МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 153–162

UDK 517.982.4
оригинални научни рад
research paper

α-TIMES INTEGRATED SEMIGROUPS ($\alpha \in \mathbf{R}^-$)

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Abstract. The α-times integrated semigroups, $\alpha \in \mathbf{R}^- = (-\infty, 0]$, are introduced and analyzed as extensions of 0-integrated semigroups.

0. Introduction

We introduce and analyze α-times integrated semigroups, $\alpha \in \mathbf{R}^-$. With $\alpha \in \mathbf{N}$ this type of semigroups is extensively investigated in many papers, see for example [1], [2], [4], [5], [6], [11], [17]; for $\alpha > 0$ we refer to [6], [10].

In this paper we apply results concerning 0-integrated semigroups [9] and analyze families of operators on the test space $\mathcal{K}_1$ with values in $L(E,E)$ which are n-th distributional derivatives of α-times integrated semigroup for $\alpha > 0$ sufficiently large and $n > \alpha$.

As an application, we consider the Cauchy problem $u' = Au + T$, $T \in \mathcal{K}_1'$ in the setting of α-times integrated semigroups $\alpha < 0$.

1. Preliminaries

By $L(E) = L(E,E)$ is denoted the space of bounded linear operators from a Banach space $(E, \|\cdot\|)$ into itself and $C(\mathbf{R}, L(E))$ is the space of continuous mappings from $\mathbf{R}$ into $L(E)$. We refer to [15] and [18] for the definitions of spaces $\mathcal{D}(\mathbf{R})$, $\mathcal{E}(\mathbf{R})$, $\mathcal{S}(\mathbf{R})$, their strong duals $\mathcal{S}'(E) = L(\mathcal{S}(\mathbf{R}), E)$ and to [20] for the space $\mathcal{S}_+ = \{\varphi;\ |t^k \varphi^{(\nu)}(t)| < C_{k,\nu}, t \in [0,\infty), k, \nu \in \mathbf{N}_0\}$ ($\mathbf{N}_0 = \mathbf{N} \cup \{0\}$) and its dual $\mathcal{S}_+'$ which consists of tempered distributions supported by $[0,\infty)$.

The space of exponentially decreasing test functions on the real line $\mathbf{R}$ is defined by $\mathcal{K}_1(\mathbf{R}) = \{\varphi;\ |e^{k|t|} \varphi^{(\nu)}(t)| < C_{k,\nu}, t \in \mathbf{R}, k, \nu \in \mathbf{N}_0\}$ ([3]). This space has the same topological properties as $\mathcal{S}(\mathbf{R})$. The space $\mathcal{K}_1(\mathbf{R}^2)$ is defined in an appropriate way. The strong dual of $\mathcal{K}_1(\mathbf{R})$, $\mathcal{K}_1'(\mathbf{R})$ is the sapce of exponential distributions. The space $\mathcal{K}_{1+}' \subset \mathcal{K}_1'(\mathbf{R})$ consists of distributions which are supported by $[0,\infty)$.

AMS Subject Classification: 46 F 99

Communicated at the 4th Symposium on Mathematical Analysis and Its Applications, Aranđelovac 1997.

It is the dual space to $\mathcal{K}_{1+} = \{\varphi;\ |e^{k|t|}\varphi^{(\nu)}(t)| < C_{k,\nu}, t \in [0,\infty), k,\nu \in \mathbf{N}_0\}$ which has the same topological properties as $\mathcal{S}_+$. Note,

$$f \in \mathcal{K}_1'(\mathbf{R}) \text{ if and only if } e^{-r|x|} f \in \mathcal{S}'(\mathbf{R}) \text{ for some } r \in \mathbf{R}. \tag{1}$$

The space $\mathcal{K}_1'(E)$ consists of continuous linear mappings $S : \mathcal{K}_1 \to E$ with the strong topology. Similarly $\mathcal{K}_{1+}'(E)$ is defined; we have $\mathcal{K}_{1+}'(E) \subset \mathcal{K}_1'(E)$.

The convolution of $f \in \mathcal{K}_{1+}'(E)$ and $g \in \mathcal{K}_{1+}'$ is defined by $\langle f * g, \varphi\rangle = \langle f, \check{g} * \varphi\rangle$, $\varphi \in \mathcal{K}_1(\mathbf{R})$ $(\check{g}(t) = g(-t))$. One can prove easily that $f * g = g * f \in \mathcal{K}_{1+}'(E)$.

