МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 99–107

UDK 519.633.6
оригинални научни рад
research paper

APPLICATION OF INTERPOLATION THEORY TO THE ANALYSIS OF THE CONVERGENCE RATE FOR FINITE DIFFERENCE SCHEMES OF PARABOLIC TYPE

Dejan Bojović and Boško S. Jovanović

Abstract. In this paper we show how the theory of interpolation of function spaces can be used to establish convergence rate estimates for finite difference schemes. As a model problem we consider THE first initial-boundary value problem for the heat equation with variable coefficients. We assume that the solution of the problem and the coefficients of the equation belong to the corresponding Sobolev spaces. Using interpolation theory we construct a fractional-order convergence rate estimate which is consistent with the smoothness of the data.

1. Introduction

For a class of finite difference schemes for parabolic initial-boundary value problems, the estimate of the convergence rate consistent with the smoothness of the data, are of major interest, i.e.

$$\|u-v\|_{W_2^{r,r/2}(Q_{h\tau})} \le C(h+\sqrt{\tau})^{s-r}\|u\|_{W_2^{s,s/2}(Q)}, \quad s \ge r. \tag{1}$$

Here $u=u(x,t)$ denotes the solution of the original initial-boundary value problem, v denotes the solution of the corresponding finite difference scheme, h and τ are discretisation parameters, $W_2^{s,s/2}(Q)$ denotes a Sobolev space, $W_2^{s,s/2}(Q_{h\tau})$ denotes the discrete Sobolev space, and C is a positive generic constant, independent of h, τ and u. For problems with variable coefficients the constant C depends on the norms of the coefficients.

A standard technique for the derivation of such estimates is based on the Bramble–Hilbert lemma [2]. In this paper we expose an alternative technique, based on the theory of interpolation of Banach spaces.

AMS Subject Classification: 65 M 15, 46 B 70

Keywords and phrases: Initial-boundary Value Problems, Finite Differences, Interpolation of Function Spaces, Sobolev Spaces, Convergence Rate Estimates.

Supported by MST of Serbia, grant number 04M03 / C

2. Interpolation of Banach spaces

Let A_1 and A_2 be two Banach spaces, linearly and continuosly imbedded in a topological linear space $\mathcal{A}$. Two such spaces are called *interpolation pair* $\{A_1, A_2\}$ (see [11]). Consider also the space $A_1 \cap A_2$, with the norm

$$\|a\|_{A_1\cap A_2} = \max\{\|a\|_{A_1}, \|a\|_{A_2}\},$$

and the space $A_1 + A_2 = \{a \in A : a = a_1 + a_2,\ a_i \in A_i,\ i = 1, 2\}$, with the norm

$$\|a\|_{A_1+A_2} = \inf_{\substack{a=a_1+a_2\\ a_i\in A_i}} \{\|a_1\|_{A_1} + \|a_2\|_{A_2}\}.$$

Obviously, $A_1 \cap A_2 \subset A_i \subset A_1 + A_2$, $i = 1, 2$.

Let us introduce category $\mathcal{C}_1$ whose objects $A, B, C, \ldots$ are Banach spaces, and morphisms—bounded linear operators $L \in \mathcal{L}(A, B)$. Let, also, $\mathcal{C}_2$ be a category whose objects are interpolation pairs $\{A_1, A_2\}$, $\{B_1, B_2\}, \ldots$ while morphisms are $L \in \mathcal{L}(\{A_1, A_2\}, \{B_1, B_2\})$. Here $\mathcal{L}(\{A_1, A_2\}, \{B_1, B_2\})$ denotes the set of bounded linear operators from $A_1 + A_2$ into $B_1 + B_2$, whose restrictions on A_i belong to the set $\mathcal{L}(A_i, B_i)$, $i = 1, 2$.

