are polynomials whose zeros lie in $\Pi_+ = \{\, z \,:\, \operatorname{Im} z \geqslant 0 \,\}$ and $\Pi_- = \{\, z \,:\, \operatorname{Im} z < 0 \,\}$ respectively. Then*

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a) If $\int_{-\infty}^{\infty}|f(x)|^r\,dx<+\infty$ *(*$r>0$*), then*

$$\int_{-\infty}^{\infty}|P(d/dx)f(x)|^r\;dx\leqslant|P_+(-i\sigma)P_-(i\sigma)|^r\int_{-\infty}^{\infty}|f(x)|^r\,dx. \tag{4}$$

b) If $\sup_{x\in\mathbf{R}}|f(x)|<+\infty$*, then*

$$\sup_{x\in\mathbf{R}}|P(d/dx)f(x)|\leqslant|P_+(-i\sigma)P_-(i\sigma)|\sup_{x\in\mathbf{R}}|f(x)|. \tag{5}$$

THEOREM 2. *Let* f *be an entire function of exponential type* σ *such that* $\int_{-\infty}^{\infty}|f(x)|^p\,dx<+\infty$ *(*$p>0$*). Then*

$$|f(x)|^p\leqslant\frac{2\sigma pm}{\pi}\int_{-\infty}^{\infty}|f(u)|^p\,du\quad(x\in\mathbf{R}), \tag{6}$$

where $m=\min_{t>0}(e^t-1)/t^2$.

It was proved in [1] that if f is an entire function of exponential type such that $\int_{-\infty}^{\infty}|f(x)|^p\,dx<\infty$ for some $p\geqslant1$, then it is bounded on $\mathbf{R}$. In [4] this was extended to all $p>0$, but the method did not give an estimate from above for $\|f\|_\infty/\|f\|_p$, where $\|f\|_\infty=\sup_{x\in\mathbf{R}}|f(x)|$, $\|f\|_p=(\int_{\mathbf{R}}|f(x)|^p\,dx)^{1/p}$.

Theorem 2 asserts that $\|f\|_\infty\leqslant C(p,\sigma)\|f\|_p$ where $C(p,\sigma)=(2\sigma mp/\pi)^{1/p}$. This constant (in the case $p\geqslant1$) is better than the one given in [1]. It is an open question what is the best constant in (6).

3. Proofs

Theorem 1 (a) is a direct consequence of (3). Indeed, from (3) we obtain

$$\int_{\mathbf{R}}|\alpha f(x)+f'(x)|^r\,dx\leqslant|\alpha+i\sigma|^r\int_{\mathbf{R}}|f(x)|^r\,dx,$$

provided $\operatorname{Im}\alpha\geqslant0$. Let $P(z)=\prod_{i=1}^m(z+\alpha i)$, $\operatorname{Im}\alpha_i\geqslant0$, $i=1,2,\dots,m$. Then by successive applications of the last inequality, we get

$$\begin{aligned}\int_{\mathbf{R}}|P(d/dx)f(x)|^r\;dx&=\int_{\mathbf{R}}\left|\left(\prod_{i=1}^m\left(\alpha_i+\frac{d}{dx}\right)\right)f(x)\right|^r\,dx\\&\leqslant|\alpha_m+i\sigma|^r\int_{\mathbf{R}}\left|\left(\prod_{i=1}^{m-1}\left(\alpha_i+\frac{d}{dx}\right)\right)f(x)\right|^r\,dx\leqslant\cdots\\&\leqslant|\alpha_m+i\sigma|^r|\alpha_{m-1}+i\sigma|^r\cdots|\alpha_1+i\sigma|^r\int_{\mathbf{R}}|f(x)|^r\,dx\\&=|P(\sigma i)|^r\int_{\mathbf{R}}|f(x)|^r\,dx.\end{aligned}$$

In the case when all the zeros of P lie in Π_+, an application of the preceding inequality to the polynomial $P_1(z) = P(-z)$ and the entire function $f_1(z) = f(-z)$ shows that

$$\int_{\mathbf{R}} |P(d/dx)f(x)|^r\,dx \leqslant |P(-i\sigma)|^r \int_{\mathbf{R}} |f(x)|^r\,dx.$$

Finally, if $P = P_+P_-$, then combining the last two inequalities we obtain (4).

Inequality (5) cannot be obtained from (4) as the limit when $r \to \infty$, because the interval of integration is unbounded. In proving (5) we shall use Levitan polynomials [1], [3] as well as the following theorem [3].

Let the roots of an algebraic polynomial P lie in Π_+ and let

$$S(\theta) = \sum_{\nu=-n}^{n} b_\nu e^{i\nu\theta}, \quad T(\theta) = \sum_{\nu=-n}^{n} a_\nu e^{i\nu\theta}, \quad a_{-n} \neq 0.$$

If $T(\theta) \neq 0$ in Π_+ and $|S(\theta)| < |T(\theta)|$ for all $\theta \in \mathbf{R}$, then $|P(d/d\theta)S(\theta)| < |P(d/d\theta)T(\theta)|$ for all $\theta \in \Pi_+$.

