МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 221–223

UDK 517.982.2
оригинални научни рад
research paper

SOME PROPERTIES OF HAUSDORFF MEASURE OF NONCOMPACTNESS ON LOCALLY BOUNDED TOPOLOGICAL VECTOR SPACES

Ivan D. Aranđelović and Marina M. Milovanović-Aranđelović

Abstract. In this note we present some properties of Hausdorff measure of noncompactness on locally bounded topological vector space.

Introduction

Let X be a Hausdorff topological vector space. A set $A \subseteq X$ is *bounded* if for each neighborhood of zero U there is a scalar α such that $A \subseteq \alpha U$. The space X is *locally bounded* if it contains a bounded neighborhood of zero.

EXAMPLES. Normed spaces, ℓ^p $(p > 0)$ and L^p $(p > 0)$ are locally bounded spaces.

PROPOSITION 0.1 (see Rolewicz [4], Wilansky [5]) *Hausdorff topological vector space X is locally bounded if and only if there exist a real number p $(0 < p \leq 1)$ and a function $|\cdot\|\colon X \to [0, +\infty)$ such that:*

1) $|x\| = 0$ if and only if $x = 0$;

2) $|\alpha x\| = |\alpha|^p |x\|$;

3) $|x + y\| \leq |x\| + |y\|$;

4) function $d' : X^2 \to [0, \infty)$ defined by $d'(x, y) = |x - y\|$ is a metric on X;

5) original topology on X is equivalent with the topology of metric space (X, d).

The theory of measures of noncompactness has many applications in Functional analaysis and Operator theory (see [1],[3]). If Q is a bounded subset of a metric space X, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

AMS Subject Classification: 46 A 16, 46 A 50

Communicated at the 4th Symposium on Mathematical Analysis and Its Applications, Aranđelovac 1997.

In the proofs of our results we need the following well known properties of Hausdorff measure of noncompactness.

Proposition 0.2 (see Banás and Goebel [1], Rakočević [3]) *If Q, Q_1 and Q_2 are bounded subsets of a metric spaces (X,d) then*

1) $\chi(Q)=0$ if and only if Q is a totally bounded set;

2) $Q_1\subseteq Q_2$ implies $\chi(Q_1)\le\chi(Q_2)$;

3) $\chi(Q_1\cup Q_2)=\max\{\chi(Q_1),\chi(Q_2)\}$.

In this paper we investigate some properties of measure of noncompactness on arbitrary locally bounded Hausdorff topological vector space. Corresponding results for ℓ^p $(0<p)$ spaces were obtained by I. Jovanović and V. Rakočević [2].

Results

Let (X,d) be a metric space, $x\in X$ and $r>0$. By $B(x,r)$ we denote $\{y\in X: d(x,y)\le r\}$.

Proposition 1. *If Q, Q_1 and Q_2 are bounded subsets of an arbitrary metric linear space X and $x\in X$, then:*

1) $\chi(Q_1+Q_2)\le\chi(Q_1)+\chi(Q_2)$;

2) $\chi(x+Q)=\chi(Q)$.

Proof. Let $\delta>0$ be an arbitrary positive real number. From

$$Q_1+Q_2\subseteq\bigcup_{i=1}^{n}\bigcup_{j=1}^{m}[B(x_i,\chi(Q_1)+\delta)+B(y_j,\chi(Q_2)+\delta)]$$

it follows that $z\in Q_1+Q_2$ implies that there exist $z_1\in Q_1$ and $z_2\in Q_2$ such that $z=z_1+z_2$, $d(z_1,x_i)<\chi(Q_1)+\delta$ and $d(z_2,y_j)<\chi(Q_2)+\delta$, for some i,j $(1\le i\le n;\ 1\le j\le m)$. Since

$$\begin{aligned} d(z,x_i+y_j)&=d(z_1+z_2,x_i+y_j)=d(z_1-x_i,y_j-z_2)\\ &\le d(z_1-x_i,0)+d(z_2-y_j,0)=d(z_1,x_i)+d(z_2,y_j)\\ &\le\chi(Q_1)+\chi(Q_2)+2\delta, \end{aligned}$$

when $\delta\to0^+$, we have $\chi(Q_1+Q_2)\le\chi(Q_1)+\chi(Q_2)$.