A functor $\mathcal{F}: \mathcal{C}_2 \to \mathcal{C}_1$ is called an interpolation functor if

$$A_1 \cap A_2 \subset \mathcal{F}(\{A_1, A_2\}) \subset A_1 + A_2$$

for every interpolation pair $\{A_1, A_2\}$, while for every morphism $L \in \mathcal{L}(\{A_1, A_2\}, \{B_1, B_2\})$, $\mathcal{F}(L)$ is the restriction of the operator L on $\mathcal{F}(\{A_1, A_2\})$.

The corresponding Banach space $A = \mathcal{F}(\{A_1, A_2\})$ is called *interpolation space.*

Note that $A_1 \cap A_2$ and $A_1 + A_2$ are interpolation spaces.

If the inequality

$$\|L\|_{\mathcal{F}(\{A_1,A_2\})\to\mathcal{F}(\{B_1,B_2\})} \le C \|L\|^{1-\theta}_{A_1\to B_1} \|L\|^{\theta}_{A_2\to B_2},$$

where $0 < \theta < 1$ and $C = \text{const} \ge 1$, is satisfied for every morphism L of category $\mathcal{C}_2$ the interpolation functor is said to be of the *type* θ. (Here $\|L\|_{A_i\to B_i}$ denotes the standard operator norm of $L: A_i \to B_i$, $i = 1, 2$).

One of the most often used interpolation methods is so called K-method [9,11]. Let $\{A_1, A_2\}$ be an interpolation pair. Define the functional

$$K(t, a) = K(t, a, A_1, A_2) = \inf_{\substack{a\in A_1+A_2\\ a=a_1+a_2\\ a_i\in A_i}} \{\|a_1\|_{A_1} + t\|a_2\|_{A_2}\}$$

It is obvious, that for a fixed $t \in (0, \infty)$, $K(t, a)$ is a norm in $A_1 + A_2$, equivalent to the standard norm $\|a\|_{A_1+A_2}$. For $0 < \theta < 1$, $1 \le q < \infty$, let us define the space $(A_1, A_2)_{\theta,q}$ as follows:

$$(A_1, A_2)_{\theta,q} = \left\{ a \in A_1 + A_2 : \|a\|_{(A_1,A_2)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t, a)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\},$$

and for $q = \infty$

$$(A_1, A_2)_{\theta,\infty} = \left\{ a \in A_1 + A_2 : \|a\|_{(A_1,A_2)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\} .$$

The space $(A_1, A_2)_{\theta,q}$ defined in such a way is an interpolation space. The following relations hold:

$$\begin{aligned} &(A_1, A_2)_{\theta,q} = (A_2, A_1)_{1-\theta,q} \, , \\ &(A, A)_{\theta,q} = A \, , \\ &\|a\|_{(A_1,A_2)_{\theta,q}} \le C_{\theta,q} \|a\|_{A_1}^{1-\theta} \|a\|_{A_2}^{\theta} \, , \ \forall a \in A_1 \cap A_2. \end{aligned}$$

The corresponding interpolation functor $\mathcal{F}(\{A_1, A_2\}) = (A_1, A_2)_{\theta,q}$ is of the type θ, i.e.

$$\|L\|_{(A_1,A_2)_{\theta,q} \to (B_1,B_2)_{\theta,q}} \le \|L\|_{A_1 \to B_1}^{1-\theta} \|L\|_{A_2 \to B_2}^{\theta} .$$

An analogous assertion holds true for bilinear operators:

Lemma 1. *Let $A_1 \subset A_2$, $B_1 \subset B_2$ and $C_1 \subset C_2$ and let $L: A_2 \times B_2 \to C_2$ be a continuous bilinear form whose restriction on $A_1 \times B_1$ is a continuous mapping with values in C_1. Than L is a continuous mapping from $(A_1, A_2)_{\theta,p} \times (B_1, B_2)_{\theta,q}$ into $(C_1, C_2)_{\theta,r}$, $0 < \theta < 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \ge 0$, and*

$$\|L\|_{(A_1,A_2)_{\theta,p} \times (B_1,B_2)_{\theta,q} \to (C_1,C_2)_{\theta,r}} \le \|L\|_{A_1 \times B_1 \to C_1}^{1-\theta} \|L\|_{A_2 \times B_2 \to C_2}^{\theta} .$$

As an example of interpolation spaces, let us consider the Sobolev spaces W_p^s [1]. For noninteger positive s one sets

$$W_p^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n),$$

where B_{pp}^s is a Besov space [11].

