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BOUNDARY VALUE PROBLEM WITH SHIFT FOR TWO SIMPLY CONNECTED REGIONS

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Abstract. In this paper a boundary value problem for two analytic functions in different simply connected regions is considered. If index corresponeded to the problem is not greater than zero, then the solavability of that problem is shown without any further conditions and its solution is obtained in an explicit form (4). However, if the index is greater than zero, then the problem is solvable and its solution is given by (5) if real conditions of solvability (6) hold.

Let S and D be finite simply-connected regions in the complex plane, bounded by closed Ljapunov curves L and Γ respectively. Suppose that the contours L and Γ are oriented in the positive sense with respect to their interiors S and D, respectively. Let $\alpha(t)$ be a function defined on L satisfying the following conditions:

a) it transforms homeomorphically the closed contour L onto intself, changing the orientation;

b) function $\alpha(t)$ has continuous derivatives which are distinct from zero at all points of the contour L.

Let $\alpha^{-1}(t)$ be the inverse function of $\alpha(t)$, $t \in L$.

We shall determine the functions $\Phi_1(z)$ and $\Phi_2(z)$ which are analytic in S and D respectively, whose boundary values on the appropriate contours satisfy the following boundary condition:

$$\Phi_2(\alpha(t)) = G(t) * \Phi_1(t) + g(t), \quad t \in L, \tag{1}$$

where $G(t)$ and $g(t)$ are given continuous functions on L in the sense of Hölder [5].

In the case $D = \text{exterior}(S)$ many authors (cf. [4]) considered the problem of finding functions $\Phi_1(z)$ and $\Phi_2(z)$ which are analytic in S and D, respectively, whose boundary values on appropriate contours satisfy the boundary condition analogous to (1).

Assume that the origin belongs to the region S and define the function $G_0(t)$ in the following way: $G_0(t) = t^{-k}G(t)$, $t \in L$, where $k = \frac{1}{2}[\arg G(t)]_L$ is the index corresponded to the function $G(t)$.

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Now, $k = \frac{1}{2}[\arg G_0(t)]_L = 0$. From the paper [2] it follows that for the homogeneous boundary value problem with the coefficient $G_0(t)$ there exist functions $X_0(t)$ and $X_{0,1}(t)$, being analytic in S and D respectively, and distinct from zero on $S \cup L$ and $D \cup \Gamma$, respectively, and which, on the appropriate contours L and Γ, have limits $X_0(t) \in H(L)$ and $X_{0,1} \in H(L)$ satisfying the following boundary value condition: $X_{0,1}(\alpha(t)) = G_0(t) * X_0(t)$, $t \in L$.

These functions are determined by the formulae:

$$X_0(z) = \exp\left(-\frac{1}{2\pi i}\int_L \frac{h(t)}{t-z}\,dt\right), \quad z \in S,$$
$$X_{0,1}(z) = \exp\left(\frac{1}{2\pi i}\int_L \frac{h(\alpha^{-1}(t))}{t-z}\,dt\right), \quad z \in D,$$

where $h(t)$, $t \in L$ is the solution of the equation

$$(Fh)(t) \equiv h(t) - \frac{1}{2\pi i}\int_L \left(\frac{1}{t-\tau} - \frac{h'(\tau)}{h(\tau)-h(t)}\right) h(\tau)\,d\tau = \ln G_0(t), \quad t, \tau \in L.$$

According to these conclusions, it follows that the coefficient $G(t)$ of the problem (1) for the contour L can be represented in the form

$$G(t) = \frac{X_{0,1}(\alpha(t))}{t^{-k}X_0(t)}, \quad t \in L, \tag{2}$$

In such a way, the boundary condition (1) can be written as

$$\frac{\Phi_2(\alpha(t))}{X_{0,1}(\alpha(t))} - \frac{\Phi_1(t)}{t^{-k}X_0(t)} = \frac{g(t)}{X_{0,1}(\alpha(t))}, \quad t \in L. \tag{3}$$

Let us denote by $f_2(z)$ the function $\dfrac{\Phi_2(z)}{X_{0,1}(z)}$, $z \in D$. In the case $k < 0$ the function $\dfrac{\Phi_1(z)}{z^{-k}X_0(z)}$, $z \in S$ has the point $z = 0$ as a pole of order $-k$, so that it can be represented in the form

$$\frac{\Phi_1(z)}{z^{-k}X_0(z)} = \sum_{i=1}^{-k}\frac{c_i}{z^i} + f_1(z), \quad z \in S,$$

where $f_1(z)$ is indefinite analytic function in S and c_i, $i = 1, 2, \dots, -k$, are complex constants.