Hence, by taking $T(\theta) = e^{-in\theta}\max_{\theta\in\mathbf{R}}|S(\theta)|$ we obtain the following

Lemma. *If all the zeros of a polynomial P lie in Π_+ and S is a trigonometric polynomial of degree n, then*

$$\max_{\theta\in\mathbf{R}}|P(d/d\theta)S(\theta)| \leqslant |P(-in)| \max_{\theta\in\mathbf{R}}|S(\theta)|.$$

Let $\varphi(x) = (\sin \pi x/\pi x)^2$ and f be an entire function of exponential type σ such that $\|f\|_\infty = \sup_{x\in\mathbf{R}}|f(x)| = M < +\infty$. For $h > 0$ we define

$$f_h(z) = \sum_{\nu=-\infty}^{\infty} \varphi(hz+\nu) f\left(z + \frac{\nu}{h}\right).$$

It turns out [3] that f_h has the following properties:

1° f_h is a trigonometric polynomial, $f_h(z) = \sum_{\nu=-N}^{N} a_\nu e^{2\pi i\nu hz}$, $a_\nu \in \mathbf{C}$, $N = 1 + [\sigma/2\pi h]$ (Levitan polynomial).

2° $\|f_h\|_\infty = \sup_{x\in\mathbf{R}}|f_h(x)| \leqslant M$.

3° $\lim_{h\to+0} f_h(z) = f(z)$, the convergence being uniform on compact subsets. (The same holds for the derivatives.)

Consider first the case of a polynomial $P(z) = \sum_{k=0}^{m} d_k z^k$ with zeros in Π_+. Applying Lemma to the trigonometric polynomial $f_h(\theta/2\pi h)$ (of degree $N = 1 + [\sigma/2\pi h]$) and the polynomial $P_h(z) = P(2\pi hz)$, it follows

$$\max_{\theta\in\mathbf{R}}|P_h(d/d\theta)f_h(\theta/2\pi h)| \leqslant |P_h(-iN)| \max_{\theta\in\mathbf{R}}|f_h(\theta/2\pi h)| \leqslant |P_h(-iN)| \sup_{x\in\mathbf{R}}|f(x)|,$$

i.e.

$$\sup_{\theta\in\mathbf{R}}\left|\sum_{k=0}^{m} d_k f_h^{(k)}(\theta/2\pi h)\right| \leqslant \sup_{x\in\mathbf{R}}|f(x)|\cdot|P_h(-iN)|.$$

Hence, $\sup_{x\in\mathbf{R}}|\sum_{k=0}^{m} d_k f_h^{(k)}(x)| \leqslant \sup_{x\in\mathbf{R}}|f(x)|\cdot|P_h(-iN)|$.

When $h \to 0+$ we obtain $P_h(-iN) = P(-2\pi hiN) \to P(-i\sigma)$ and, since $f_h^{(k)} \to f^{(k)}$, we have $\sup_{x\in\mathbf{R}} |\sum_{k=0}^m d_k f^{(k)}(x)| \leqslant |P(-i\sigma)| \cdot \|f\|_\infty$, i.e.

$$|P(d/dx)f(x)|_\infty \leqslant |P(-i\sigma)| \cdot \|f\|_\infty. \tag{7}$$

If the zeros of P lie in Π_-, then we only have to apply the preceding inequality to $P_1(z) = P(-z)$ and $f_1(z) = f(-z)$. So we obtain, for such polynomials, the inequality $|P(d/dx)f(x)|_\infty \leqslant |P(i\sigma)| \cdot \|f\|_\infty$. By successive applications of the last inequality and (7) we obtain (5).

Proof of Theorem 2. In [2], p. 98 it is proved that if f is an entire function of exponential type σ such that $\int_{-\infty}^{\infty} |f(x)|^p\,dx < \infty$ $(p > 0)$ then

$$\int_{-\infty}^{\infty} |f(x+iy)|^p\,dx \leqslant e^{p\sigma|y|} \int_{-\infty}^{\infty} |f(x)|^p\,dx.$$

Hence

$$\int_{-h}^{h} dy \int_{-\infty}^{\infty} |f(x+iy)|^p\,dx \leqslant \|f\|_p^p \int_{-h}^{h} e^{p\sigma|y|}\,dy = 2\|f\|_p^p \frac{e^{p\sigma h}-1}{p\sigma},$$

i.e.

$$\int_{|\operatorname{Im} z|\leqslant h} |f(z)|^p\,dA(z) \leqslant 2\|f\|_p^p \frac{e^{p\sigma h}-1}{p\sigma}.$$

(Here $dA(z) = dx\,dy$, $z = x+iy$.)

Since $|f|^p$ is a subharmonic function we have that

$$|f(t)|^p \leqslant \frac{1}{\pi h^2} \int_{|z-t|\leqslant h} |f|^p\,dA \leqslant 2\|f\|_p^p \frac{e^{p\sigma h}-1}{p\sigma} \frac{1}{\pi h^2}, \quad t \in \mathbf{R},$$

i.e.

$$|f(t)|^p \leqslant \frac{2p\sigma\|f\|_p^p}{\pi} \frac{e^{p\sigma h}-1}{(p\sigma h)^2}, \quad t \in \mathbf{R}.$$

The last inequality holds for all $h > 0$ and hence

$$|f(t)|^p \leqslant \frac{2p\sigma}{\pi} \min_{x>0} \frac{e^x-1}{x^2} \|f\|_p^p, \quad t \in \mathbf{R}.$$

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