For $0 \le s_1, s_2 < \infty$, $s_1 \ne s_2$, $0 < \theta < 1$, $1 \le q < \infty$ we have [11]:

$$\left(W_p^{s_1}(\mathbb{R}^n), W_p^{s_2}(\mathbb{R}^n) \right)_{\theta,q} = B_{p,q}^s(\mathbb{R}^n) \, , \ s = (1-\theta)s_1 + \theta s_2 .$$

In such a way, for $q = p$ and noninteger $s = (1-\theta)s_1 + \theta s_2$, we obtain

$$\left(W_p^{s_1}(\mathbb{R}^n), W_p^{s_2}(\mathbb{R}^n) \right)_{\theta,p} = W_p^s(\mathbb{R}^n) \, , \ s = (1-\theta)s_1 + \theta s_2 \, .$$

For $p = 2$ this relation holds without restrictions, i.e.:

$$(W_2^{s_1}(\mathbb{R}^n), W_2^{s_2}(\mathbb{R}^n))_{\theta,2} = W_2^s(\mathbb{R}^n) \quad \text{for all } s \in (s_1, s_2).$$

Hence, $W_2^s(\mathbb{R}^n)$ are interpolation spaces. The same result holds for the Sobolev spaces in a domain Ω with sufficiently smooth boundary.

Let us define anisotropic Sobolev spaces $W_2^{s,s/2}(Q)$, $Q = \Omega \times I$, $I = (0, T)$, as follows [4]:

$$W_2^{s,s/2}(Q) = L_2(I, W_2^s(\Omega)) \cap W_2^{s/2}(I, L_2(\Omega)),$$

with the norm

$$\|f\|_{W_2^{s,s/2}(Q)} = \left(\int_0^T \|f(t)\|^2_{W_2^s(\Omega)} dt + \|f\|^2_{W_2^{s/2}(I,L_2(\Omega))} \right)^{\frac{1}{2}} .$$

These spaces are interpolation spaces, too. For $s_1, s_2, r_1, r_2 \geq 0$, $0 < \theta < 1$, we have [7,11]

$$(W_2^{s_1,r_1}(Q), W_2^{s_2,r_2}(Q))_{\theta,2} = W_2^{s,r}(Q)\,,\ s = (1-\theta)s_1 + \theta s_2,\ r = (1-\theta)r_1 + \theta r_2.$$

3. Initial-boundary value problem and its aproximation

Let us consider, as a model problem, the first initial-boundary value problem for a parabolic equation with variable coefficient in the rectangular domain $Q = \Omega \times (0,T] = (0,1) \times (0,T]$:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}) = f\,, \qquad & (x,t) \in Q\ , \\ u = 0\,, \qquad & (x,t) \in \partial\Omega \times [0,T]\ , \\ u(x,0) = u_0(x)\,, \quad & x \in \Omega\ , \end{aligned} \tag{2}$$

We assume that the generalized solution of problem (2) belongs to the Sobolev space $W_2^{s,s/2}(Q)$, $2 \leq s \leq 4$, with right-hand side $f(x,t)$ which belongs to $W_2^{s-2,s/2-1}(Q)$. Consequently, the coefficient $a = a(x)$ belongs to the space of multipliers $M\left(W_2^{s-1,(s-1)/2}(Q)\right)$, i.e. it is sufficient that a belongs to the space $W_2^{s-1}(\Omega)$ [8].