Let $T\colon [0,\infty) \to L(E)$ be strongly continuous. Then it is exponentially bounded at infinity if there exist $M \geqslant 0$ and $\omega \geqslant 0$ such that

$$\|T(t)\| \leqslant M e^{\omega t}, \qquad t \geqslant 0. \tag{2}$$

In this case $\varphi \mapsto \int_0^\infty T(t)\varphi(t)\,dt$, $\varphi \in \mathcal{K}_1(\mathbf{R})$, defines an element of $\mathcal{K}_{1+}'(L(E))$.

The structure of $\mathcal{K}_{1+}'(L(E))$ is given in the following theorem.

Theorem 1. [9] *Let* $S \in \mathcal{K}_{1+}'(L(E))$.

a) There exists $n_0 \in \mathbf{N}$ *such that for every* $n \geqslant n_0$ *there exist a strongly continuous function* $F_n\colon \mathbf{R} \to L(E)$, $\operatorname{supp} F_n \subset [0,\infty)$ *and positive constants* m_n *and* C_n, *such that*

$$\|F_n(t)\| \leqslant C_n e^{m_n t}, \quad t \geqslant 0,\ S = F_n^{(n)} \quad ({}^{(n)} \text{ is the distributional } n\text{-th derivative}).$$

b) Let $S \in \mathcal{K}_{1+}'(L(E))$ *and* $\psi, \varphi \in \mathcal{K}_1(\mathbf{R})$. *Then*

$$\langle S(t, \langle S(s,x), \psi(s)\rangle), \varphi(t)\rangle = \int F_{n_0}(t, F_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0)}(t)\,ds\,dt. \tag{3}$$

c) Let $\varphi(t,s) \in \mathcal{K}_1(\mathbf{R}^2)$ *and* $\varphi_\nu(t)$, $\psi_\nu(s)$ *be sequences in* $\mathcal{D}(\mathbf{R})$ *such that the product sequence* $\varphi_\nu(t) \cdot \psi_{(\nu)}(s)$ *converges to* $\varphi(t,s)$ *in* $\mathcal{K}_1(\mathbf{R}^2)$ *as* $\nu \to \infty$. *Then the limit*

$$\lim_{\nu\to\infty} \langle S(t, \langle S(s,x), \psi_\nu(s)\rangle), \varphi_\nu(t)\rangle$$

exists and defines an element of $\mathcal{K}_1'(\mathbf{R}^2)$ *which we denote by* $S(t, S(s,x))$, *i.e.*

$$\langle S(t, S(s,x)), \varphi(t,s)\rangle = \lim_{\nu\to\infty} \langle S(t, \langle S(s,x), \psi_\nu(s)\rangle), \varphi_\nu(t)\rangle, \quad \varphi \in \mathcal{K}_1(\mathbf{R}^2). \tag{4}$$

d) Also, for $\varphi \in \mathcal{K}_1(\mathbf{R}^2)$ *and* $r, p \in \mathbf{N}$, *we have*

$$(i) \quad \left\langle \frac{\partial^r}{\partial t^r} S(t, S(s,x)), \varphi(t,s) \right\rangle = (-1)^r \left\langle S(t, S(s,x)), \frac{\partial^r}{\partial t^r}\varphi(t,s) \right\rangle;$$

$$(ii) \quad \left\langle \frac{\partial^p}{\partial s^p} S(t, S(s,x)), \varphi(t,s) \right\rangle = \left\langle S\left(t, \frac{\partial^p}{\partial s^p} S(s,x)\right), \varphi(t,s) \right\rangle = (-1)^p \left\langle S(t, S(s,x)), \frac{\partial^p}{\partial s^p}\varphi(t,s) \right\rangle.$$

As in the case of ordinary distributions (1) we have

$$f \in \mathcal{K}'_{1+}(L(E)) \text{ if and only if } e^{-r|x|} f \in \mathcal{S}'_{+}(L(E)) \text{ for some } r \geqslant 0. \tag{5}$$

The Laplace transformation of an f satisfying (5) is defined by

$$\mathcal{L}(f)(\lambda) = \hat{f}(\lambda) = \left\langle f(t), e^{-\lambda t}\eta(t) \right\rangle, \quad \operatorname{Re}\lambda > r,$$

where $\eta \in C^\infty(\mathbf{R})$, $\operatorname{supp}\eta = [-\varepsilon, \infty)$, $\varepsilon > 0$ and $\eta \equiv 1$ on $[0,\infty)$. This definition does not depend on η (cf. [20]). If $f \in L^1([0,\infty), E)$ (which means $\| \int_0^\infty f(t)\,dt\|_E < \infty$), then

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt = \left\langle f(t), e^{-\lambda t}\right\rangle, \qquad \operatorname{Re}\lambda > 0,$$

where the integral is taken in Bochner's sense.

