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SIMPLE SUFFICIENT CONDITIONS FOR UNIVALENCE

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Abstract. For a function $f(z) = z + a_2z^2 + \cdots$, analytic in the unit disc, we find $\lambda > 0$ such that $|f''(z)| \leqslant \lambda$ implies starlikeness (Mocanu's problem [2]) or convexity. The given results are sharp.

As usual, let A denote the class of functions f which are analytic in the unit disc $U = \{z : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$. Let $S^* \subset A$ be the class of *starlike* functions in U defined by the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U,$$

and let $K \subset A$ be the class of *convex* functions defined by the condition

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0, \quad z \in U.$$

Let f and g be analytic in U. We say that f is *subordinate* to g, written $f(x) \prec g(z)$ or $f \prec g$, if there exists a function ω analytic in U which satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$. If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

In his paper [2] Mocanu considered the problem of finding $\lambda > 0$ such that the condition $|f''(z)| \leqslant \lambda$, $z \in U$, implies $f \in S^*$. He found that $\lambda = 2/3$ is sufficient for that problem. Later, Ponnusamy and Singh found a better constant $\lambda = 2/\sqrt{5}$. In the next theorem we give a more precise result.

THEOREM 1. *If $f \in A$ and $|f''(z)| \leqslant 1$, $z \in U$, then $f \in S^*$. The result is sharp.*

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For sharpness we may consider the function $f(z)=z+\frac{1+\varepsilon}{2}z^2$, $\varepsilon>0$. For this function we have $|f''(z)|=1+\varepsilon>1$, but $f'(z)=1+(1+\varepsilon)z$ vanishes at the point $z=-1/(1+\varepsilon)\in U$; that means f is not univalent in U.

For the proof of Theorem 1 (and others) we need the following two lemmas.

LEMMA A. *If f, g are analytic in U, $g'(0)\neq 0$, and g is convex (univalent) in U, then*

$$f\prec g \implies \frac{1}{z}\int_0^z f(t)\,dt \prec \frac{1}{z}\int_0^z g(t)\,dt.$$

LEMMA B. *If $f(z)=\sum_{k=1}^{\infty} a_k z^k$, $z\in U$, and g is convex (univalent) in U, then*

$$zf'(z)\prec zg'(z) \implies f\prec g.$$

A more general result than the one in Lemma A one can find in [1]. Lemma B is due to [4].

Proof of Theorem 1. We can write the condition of the theoem as

$$zf''(z)\prec z. \tag{1}$$

By Lemma A, from (1) we obtain $f'(z)-\dfrac{f(z)}{z}\prec\dfrac{1}{2}z$. We can arrange the last relation in the following two ways:

$$z\left(\frac{f(z)}{z}\right)'\prec z\left(1+\frac{z}{2}\right)' \tag{2}$$

and

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1}{2}z. \tag{3}$$

From (2), by Lemma B we have $\dfrac{f(z)}{z}\prec 1+\dfrac{z}{2}$, which implies $\dfrac{1}{2}<\left|\dfrac{f(z)}{z}\right|<\dfrac{3}{2}$, $z\in U$. From the last relation and (3) we get

$$\frac{1}{2}\left|\frac{zf'(z)}{f(z)}-1\right|\leqslant\left|\frac{f(z)}{z}\right|\,\left|\frac{zf'(z)}{f(z)}-1\right|<\frac{1}{2},\quad z\in U,$$

which finally gives $\left|\dfrac{zf'(z)}{f(z)}-1\right|<1$, $z\in U$, i.e. $f\in S^*$. ∎

We can prove that the condition in Theorem 1 may be weaker if we have some additional condition as the following theorem shows.

THEOREM 2. *Let $f\in A$ and let $|f''(z)|\leqslant a$, $\left|\dfrac{f(z)}{z}\right|\geqslant\dfrac{a}{2}$, for some $0\leqslant a\leqslant 2$ and for every $z\in U$. Then $f\in S^*$.*

Proof. As in the proof of Theorem 1 we have that

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{a}{2}z. \tag{4}$$

Suppose that $\operatorname{Re}\left\{\frac{z_0f'(z_0)}{f(z_0)}\right\}=0$ for some z_0, $|z_0|<1$, i.e. let $\frac{z_0f'(z_0)}{f(z_0)}=ix$ (x is real). Then for such z_0 we obtain that

$$\left|\frac{f(z_0)}{z_0}\left(\frac{z_0f'(z_0)}{f(z_0)}-1\right)\right|=\left|\frac{f(z_0)}{z_0}\right|\,|ix-1|\geqslant\left|\frac{f(z_0)}{z_0}\right|\geqslant\frac{a}{2},$$

which is a contradiction to (4). It means that $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$, $z\in U$, i.e. $f\in S^*$. ∎

Example. For the function $f(z)=z+\frac{3}{40}z^5$ we have $f''(z)=\frac{3}{2}z^3$, which implies $|f''(z)|<3/2$, $z\in U$, while $|f(z)/z|\geqslant 1-\frac{3}{40}|z|^4>37/40>3/4$, $z\in U$. By theorem 2 it means that $f\in S^*$.

Remark. If $a=1$ in Theorem 2, then the condition $|f(z)/z|\geqslant 1/2$, $z\in U$, is satisfied (see the proof of Theorem 1), and the statement of Theorem 1 easily follows, but we cannot conclude that $\left|\frac{zf'(z)}{f(z)}-1\right|<1$, $z\in U$, as in Theorem 1.

Finally, we give the convexity condition for the same kind of problem.

Theorem 3. *If $f\in A$ and $|f''(z)|\leqslant 1/2$, $z\in U$, then $f\in K$. The result is sharp.*

Proof. Since by the condition of the theorem

$$zf''(z)\prec\frac{1}{2}z, \tag{5}$$

then, by applying Lemma B, we obtain

$$f'(z)\prec 1+\frac{1}{2}z. \tag{6}$$

If we put $\frac{zf''(z)}{f'(z)}+1=p(z)$, then from (5) we have

$$(p(z)-1)f'(z)\prec\frac{1}{2}z, \tag{7}$$

and we want to show that $\operatorname{Re}\{p(z)\}>0$, $z\in U$. If not, then suppose that there exists a z_0, $|z_0|<1$, such that $p(z_0)=ix$, where x is real. Hence by (6): $|f'(z_0)|>1/2$, then we have

$$|(p(z_0)-1)f'(z_0)|^2-\frac{1}{4}=|ix-1|^2|f'(z_0)|^2-\frac{1}{4}>\frac{1}{4}(x^2+1)-\frac{1}{4}=\frac{1}{4}x^2\geqslant 0,$$

which is a contradiction to (7). Therefore, $\operatorname{Re}\{p(z)\} > 0$, $z \in U$, i.e. f is a convex function.

If we consider the function $f(z) = z + \frac{1+\varepsilon}{4} z^2$, $0 < \varepsilon < 1$, then we have that $|f''(z)| = \dfrac{1+\varepsilon}{2} > \dfrac{1}{2}$, but $\dfrac{zf''(z)}{f'(z)} + 1 = \dfrac{1+(1+\varepsilon)z}{1+\frac{1+\varepsilon}{2}z}$ becomes negative for z real close to -1, implying that f is not convex. ∎

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