Let ω be a uniform mesh in $\Omega = (0,1)$ with the step size h, $\bar\omega = \omega \cup \{0,1\} = \omega \cup \gamma$. Let θ_τ be a uniform mesh in $(0,T)$ with step size τ, $\theta_\tau^+ = \theta_\tau \cup \{T\}$, $\bar\theta_\tau = \theta_\tau \cup \{0,T\}$. We define the following uniform mesh in Q: $Q_{h\tau} = \omega \times \theta_\tau$, $Q_{h\tau}^+ = \omega \times \theta_\tau^+$ and $\overline{Q}_{h\tau} = \bar\omega \times \bar\theta_\tau$. We assume that the condition:

$$k_1 h^2 \leq \tau \leq k_2 h^2\,, \quad k_1, k_2 = \text{const} > 0$$

is satisfied. We define finite differences in the usual manner:

$$v_x = \frac{v^+ - v}{h} = v_{\bar x}^+\ ,\ v_{x\bar x} = \frac{v^+ - 2v + v^-}{h^2}\ ,\ \text{where } v^\pm(x,t) = v(x \pm h, t)\,,$$

$$v_t(x,t) = \frac{v(x,t+\tau) - v(x,t)}{\tau} = v_{\bar t}(x, t+\tau).$$

We also define the Steklov smoothing operators:

$$T_x^+ f(x,t) = \int_0^1 f(x + hx', t)\, dx' = T_x^- f(x+h, t)\,,$$

$$T_x^2 f(x,t) = T_x^+ T_x^- f(x,t) = \int_{-1}^1 (1 - |x'|) f(x + hx', t)\, dx'\,,$$

$$T_t^+ f(x,t) = \int_0^1 f(x, t + \tau t')\, dt' = T_t^- f(x, t+\tau)\,.$$

These operators commute with derivatives and transform derivatives into differences:

$$T_x^2\left(D_x^2 f(x,t)\right) = D_x^2\left(T_x^2 f(x,t)\right) = f_{x\bar{x}}(x,t)\,,$$
$$T_t^-\left(D_t f(x,t)\right) = D_t\left(T_t^- f(x,t)\right) = f_{\bar{t}}(x,t)\,,\quad \text{etc.}$$

We approximate problem (2) with the following finite–difference scheme:

$$\begin{aligned} v_{\bar{t}} + L_h v &= T_x^2 T_t^- f\,, && \text{in } \ Q_{h\tau}^+\,,\\ v &= 0\,, && \text{on } \ \gamma\times\bar{\theta}_\tau\,,\\ v &= u_0\,, && \text{on } \ \omega\times\{0\}\,, \end{aligned} \tag{3}$$

where

$$L_h v = -\frac{1}{2}((av_x)_{\bar{x}} + (av_{\bar{x}})_x)\,.$$

The finite-difference scheme (3) is the the standard symmetric scheme with the averaged right-hand side. Note that for $s \le 3.5$ the right-hand side may be discontinuous function, so without averaging the scheme is not well defined.

4. Convergence of the finite-difference scheme

Let u be the solution of the initial-boundary value problem (2) and v—the solution of the finite difference scheme (3). The error $z = u - v$ satisfies the conditions

$$\begin{aligned} z_{\bar{t}} + L_h z &= \eta + \varphi\,, && \text{in } \ Q_{h\tau}^+\,,\\ z &= 0\,, && \text{on } \ \omega\times\{0\}\,,\\ z &= 0\,, && \text{on } \ \gamma\times\bar{\theta}_\tau\,, \end{aligned} \tag{4}$$

where

$$\eta = T_x^2 T_t^-\left(D_x(aD_x u)\right) - \frac{1}{2}\left((au_x)_{\bar{x}} + (au_{\bar{x}})_x\right) \quad \text{and} \quad \varphi = u_{\bar{t}} - T_x^2 u_{\bar{t}}\,.$$