2. α-times integrated semigroup

Let $T\colon (0,\infty) \to L(E)$ be strongly continuous, integrable in a neighborhood of 0 (i.e. integrable on $(0,\varepsilon)$ for some $\varepsilon > 0$) and exponentially bounded at infinity, which means that (2) holds on (ε,∞) for some $\varepsilon > 0$. The operator $R\colon \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > \omega\} \to L(E)$ defined by

$$R(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\,dt, \qquad \operatorname{Re}\lambda > \omega,$$

where the integral is understood in Bochner's sense, is the Laplace transformation of T.

The family $R(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\,dt$, $\operatorname{Re}\lambda > \omega$, where $T\colon [0,\infty) \to L(E)$ is a strongly continuous and exponentially bounded function, is a pseudoresolvent iff $T(t)T(s) = T(t+s)$, $t, s \geqslant 0$. Let $\alpha > 0$ and $S\colon (0,\infty) \to L(E)$ be strongly continuous, integrable in a neighborhood of 0, exponentially bounded at infinity and

$$R(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)\,dt, \qquad \Re\lambda > \omega.$$

Then $(R(\lambda))_{\operatorname{Re}\lambda > \omega}$ is a pseudoresolvent iff

$$S(t)S(s) = \frac{1}{\Gamma(\alpha)}\left[\int_t^{t+s} (t+s-r)^{\alpha-1} S(r)\,dr - \int_0^s (t+s-r)^{\alpha-1} S(r)\,dr\right], \quad t, s \geqslant 0 \tag{6}$$

(cf. [2], [10]).

Recall,

$$f_\alpha(t) = \begin{cases} \dfrac{H(t)t^{\alpha-1}}{\Gamma(\alpha)}, & t \in \mathbf{R}, \alpha > 0, \\ f_{\alpha+n}^{(n)}, & t \in \mathbf{R}, \alpha \leqslant 0, \alpha + n > 0, n \in \mathbf{N}, \end{cases} \tag{7}$$

(H is Heaviside's function).

Theorem 2. *Let* $\alpha \in \mathbf{R}^-$, $S_\alpha \in \mathcal{K}'_{1+}(L(E))$ *and* $R(\lambda) = \lambda^\alpha \mathcal{L}(S_\alpha)(\lambda)$. *Then* $(R(\lambda))_{\mathrm{Re}\,\lambda>\omega}$ *is a pseudoresolvent iff there exists* $n_0 \in \mathbf{N}$ *such that* $n_0 + \alpha > 0$ *and*

$$S_{n_0+\alpha}(t,\cdot) = (S_\alpha * f_{n_0})(t,\cdot), \quad t \geqslant 0,$$

is continuous, $\operatorname{supp} S_{n_0+\alpha} \subset [0,\infty)$ *and satisfies*

$$\begin{aligned}\langle S_\alpha(t, S_\alpha(s,x)), \varphi(t)\psi(s)\rangle &= \left\langle (S_{n_0+\alpha}(t, S_{n_0+\alpha}(s,x)))^{(n_0,n_0)}, \varphi(t)\psi(s)\right\rangle \\ &= \left\langle \frac{1}{\Gamma(n_0+\alpha)} \left(\int_t^{t+s} (t+s-r)^{n_0+\alpha-1} S_{n_0+\alpha}(r,x)\,dr - \right.\right. \\ &\qquad \left.\left. - \int_0^s (t+s-r)^{n_0+\alpha-1} S_{n_0+\alpha}(r,x)\,dr \right)^{(n_0,n_0)}, \varphi(t)\psi(s) \right\rangle \end{aligned} \tag{8}$$

for every $\varphi, \psi \in \mathcal{K}_1(\mathbf{R})$.

Moreover, (8) holds with $S_{n+\alpha} = S_\alpha * f_n$, *for every* $n \geqslant n_0$.