If we denote $B_{2j-1} = \operatorname{Re} c_j$, $B_{2j} = \operatorname{Im} c_j$, $q_{2j-1}(t) = 1/t^j$, $t \in L$, $q_{2j}(t) = i/t^j$, $t \in L$, we shall get the boundary value problem

$$f_2(\alpha(t)) - f_1(t) = \sum_{j=1}^{-2k} B_j q_j(t)$$

with solution

$$f_1(t) = B_0 - \sum_{j=1}^{-2k} \frac{B_j}{2\pi i} \int_L \frac{\varphi_j(t)}{t-z}\, dt, \quad z \in S,$$
$$f_2(t) = B_0 + \sum_{j=1}^{-2k} \frac{B_j}{2\pi i} \int_L \frac{\varphi_j(\alpha^{-1}(t))}{t-z}\, dt, \quad z \in D,$$

where B_0 is an arbitrary complex constant and $\varphi_j(t)$ are the solutions of Fredholm's integral equations $(F\varphi_j)(t) = g_j(t)$, $j = 1, 2, \ldots, -2k$.

Let us assume that $B_{-2k+1} = \operatorname{Re} B_0$, $B_{-2k+2} = \operatorname{Im} B_0$ and let us define the functions

$$\left.\begin{aligned} W_{2j-1}(z) &= \frac{1}{z^j} - \frac{1}{2\pi i} \int_L \frac{\varphi_{2j-1}(t)}{t-z}\, dt, \quad z \in S, \\ W_{2j}(z) &= \frac{i}{z^j} - \frac{1}{2\pi i} \int_L \frac{\varphi_{2j}(t)}{t-z}\, dt, \quad z \in S, \end{aligned}\right. \quad j = 1, 2, \ldots, -k,$$
$$W_{-2k+1}(z) = 1, \quad z \in S, \qquad W_{-2k+2}(z) = i, \quad z \in S,$$
$$V_j(z) = \frac{1}{2\pi i} \int_L \frac{\varphi_j(\alpha^{-1}(t))}{t-z}, \quad z \in D,$$
$$V_{-2k+1}(z) = 1, \quad z \in D, \qquad V_{-2k+2}(z) = i, \quad z \in D.$$

Now we can give the general solution of the boundary value problem (1) in the case $k < 0$ in the following way:

$$\begin{aligned} \Phi_1(z) &= z^{-k} X_0(z) \Big(\sum_{j=1}^{-2k+2} B_j W_j(z) - \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z}\, dt \Big), \quad z \in S, \\ \Phi_2(z) &= z^{-k} X_{0,1}(z) \Big(\sum_{j=1}^{-2k+2} B_j V_j(z) + \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z}\, dt \Big), \quad z \in D, \end{aligned} \tag{4}$$

where $\varphi(t)$ is the solution of the integral equation

$$(F\varphi)(t) = \frac{g(t)}{X_{0,1}(\alpha(t))}.$$

In the case $k > 0$, the solution, if it exists at all, may be represented in the form

$$\begin{aligned} \Phi_1(z) &= z^{-k} X_0(z) \Big(C_0 - \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t-z}\, dt \Big), \quad z \in S, \\ \Phi_2(z) &= X_{0,1}(z) \Big(C_0 + \frac{1}{2\pi i} \int_L \frac{\varphi(\alpha^{-1}(t))}{t-z}\, dt \Big), \quad z \in D, \end{aligned} \tag{5}$$

where C_0 is an arbitrary complex constant.

For the purpose of finding the conditions of solvability, we shall expand the function

$$\Phi_*(z) = C_0 - \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}\,dt$$

in Taylor's series in the neighborhood of the point $z = 0$. Thus, the function

$$\Phi_1(z) = z^{-k}X_0(z)\biggl(C_0 - \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt - \sum_{j=1}^{\infty}\frac{z^j}{2\pi i}\int_L \frac{\varphi(t)}{t^{j+1}}\,dt\biggr), \quad z \in S,$$

will be analytic in S if all the coefficients at $z^0, z^1, \dots, z^{k-1}$ in this expansion are equal to zero. If we choose

$$C_0 = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt$$

and separate the real and imaginary parts in the conditions

$$\int_L \frac{\varphi(t)}{t^{j+1}}\,dt = 0, \quad j = 1, 2, \dots, k-1,$$

we get the following $2(k-1)$ real conditions of solvability:

$$\operatorname{Re}\int_L \frac{\varphi(t)}{t^{j+1}}\,dt = 0, \quad \operatorname{Im}\int_L \frac{\varphi(t)}{t^{j+1}}\,dt = 0, \quad j = 1, 2, \dots, k-1. \tag{6}$$

The unique solution in this case is given by (5). In such a way we have proved the following theorem.

THEOREM. *If the index $k = \frac{1}{2}[\arg G(t)]_L > 0$, then the problem (1) is solvable and its unique solution is given by (5) if and only if all the $2(k-1)$ real conditions of solvability (6) hold. If, however, index $k \leqslant 0$, then the boundary value problem (1) is solvable and its solution, represented by the formula (4), contains $2(-k+1)$ arbitrary real constants.*

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