We define the discrete inner products

$$(v,w)_\omega = (v,w)_{L_2(\omega)} = h\sum_{x\in\omega} v(\cdot,t)w(\cdot,t)\,,$$

where $v(\cdot,t) = v(x,t),\ (x,t)\in\omega\times\{t\},\ t\in\theta_\tau^+$–fixed,

$$(v,w)_{Q_{h\tau}} = (v,w)_{L_2(Q_{h\tau})} = h\tau\sum_{x\in\omega}\sum_{t\in\theta_\tau^+} v(x,t)w(x,t) = \tau\sum_{t\in\theta_\tau^+}(v,w)_\omega\,,$$

and the discrete Sobolev norms

$$\|v\|_\omega^2 = (v,v)_\omega\,,\quad \|v\|_{Q_{h\tau}}^2 = (v,v)_{Q_{h\tau}}\,,$$
$$\|v\|_{W_2^{2,1}(Q_{h\tau})}^2 = \|v\|_{Q_{h\tau}}^2 + \|v_x\|_{Q_{h\tau}}^2 + \|v_{x\bar{x}}\|_{Q_{h\tau}}^2 + \|v_{\bar{t}}\|_{Q_{h\tau}}^2\,.$$

The following assertion holds true:

LEMMA 2. *Finite-difference scheme (4) satisfies a priori estimate*

$$\|z\|_{W_2^{2,1}(Q_{h\tau})} \le \|\eta\|_{Q_{h\tau}} + \|\varphi\|_{Q_{h\tau}} \,. \tag{5}$$

In such a way, the problem of deriving a convergence rate estimate for the finite-difference scheme (3) is now reduced to estimating the right-hand side terms in (5).

Let us derive an estimate (1) for $s = r = 2$. We decompose η in the following manner:

$$\begin{aligned}\eta &= T_x^2 T_t^-(D_x(aD_x)) - \frac{1}{2}((au_{\bar{x}})_x + (au_x)_{\bar{x}}) \\ &= T_x^2(aT_t^- D_x^2 u) + T_x^2(D_x a T_t^- D_x u) - au_{x\bar{x}} - \frac{1}{2}(a_{\bar{x}} u_x^- + a_x u_{\bar{x}}^+) = \sum_{k=1}^{4} \eta_k\end{aligned}$$

where:

$$\begin{aligned}&\eta_1 = T_x^2(aT_t^- D_x^2 u)\,, \quad \eta_2 = T_x^2(D_x a T_t^- D_x u)\,, \\ &\eta_3 = -au_{x\bar{x}}\,, \quad \eta_4 = -\frac{1}{2}(a_{\bar{x}} u_x^- + a_x u_{\bar{x}}^+)\,.\end{aligned}$$

The value η_1 in the node $(\cdot, t) \in \omega \times \{t\}$ can be represented in the form

$$\eta_1(\cdot, t) = \frac{1}{h} \int_{x-h}^{x+h} \left(1 - \frac{|\xi - x|}{h}\right) a(\xi) T_t^- \frac{\partial^2 u(\xi, t)}{\partial x^2}\, d\xi \,.$$

Applying the Cauchy-Schwartz inequality we obtain

$$|\eta_1(\cdot, t)| \le \frac{C}{h^{1/2}} \left(\int_{x-h}^{x+h} \left| a(\xi) T_t^- \frac{\partial^2 u(\xi, t)}{\partial x^2} \right|^2 d\xi \right)^{\frac{1}{2}} .$$

From here, summing over the mesh ω, we obtain:

$$\|\eta_1(\cdot, t)\|_\omega \le C \|a\|_{C(\overline{\Omega})} \|T_t^- u(\cdot, t)\|_{W_2^2(\Omega)} \,.$$

Using the imbedding $W_2^1(\Omega) \subseteq C(\overline{\Omega})$ we have

$$\|\eta_1(\cdot, t)\|_\omega \le C \|a\|_{W_2^1(\Omega)} \|T_t^- u(\cdot, t)\|_{W_2^2(\Omega)} \,.$$

Summation over the mesh θ_τ^+ yields:

$$\|\eta_1\|_{Q_{h\tau}} \le C \|a\|_{W_2^1(\Omega)} \|u\|_{W_2^{2,1}(Q)} \,.$$

Analogous estimates hold true also for other terms η_k and for term φ. In such a way we obtain the estimates:

$$\|\eta\|_{Q_{h\tau}} \le C \|a\|_{W_2^1(\Omega)} \|u\|_{W_2^{2,1}(Q)} \,, \quad \text{and} \tag{6}$$

$$\|\varphi\|_{Q_{h\tau}} \le C \|u\|_{W_2^{2,1}(Q)} \,. \tag{7}$$

From (5), (6) and (7) we obtain estimate (1) for $s = r = 2$.