Remark. [9] If $\alpha = 0$, then (8) is equivalent to

$$\langle S_0(t, S_0(s,x)), \varphi(t,s)\rangle = \langle S_0(t+s,x), \varphi(t,s)\rangle, \quad \varphi \in \mathcal{K}_1(\mathbf{R}^2).$$

Proof. We have $S_\alpha = S_{n_0+\alpha}^{(n_0)}$. Let $x \in E$. Relation (8) implies

$$\begin{aligned}(S_\alpha(t, S_\alpha(s,x))) &= (S_{n_0+\alpha}(t, S_{n_0+\alpha}(s,x)))^{(n_0,n_0)} = \\ &\left(\frac{H(t)H(s)}{\Gamma(n_0+\alpha)} \left[\int_t^{t+s} (t+s-r)^{n_0+\alpha-1} S_{n_0+\alpha}(r,x)\,dr - \right.\right. \\ &\qquad \left.\left. \int_0^s (t+s-r)^{n_0+\alpha-1} S_{n_0+\alpha}(r,x)\,dr \right] \right)^{(n_0,n_)}, \end{aligned} \tag{9}$$

$t, s > 0$, in the distributional sense. Since both sides are supported by $[0,\infty) \times [0,\infty)$, it follows that

$$\begin{aligned}(S_\alpha(t, S_\alpha(s,x))) &= S_{n_0+\alpha}(t, S_{n_0+\alpha}(s,x)) = \\ &\frac{1}{\Gamma(n_0+\alpha)} \left[\int_t^{t+s} (t+s-r)^{n_0+\alpha-1} S_{n_0+\alpha}(r,x)\,dr - \right. \\ &\qquad \left. \int_0^s (t+s-r)^{n_0+\alpha-1} S_{n_0+\alpha}(r,x)\,dr \right]\end{aligned}$$

holds true for every $t, s \geqslant 0$. Thus, $R(\lambda,\cdot) = \lambda^{n_0+\alpha}\mathcal{L}(S_{n_0+\alpha})(\lambda,\cdot)$ is a pseudoresolvent. Let $n \geqslant n_0$. Since $S_{n+\alpha} = S^{n-n_0}_{n_0+\alpha+(n-n_0)}$, it follows that (8) holds for every $n \geqslant n_0$. ∎

Definition 1. Let $(S(t))_{t\geqslant 0}$ be a strongly continuous exponentially bounded family in $L(E)$ and $\alpha > 0$. Then it is called an *α-times integrated semigroup* if (6) is satisfied and $S(0) = 0$ ([10]).

Let $S_\alpha \in \mathcal{K}'_{1+}(L(E))$ and $\alpha \in \mathbf{R}^-$. Then, S_α is called an *α-times integrated semigroup* if there exists $n_0 \in \mathbf{N}$, such that $n_0 + \alpha > 0$, $S_{n_0+\alpha} = S_\alpha * f_{n_0}$ is continuous on $\mathbf{R}$, supported by $[0,\infty)$, exponentially bounded and satisfies (8). This is equivalent to say that, for some n_0 and every $n \geqslant n_0$, it is an n-th distributional derivative of an $n+\alpha$-times integrated semigroup.

We will use the symbol $(S(t))_{t\geqslant 0}$ or $(S_\alpha(t))_{t\geqslant 0}$ for an α-times integrated semigroup if it is not specified whether $\alpha > 0$ or $\alpha \leqslant 0$, although for $\alpha \leqslant 0$ it is an element of $\mathcal{K}'_{1+}(L(E))$ and the above expression is formal.

DEFINITION 2. Let $\alpha > 0$. Then, $(S(t))_{t\geqslant 0}$ with the above properties is called *non-degenerate* if $S(t)x = 0$ for all $t \geqslant 0$, implies $x = 0$ ([10]). Let $\alpha \leqslant 0$. Then $S \in \mathcal{K}'_{1+}(L(E))$ is called *non-degenerate* if $\langle S(t,x), \varphi(t)\rangle = 0$ for all $\varphi \in \mathcal{K}_1$ implies $x = 0$.

Note, C_0-semigroup is a 0-integrated semigroup ([9]). Also, if $(S(t))_{t\geqslant 0}$ is an n-times integrated semigroup, then n-th distributional derivative $S^{(n)}$ is a 0-integrated semigroup.