From 1) we have $\chi(x+Q)\le\chi(\{x\})+\chi(Q)=\chi(Q)$, which implies $\chi(Q)=\chi(-x+x+Q)\le\chi(x+Q)$. So $\chi(x+Q)=\chi(Q)$. ∎

For proofs of the next proposition we need the following lemma.

Lemma. *If X is a locally bounded Hausdorff topological vector space, and $r,s>0$ then $B(0,rs)=r^{1/p}B(0,s)$, for some p, $(0<p\le1)$.*

Proof. From $x\in X$ and $|x\|\le s$ it follows $r|x\|\le rs$ which implies $|r^{\frac{1}{p}}x\|\le rs$, for some p, $(0<p\le1)$. ∎

PROPOSITION 2. *If X is a locally bounded Hausdorff topological vector space, $Q \subseteq X$ its bounded subset and α arbitrary scalar then*

$$\chi(\alpha Q) = |\alpha|^p \chi(Q)$$

for some p, $0 < p \le 1$.

Proof. Let $\alpha \neq 0$. From $Q \subseteq \bigcup_{i=1}^n \{x_i + B(0, \chi(Q))\}$ it follows

$$\alpha Q \subseteq \bigcup_{i=1}^n \{\alpha x_i + \alpha B(0, \chi(Q))\} = \bigcup_{i=1}^n \{\alpha x_i + B(0, |\alpha|^p \chi(Q))\}$$

which implies $\chi(\alpha Q) \le |\alpha|^p \chi(Q)$. Since $\alpha^{-1}\alpha Q = Q$, we have $\chi(Q) \le |\alpha|^{-p}\chi(\alpha Q)$. So $|\alpha|^p \chi(Q) \le \chi(\alpha Q)$. It follows $\chi(\alpha Q) = |\alpha|^p \chi(Q)$. ■

PROPOSITION 3. *If X is an infinite-dimensional locally bounded Hausdorff topological vector space, and $B(0,1)$ is its closed unit ball then*

$$\chi(B(0,1)) = 1.$$

Proof. Let us remark that clearly $\chi(K) \le 1$. If $\chi(K) = s < 1$ then we can find $\varepsilon > 0$ such that $s + \varepsilon < 1$. Now, there is an $(s+\varepsilon)$-net of $B(0,1)$, say $x_1, \dots, x_n$. Hence

$$B(0,1) \subseteq \bigcup_{i=1}^n \{x_i + B(0, s+\varepsilon)\} = \bigcup_{i=1}^n \{x_i + (s+\varepsilon)^{\frac{1}{p}} B(0,1)\},$$

and

$$\begin{aligned} s = \chi(B(0,1)) &\le \max_{1 \le i \le n} \chi(x_i + (s+\varepsilon)^{\frac{1}{p}} B(0,1)) \\ &= \chi((s+\varepsilon)^{\frac{1}{p}} B(0,1)) = (s+\varepsilon)\chi(B(0,1)) = (s+\varepsilon)s. \end{aligned}$$

From $s + \varepsilon < 1$, it follows that $s = \chi(B(0,1)) = 0$ i.e. $B(0,1)$ is totally bounded, which implies that X is a finite-dimensional space. Hence we get a contradiction, and the proof is complete. ■

COROLLARY. *If X is an infinite-dimensional locally bounded Hausdorff topological vector space, $x_0 \in X$ and $r > 0$ then $\chi(B(x_0, r)) = r$.*

Proof. $\chi(B(x_0,r)) = \chi(x_0 + B(0,r)) = \chi(B(0,r)) = \chi(r^{\frac{1}{p}} B(0,1)) = r\chi(B(0,1)) = r$. ■

REFERENCES

[**1**] J. Banás and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York and Basel, 1980.

[**2**] I. Jovanović and V. Rakočević, *Multipliers of mixed-normed sequence spaces and measures of noncompactness*, Publ. Inst. Math. (Beograd) (N.S.) **56** (1994) 61–68.

[**3**] V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd 1994.

[**4**] S. Rolewicz, *Metric Linear Spaces*, PWN, Warszawa 1972.

[**5**] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, New York 1978.

(received 20.08.1997, in revised form 06.03.1998.)

Faculty of Mehanical Engineering, 27 marta 80, 11000 Beograd, Yugoslavia