Let us derive estimate (1) for $s = 4$, $r = 2$. We decompose term η in the following manner: $\eta = \sum_{k=5}^{11} \eta_k$, where

$$\begin{aligned}
\eta_5 &= T_x^2(aT_t^- D_x^2 u) - (T_x^2 a)(T_x^2 T_t^- D_x^2 u)\,,\\
\eta_6 &= (T_x^2 a - a)(T_x^2 T_t^- D_x^2 u)\,,\\
\eta_7 &= a(T_x^2 T_t^- D_x^2 u - u_{x\bar{x}})\,,\\
\eta_8 &= T_x^2(D_x a T_t^- D_x u) - (T_x^2 D_x a)(T_x^2 T_t^- D_x u)\,,\\
\eta_9 &= (T_x^2 D_x a - 0.5(a_x + a_{\bar{x}}))(T_x^2 T_t^- D_x u)\,,\\
\eta_{10} &= 0.5(a_x + a_{\bar{x}})(T_x^2 T_t^- D_x u - 0.5(u_x^- + u_{\bar{x}}^+))\,,\\
\eta_{11} &= 0.25(a_x - a_{\bar{x}})(u_x^- - u_{\bar{x}}^+)\,.
\end{aligned}$$

The value of η_5 in the node $(\cdot, t) \in \omega \times \{t\}$ can be represented in the form

$$\eta_5(\cdot,t) = \frac{1}{h^2}\int_{x-h}^{x+h}\int_{x-h}^{x+h}\int_{\sigma}^{\xi}\int_{\sigma}^{\xi}\left(1 - \frac{|\xi - x|}{h}\right)\left(1 - \frac{|\sigma - x|}{h}\right)\times$$
$$\times\, a'(\rho)T_t^- \frac{\partial^3 u(\rho_1, t)}{\partial x^3}\, d\rho_1 d\rho d\sigma d\xi\ .$$

From here, using the Cauchy-Schwartz inequality we obtain

$$|\eta_5(\cdot,t)| \le Ch^{3/2}\|a\|_{W_\infty^1(\Omega)}\|T_t^- u(\cdot,t)\|_{W_2^3(x-h,x+h)}\,.$$

Summation over the mesh ω yields:

$$\|\eta_5(\cdot,t)\|_\omega \le Ch^2\|a\|_{W_\infty^1(\Omega)}\|T_t^- u(\cdot,t)\|_{W_2^3(\Omega)}\,.$$

Using the imbedding $W_2^3(\Omega) \subset W_\infty^1(\Omega)$ we obtain

$$\|\eta_5(\cdot,t)\|_\omega \le Ch^2\|a\|_{W_2^3(\Omega)}\|T_t^- u(\cdot,t)\|_{W_2^3(\Omega)}\,.$$

From here, summing over the mesh θ_τ^+ we obtain

$$\|\eta_5\|_{Q_{h\tau}} \le Ch^2\|a\|_{W_2^3(\Omega)}\|u\|_{W_2^{4,2}(Q)}\,.$$

The value of η_7 in the node $(x,t) \in Q_{h\tau}^+$ can be represented in the form:

$$\eta_7(x,t) = \frac{1}{h\tau}a(x)\int_{x-h}^{x+h}\int_{t-\tau}^{t}\int_{t}^{\nu}\left(1 - \frac{|\xi - x|}{h}\right)\frac{\partial^3 u(\xi,\nu_1)}{\partial x^2 \partial t}\, d\nu_1 d\nu d\xi\,.$$