DEFINITION 3. Let $\alpha \in \mathbf{R}$. An operator A is the generator of an α-times integrated semigroup $(S(t))_{t\geqslant 0}$ iff $(a,\infty) \subset \rho(A)$ for some $a \in \mathbf{R}$ and the function $\lambda \mapsto \dfrac{(\lambda I - A)^{-1}}{\lambda^\alpha} = \mathcal{L}(S_\alpha)(\lambda)$, $\operatorname{Re}\lambda > a$, is injective, where the Laplace transformation is understood in ordinary sense for $\alpha > 0$ and in distributional sense for $\alpha \leqslant 0$.

Part b) of Theorem 1 and the above definition directly imply the next Proposition.

PROPOSITION 1. *a) Let S_α, $\alpha \in \mathbf{R}$ be an α-times integrated semigroup. Then $S_\alpha * f_{-\alpha}$ is a 0-integrated semigroup.*

b) Let $\alpha < 0$. Then A is the generator of an α-times integrated semigroup S_α iff A is the generator of a 0-integrated semigroup $S_\alpha * f_{-\alpha}$.

3. The properties of A

Let A be the generator of an α-times integrated semigroup $(S(t))_{t\geqslant 0}$, $\alpha > 0$. Recall ([2], [10]), for all $x \in D(A)$ and $t \geqslant 0$, $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$, $S(t)x = \dfrac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t S(s)Ax\,ds$. Moreover, $\int_0^t S(s)x\,dx \in D(A)$ for all $x \in E$, $t \geqslant 0$ and

$$A\int_0^t S(s)x\,ds = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x.$$

THEOREM 3. *Let $\alpha \in \mathbf{R}^-$ and A be a generator of an α-times integrated semigroup $(S(t))_{t\geqslant 0}$, $S \in \mathcal{K}'_{1+}(L(E))$. Then, for all $\varphi \in \mathcal{K}_1$, we have*

a) $A\langle S(t,x), \varphi(t)\rangle = \langle S(t,Ax), \varphi(t)\rangle$ for every $x \in D(A)$.

b) $\langle S(t,x), \varphi(t)\rangle \in D(A)$ for every $x \in E$.

c) $\langle S(t,x),\varphi(t)\rangle = \langle f_{\alpha+1}(t,x),\varphi(t)\rangle + \langle (f_1 * S)(t,Ax),\varphi(t)\rangle$, $x \in D(A)$ *and*

$$A\left\langle (f_1 * S)(t,x),\varphi(t)\right\rangle = \langle S(t,x),\varphi(t)\rangle - \langle f_{\alpha+1}(t,x),\varphi(t)\rangle\,, \quad x \in E. \tag{10}$$

Remark. if $\alpha = -1$, then (10) with $S = S_{-1}$, implies

$$A\left\langle (f_1 * S_{-1})(t,x),\varphi(t)\right\rangle = \langle S_{-1}(t,x),\varphi(t)\rangle - \langle \delta(t,x),\varphi(t)\rangle\,,$$

i.e.

$$A\left\langle S_0(t,x),\varphi(t)\right\rangle = \langle S_0(t,x),\varphi'(t)\rangle - \varphi(0)x, \quad x \in E, \varphi \in \mathcal{K}_1.$$

We will use also the notation $A\langle S(t,x),\varphi(t)\rangle = \langle AS(t,x),\varphi(t)\rangle$.

Proof. We will also use notation S_α for S. Let $\varphi \in \mathcal{D}(\mathbf{R})$ and $x \in D(A)$. Then

$$\langle S_\alpha(t,x),\varphi(t)\rangle = (-1)^{n_0}\left\langle S_{n_0+\alpha}(t,x),\varphi^{(n_0)}(t)\right\rangle, \quad n_0+\alpha > 0, n_0 \in \mathbf{N}$$

and Proposition 3.3 in [2] implies $S_{n_0+\alpha}(t,x) \in D(A)$ and $AS_{n_0+\alpha}(t,x) = S_{n_0+\alpha}(t,Ax)$. This and the continuity of A imply

$$A\langle S_\alpha(t,x),\varphi(t)\rangle = (-1)^{n_0} A\left\langle S_{n_0+\alpha}(t,x),\varphi^{(n_0)}(t)\right\rangle$$

$$= (-1)^{(n_0)} A\int S_{n_0+\alpha}(t,x)\varphi^{(n_0)}(t)\,dt = (-1)^{n_0} A\lim_{\nu\to\infty}\sum_{j=1}^{\nu} S_{n_0+\alpha}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j$$

$$= (-1)^{n_0}\lim_{\nu\to\infty}\sum_{j=1}^{\nu} AS_{n_0+\alpha}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j$$

$$= (-1)^{n_0}\left\langle AS_{n_0+\alpha}(t,x),\varphi^{(n_0)}(t)\right\rangle = \langle S_\alpha(t,Ax),\varphi(t)\rangle\,, \quad x \in E, \varphi \in \mathcal{D},$$

where $(\sum_{j=1}^{\nu} S_{n_0+\alpha}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j)$ is a sequence of integral sums for $\int S_{n_0+\alpha}(t,x)\varphi^{(n_0)}(t)\,dt$.