From here, using the Cauchy-Schwartz inequality we have

$$|\eta_7(x,t)| \le C\left(\frac{\tau}{h}\right)^{1/2}\|a\|_{C(\overline{\Omega})}\left(\int_{x-h}^{x+h}\int_{t-\tau}^{t}\left|\frac{\partial^3 u(\xi,\nu)}{\partial x^2\partial t}\right|^2 d\nu d\xi\right)^{\frac{1}{2}}\,.$$

Summation over the mesh $Q_{h\tau}^+$ yields:

$$\|\eta_7\|_{Q_{h\tau}} \le C\tau\|a\|_{C(\overline{\Omega})}\left\|\frac{\partial^3 u}{\partial x^2\partial t}\right\|_{L_2(Q)}\,.$$

Using the imbedding $W_2^3(\Omega) \subseteq C(\overline{\Omega})$ and the imbedding theorems for anisotropic spaces $W_2^{s,s/2}(Q)$ [4] we have

$$\|\eta_7\|_{Q_{h\tau}} \le Ch^2 \|a\|_{W_2^3(\Omega)} \|u\|_{W_2^{4,2}(Q)} \,.$$

Analogous estimates hold true also for other terms η_k and for term φ. In such a way we obtain the estimates:

$$\|\eta\|_{Q_{h\tau}} \le Ch^2 \|a\|_{W_2^3(\Omega)} \|u\|_{W_2^{4,2}(Q)} \,, \quad \text{and} \tag{8}$$

$$\|\varphi\|_{Q_{h\tau}} \le Ch^2 \|u\|_{W_2^{4,2}(Q)} \,. \tag{9}$$

From (5), (8) and (9) we obtain estimate (1) for $s = 4$, $r = 2$.

Let us define the operators A_1 and A_2 as follows:

$$\eta = A_1(a,u) \quad \text{and} \quad \varphi = A_2(u)\,.$$

The operator A_1 is, obviously, bilinear. From (6) it follows that A_1 is a bounded bilinear operator from $W_2^1(\Omega) \times W_2^{2,1}(Q)$ to $L_2(Q_{h\tau})$, and

$$\|A_1\|_{W_2^1(\Omega)\times W_2^{2,1}(Q)\to L_2(Q_{h\tau})} \le C\,. \tag{10}$$

From (8) it follows that A_1 is a bounded bilinear operator from $W_2^3(\Omega) \times W_2^{4,2}(Q)$ to $L_2(Q_{h\tau})$, and

$$\|A_1\|_{W_2^3(\Omega)\times W_2^{4,2}(Q)\to L_2(Q_{h\tau})} \le Ch^2\,. \tag{11}$$

Applying lemma 1, from (10) and (11) it follows that A_1 is a bounded bilinear operator from

$$\big(W_2^3(\Omega), W_2^1(\Omega)\big)_{\theta,2} \times \big(W_2^{4,2}(Q), W_2^{2,1}(Q)\big)_{\theta,2} = W_2^{3-2\theta}(\Omega) \times W_2^{4-2\theta,2-\theta}(Q)$$

to

$$\big(L_2(Q_{h\tau}), L_2(Q_{h\tau})\big)_{\theta,\infty} = L_2(Q_{h\tau})\,,$$

and

$$\|A_1\|_{W_2^{3-2\theta}(\Omega)\times W_2^{4-2\theta,2-\theta}(Q)\to L_2(Q_{h\tau})} \le Ch^{2-2\theta}\,, \quad 0<\theta<1. \tag{12}$$