Let $\varphi \in \mathcal{K}_1$ and φ_ν be a sequence in $\mathcal{D}$ which converges to φ in $\mathcal{K}_1$. Then

$$A\langle S(t,x),\varphi(t)\rangle = \lim_{\nu\to\infty}\langle S(t,Ax),\varphi_\nu(t)\rangle = \langle S(t,Ax),\varphi(t)\rangle\,.$$

This implies the assertion.

b) Proposition 3.3 in [2] implies $\int_0^t S_{n_0}(s,x)\,ds \in D(A)$ for every $x \in E$. Thus, $\left\langle \int_0^t S_{n_0}(s,x)\,ds,\varphi(t)\right\rangle \in D(A)$ for every $\varphi \in \mathcal{K}_1$ and $x \in E$. We know that $\langle S_{n_0+\alpha},\varphi(t)\rangle \in D(A)$ for every $\varphi \in \mathcal{K}_1$. By putting $\varphi^{(n_0)}$ instead of φ, we obtain $\langle S(\cdot,x),\varphi\rangle \in D(A)$ for every $\varphi \in \mathcal{K}_1$.

c) Similarly, using Proposition 3.3 in [2], we obtain

$$\langle S_\alpha(t,x),\varphi(t)\rangle = (-1)^{n_0}\left\langle S_{n_0+\alpha}(t,x),\varphi^{(n_0)}(t)\right\rangle$$

$$= (-1)^{(n_0)}\left\langle f_{n_0+\alpha+1}(t,x),\varphi^{(n_0)}(t)\right\rangle + (-1)^{n_0}\left\langle (f_1 * S_{n_0+\alpha})(t,Ax),\varphi^{(n_0)}(t)\right\rangle$$

$$= \langle f_{\alpha+1}(t,x),\varphi(t)\rangle + \left\langle (f_1 * S_{n_0+\alpha}^{(n_0)})(t,Ax),\varphi(t)\right\rangle$$

$$= \langle f_{\alpha+1}(t,x),\varphi(t)\rangle + \langle (f_1 * S_\alpha)(t,Ax),\varphi(t)\rangle\,, \quad x \in D(A), \varphi \in \mathcal{K}_1,$$

which gives the first assertion.

Again by using the quoted Proposition 3.3 in [2], it follows

$$A\left\langle (f_1 * S_\alpha)(t,x), \varphi(t)\right\rangle = (-1)^{(n_0)}\left\langle A(f_1 * S_{n_0+\alpha})(t,x), \varphi^{(n_0)}(t)\right\rangle$$
$$= (-1)^{(n_0)}\left\langle S_{n_0+\alpha}(t,x), \varphi^{(n_0)}(t)\right\rangle - (-1)^{n_0}\left\langle f_{n_0+\alpha+1}(t,x), \varphi^{(n_0)}(t)\right\rangle$$
$$= \left\langle S^{(n_0)}_{n_0+\alpha}(t,x), \varphi(t)\right\rangle - \left\langle f_{\alpha+1}(t,x), \varphi(t)\right\rangle = \left\langle S_\alpha(t,x), \varphi(t)\right\rangle - \left\langle f_{\alpha+1}(t,x), \varphi(t)\right\rangle$$

which gives (10). ∎

Arendt ([2]) has obtained the characterization of a generator A of an $(n+1)$-times integrated semigroup $(S(t))_{t\geqslant 0}$, $n \in \mathbf{N}$ if A is a non-densely defined linear operator.