Finally, from (12) and the inequality

$$\|\eta\|_{Q_{h\tau}} \le \|A_1\|_{W_2^{3-2\theta}(\Omega)\times W_2^{4-2\theta,2-\theta}(Q)\to L_2(Q_{h\tau})} \|a\|_{W_2^{3-2\theta}(\Omega)} \|u\|_{W_2^{4-2\theta,2-\theta}(Q)}\,,$$

we obtain the estimate

$$\|\eta\|_{Q_{h\tau}} \le Ch^{2-2\theta} \|a\|_{W_2^{3-2\theta}(\Omega)} \|u\|_{W_2^{4-2\theta,2-\theta}(Q)}\,, \quad 0<\theta<1\,. \tag{13}$$

Analogously, we obtain the following estimate of term φ:

$$\|\varphi\|_{Q_{h\tau}} \le Ch^{2-2\theta} \|u\|_{W_2^{4-2\theta,2-\theta}(Q)}\,, \quad 0<\theta<1\,. \tag{14}$$

Setting $4-2\theta = s$, we obtain the estimates:

$$\|\eta\|_{Q_{h\tau}} \le Ch^{s-2} \|a\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^{s,s/2}(Q)}\,, \tag{15}$$

$$\|\varphi\|_{Q_{h\tau}} \le Ch^{s-2} \|u\|_{W_2^{s,s/2}(Q)}\,, \quad 2<s<4\,. \tag{16}$$

Finally, from (6)–(9), (15), (16) and (5) we obtain the main result of this paper:

THEOREM. *Finite-difference scheme (3) converges in the norm of the space* $W_2^{2,1}(Q_{h\tau})$ *and, with condition* $k_1 h^2 \le \tau \le k_2 h^2$*, the following estimate holds true:*

$$\|u-v\|_{W_2^{2,1}(Q_{h\tau})} \le C h^{s-2}(\|a\|_{W_2^{s-1}(\Omega)}+1)\|u\|_{W_2^{s,s/2}(Q)}, \qquad 2 \le s \le 4 .$$

This estimate is consistent with the smoothness of data.

REFERENCES

[**1**] R. A. Adams, *Sobolev Spaces.* Academic Press, New York 1975.

[**2**] J. H. Bramble, S. R. Hilbert, *Bounds for a class of linear functionals with application to Hermite interpolation.* Numer. Math. **16** (1971), 362–369.

[**3**] M. Dražić, *Convergence rates of difference approximations to weak solutions of the heat transfer equation.* Oxford University Computing Laboratory, Numerical Analysis Group, Report No 86/22, Oxford 1986.

[**4**] B. S. Jovanović, *The finite difference method for boundary value problems with weak solutions.* Posebna izdanja Mat. Instituta, No **16**, Beograd 1993.

[**5**] B. S. Jovanović, *Interpolation of function spaces and the convergence rate estimates for the finite difference schemes.* Second International Colloquium on Numerical Analysis, Plovdiv 1993, (D. Bainov and V. Covachev, eds.), VPS, Utrecht 1994, 103–112.

[**6**] O. A. Ladyzhenskaya, N. N. Solonnikov, N. N. Ural'ceva, *Linear and Quasilinear Parabolic Equations.* Nauka, Moskow 1967. (in Russian)

[**7**] J. L. Lions, E. Magenes, *Problèmes aux limities non homogènes et applications.* Dunod, Paris 1968.

[**8**] V. G. Maz'ya, T. O. Shaposhnikova, *Theory of multipliers in spaces of diferentiable functions.* Monographs and Studies in Mathematics 23, Pitman, Boston, Mass. 1985.

[**9**] J. Peetre, *A Theory of Interpolation of Normed Spaces.* Lecture Notes, Brasilia (1963) [Notas de Matematica 39 (1968)].

[**10**] A. A. Samarskii, *Theory of Difference Schemes.* Nauka, Moscow 1983. (in Russian)

[**11**] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* Deutscher Verlag der Wissenschaften, Berlin 1978.

[**12**] A. A. Zlotnik, *Convergence rate estimates of projection difference methods for second-order hyperbolic equations.* Vychisl. protsessy sist. **8** (1991), 116–167. (in Russian)

(received 08.10.1996.)

University of Kragujevac, Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Yugoslavia