Theorem 4. *Let $\alpha \in \mathbf{R}$, $\omega \in \mathbf{R}$, $M \geqslant 0$ and $n \in \mathbf{N}$ such that $\alpha + n > 0$ if $\alpha \in (-\infty, 0]$. If $\alpha > 0$ we take $n = 0$.*

a) Let A be a (non-densely defined) linear operator on a Banach space E such that $(a,\infty) \subset \rho(A)$ for some $a \geqslant 0$ and $\omega \in (-\infty, a]$. The following statements are equivalent:

(i) A generates an $\alpha + n + 1$-times integrated semigroup $(S(t))_{t\geqslant 0}$ satisfying

$$\lim_{h\downarrow 0}\sup \frac{1}{h}\|S(t+h) - S(t)\| \leqslant Me^{\omega t}, \qquad t \geqslant 0.$$

(ii) $$\left\|\frac{1}{k!}\left(\frac{R(\lambda,A)}{\lambda^{\alpha+n}}\right)^{(k)}\right\| \leqslant M\left(\frac{1}{\lambda-\omega}\right)^{k+1}, \text{ for all } \operatorname{Re}\lambda > a,\ k \in \mathbf{N}_0.$$

b) If A satisfies the equivalent conditions of (a), then the part of A on $\overline{D(A)}$ is the generator of an $(\alpha+n)$-times integrated semigroup.

c) Let A in (a) be a densely defined linear operator. Then (ii) in (a) is equivalent with the following condition:

A generates an $(\alpha+n)$-times integrated semigroup $(S(t))_{t\geqslant 0}$ satisfying $\|S(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

Remark. The case $\alpha = 0$ in Theorem 2c) is the Hille-Yosida theorem.

Corollary 1. *Let $\alpha \leqslant 0$ and $\alpha + n > 0$. If a densely defined linear operator A generates an $(\alpha+n)$-times integrated semigroup, then its adjoint A^* generates an $(\alpha+n+1)$-times integrated semigroup.*

This directly follows from Theorem 4 since $R(\lambda, A)^* = R(\lambda, A^*)$ for λ real.

4. Relations with distributional semigroup

We follow the definition of an exponentially bounded distributional semigroup, SGDE, given in [7], Definition 6.1. Note, instead of $\mathcal{S}(\mathbf{R})$, we use the space $\mathcal{K}_1(\mathbf{R})$ (cf. [9]). As in [7], we put $\mathcal{D}_0 = \{\varphi \in C_0^\infty;\ \operatorname{supp}\varphi \in [0,\infty)\}$.

If $(T(t))_{t\geqslant 0}$ is a C_0-semigroup and $S_\alpha = T * f_\alpha$, $\alpha \in \mathbf{R}$ then we define

$$S_\alpha(\varphi, x) = (S_\alpha(\cdot, x) * \check{\varphi})(0) = ((T * f_\alpha(\cdot, x)) * \check{\varphi})(0), \quad x \in E, \varphi \in \mathcal{K}_1. \tag{11}$$

One can show that S_α is an α-times integrated semigroup.

THEOREM 5. *Let $(S_\alpha(t))_{t\geqslant 0}$, $\alpha \in \mathbf{R}$, be an α-times integrated semigroup. Assume that its infinitesimal generator A is densely defined. Then,*

$$S_\alpha(\varphi, x) = (S_\alpha * \check{\varphi})(0)(x), \qquad \varphi \in \mathcal{K}_1, \tag{12}$$

defines an element of $\mathcal{K}'_{1+}(L(E))$ which is an SGDE iff $\alpha = 0$.

Proof. Let $(S(t))_{t\geqslant 0}$ be an SGDE. As it was remarked by Arendt, Theorem 4.3 in [2] and Theorem 3.2 in [13] imply that there exists an n-times integrated semigroup $(S_n(t))_{t\geqslant 0}$, $n \in \mathbf{R}$ such that

$$S(\varphi, x) = \left\langle S_n^{(n)}(t, x), \varphi(t) \right\rangle = (S_n^{(n)}(\cdot, x) * \check{\varphi})(0), \qquad \varphi \in \mathcal{D}, x \in E.$$

This implies $S_n^{(n)} = S_n * f_{-n} = S_0$, where S_0 is a 0-integrated semigroup equal to S.

Now we will prove that for $\alpha \in \mathbf{R} \setminus \{0\}$, (12) does not define an SGDE. If it happened for some $\alpha \in \mathbf{R} \setminus \{0\}$, then $(S_\alpha(t))_{t\geqslant 0}$ and $((S_\alpha * f_{-\alpha})(t))_{t\geqslant 0}$ would determine different SGDE's which is impossible by the uniqueness of an SGDE with the given infinitesimal generator A. ∎

Let A be an operator on E and $T \in \mathcal{K}'_{1+}(E)$. Then $u \in \mathcal{K}'_{1+}(E)$ is a solution to

$$u' = Au + T \quad \text{in } \mathcal{K}'_1(E) \tag{13}$$

if $\langle u(t), \varphi(t)\rangle \in D(A)$ for every $\varphi \in \mathcal{K}_1(\mathbf{R})$ and (13) holds.

Let $(S_0(t))_{t\geqslant 0}$ be a 0-integrated semigroup with an infinitesimal generator which is not necessarily densely defined. We recall: if for some $x \in E$

$$S_0(\varphi, x) = \int S_0(t, x)\varphi(t)\,dt = 0 \quad \text{for every } \varphi \in \mathcal{D}_0, \tag{14}$$

then $x = 0$.

As in [7], we extend $(S_0(t))_{t\geqslant 0}$ on $T \in \mathcal{E}'(\mathbf{R})$, $\operatorname{supp} T \subset [0, \infty)$ by using δ-sequences $\{\rho_\nu\}$ in $\mathcal{D}_0$, $(\rho_\nu \to \delta)$: $S_0(T, x) = \lim_{\nu\to\infty} S_0(T^*\rho_\nu, x)$ for those $x \in E$ for which this limit exists. Because of (14), we can define the closure of $S_0(T, \cdot)$ which will be denoted by $\overline{S_0(T, \cdot)}$. Theorem 4b) implies that a 0-integrated semigroup has the same properties as an SGDE except the set $\{S_0(\varphi, x);\ \varphi \in \mathcal{D}_0, x \in E\}$ is dense in E (cf. [7]).

Let $U \in \mathcal{K}'_{1+}(L(E, D(A)))$, $V \in \mathcal{K}'_{1+}(L(D(A), E))$ and $\operatorname{supp} U \subset [a, \infty)$, $\operatorname{supp} V \subset [b, \infty)$, $a, b \in \mathbf{R}$. Then U^*V and V^*U are defined as in [15]. Moreover, they are elements of $\mathcal{K}'_{1+}(L(D(A)))$ and $\mathcal{K}'_{1+}(L(E))$, respectively, and their supports are bounded from the left by $a + b$.

THEOREM 6. *Let $\alpha \in \mathbf{R}^-$ and $S_\alpha \in \mathcal{K}'_{1+}$ be an α-times integrated semigroup with the infinitesimal generator A, such that $S_\alpha * f_{-\alpha}$ be a 0-integrated semigroup. Then*

a) $\left(-A+\frac{\partial}{\partial t}\right) * S_\alpha = f_\alpha \otimes I_{\overline{D(A)}}$, $S_\alpha * \left(-A+\frac{\partial}{\partial t}\right) = f_\alpha \otimes I_{D(A)}$, *where* $-A+\frac{\partial}{\partial t} = -\delta\otimes A + \delta'\otimes I$.

b) Let $T \in \mathcal{K}'_1(L(\overline{D(A)}))$. Then $u = S_\alpha * f_{-\alpha} * T$ is the unique solution of (13).

Proof. a) Put $S = S_\alpha * f_{-\alpha}$. Then, as in [7] Theorem 4.1, one can prove

$$\left(-A+\frac{\partial}{\partial t}\right) * S_0 = \delta\otimes I_{\overline{D(A)}}. \tag{15}$$

Since $D(A)$ is not dense in E, in general, we apply both sides of (15) on $x \in D(A)$. Then, by making convolution with f_α we obtain the first assertion of a). In a similar way we prove the second one.

b) This simply follows from a).

THEOREM 7. *Let A be an infinitesimal generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geqslant 0}$, $\alpha \in \mathbf{R}^-$. Then $S_\alpha * f_{-\alpha}$ determines an SGDE with the infinitesimal generator A on $E_0\times\mathcal{K}_1$, where $E_0 = \overline{\{S_0(\varphi,x);\ \varphi\in\mathcal{D}_0, x\in E\}}$ and*

$$\left(-A+\frac{\partial}{\partial t}\right) * S_\alpha = f_\alpha\otimes I_{E_0}, \quad S_\alpha * \left(-A+\frac{\partial}{\partial t}\right) = f_\alpha \otimes I_{D(A)\cap E_0}.$$

Let $T\in\mathcal{K}'_{1+}(E_0)$. Then $u = S_\alpha * f_{-\alpha} * T$ is the unique solution of (13).

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(received 02.09.1997.)

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