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ON THE COMPACTNESS IN FUZZY TOPOLOGICAL SPACES IN SOSTAK'S SENSE

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Abstract. In this paper, we establish new definitions of smooth closure and smooth interior of a fuzzy set [3] under the name of fuzzy closure and fuzzy interior of a fuzzy set which satisfy almost all properties of the corresponding definitions in fuzzy topological spaces in Chang's sense. As a consequence of these definitions we show that the topological concepts, especially various types of compactness, can be presented in a simple way, with no additional hypotheses needed which occured in [3] and with no counterpart in [2, 4, 6].

Sostak [10–12] extended the concept of fuzzy topology (*ft* for short) on X as a mapping $\tau\colon I^X \to I$ satisfying conditions:

(01) $\tau(0) = \tau(1) = 1$, (02) $\tau(A \wedge B) \geqslant \tau(A) \wedge \tau(B)$ for each $A, B \in I^X$, (03) $\tau(\bigvee_{i\in J} A_i) \geqslant \bigwedge_{i\in J} \tau(A_i)$ for each $\{\, A_i : i \in J \,\} \subseteq I^X$.

A pair (X, τ) is called a fuzzy topological space (*fts* for simplicity) if τ is an *ft* on X [10]. For a given *fts* (X, τ), the mapping $\tau^*\colon I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ for every $A \in I^X$, measures the grade of fuzzy subsets of X being closed, and the number $\tau^*(A)$ is called the degree of closedness of A. The mapping $\tau^*\colon I^X \to I$ satisfies the next properties [10–11]:

$(C1)$ $\tau^*(0) = \tau^*(1) = 1$, $(C2)$ $\tau^*(A \vee B) \geqslant \tau^*(A) \wedge \tau^*(B)$ for each $A, B \in I^X$, $(C3)$ $\tau^*(\bigwedge_{i\in J} A_i) \geqslant \bigwedge_{i\in J} \tau^*(A_i)$ for each $\{\, A_i : i \in J \,\} \subseteq I^X$.

In 1992, fuzzy topological spaces in Sostak's sense were independently redefined by Ramadan [9] under the name of smooth topological spaces, and the mapping τ satisfying (01–03) and the mapping τ^* satisfying $(C1$–$C3)$ were respectively called smooth topology and smooth cotopology [3, 9]. In this paper, we assume the terminology used by Sostak and his colleagues [5, 7, 10–12].

In addition to (01–03), if τ satisfies the property

(04) $\tau(I^X) \subseteq \{0, 1\}$,

then such *ft* τ in the one-to-one way corresponds to a fuzzy topology [1] in Chang's sense (CFT for short) [11]. In this case, the family $\{\, A \in I^X : \tau(A) = 1 \,\}$ is the corresponding CFT on X, and it is also represented by the symbol τ.

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For a given *fts* (X, τ) and for each $\alpha \in [0, 1]$, α-level CFT of the *ft* τ, denoted by τ_α, is defined by the family $\{ A \in I^X : \tau(A) \geqslant \alpha \}$. Various topological concepts and their gradations based on the fuzzy topologies τ_α were established in [5, 10–12]. In this paper, for a given *fts* (X, τ), considering the family $P(\tau) = \{ U \in I^X : \tau(U) \geqslant 1 - U \}$, we will show that it is possible to present the topological concepts by using the family $P(\tau)$ instead of the fuzzy topologies τ_α.

Defining the closure and the interior of a fuzzy set in *fts*, a different approach for the compactness structure of *fts* was introduced in [3]. The results in [3] were obtained under several further conditions, and if such results are compared to the corresponding results in fuzzy topological spaces in Chang's sense (CFTS for short) [2, 4, 6, 8] it is observed that these results are not as good as in CFTS.

In this paper, we establish new definitions of smooth closure and smooth interior of a fuzzy set under the name of fuzzy closure and fuzzy interior of a fuzzy set which satisfy almost all properties of the corresponding definitions in CFTS. As a consequence of these definitions we show that the topological concepts, especially various types of compactness, can be presented in a simple way, with no additional hypotheses needed which occured in [3] and with no counterpart in [2, 4, 6].

1. The basic idea

For a given *fts* (X, τ), it can be easily checked that the family $P(\tau)$ satisfies the following two properties:

$(P1)$ $0, 1 \in P(\tau)$,

$(P2)$ for each subfamily $\{ A_i : i \in J \}$ of I^X, $\bigvee_{i \in J} A_i \in P(\tau)$.

Thus $P(\tau)$ can be considered as a subbase of a CFT $T := T_{P(\tau)}$ and besides $P(\tau)$ is closed under the formation of arbitrary unions. This means that the only difference between $P(\tau)$ and T is that $P(\tau)$ generally does not satisfy the third axiom of topology:

$(P3)$ $A, B \in P(\tau) \implies A \wedge B \in P(\tau)$.

For an *fts* (X, τ), defining the family $R(\tau) = \{ A \in I^X : \tau^*(A) \geqslant A \}$, it is straightforward that $R(\tau) = \{ A \in I^X : A^c \in P(\tau) \}$ and it satisfies the properties: $(R1)$ $0, 1 \in R(\tau)$, $(R2)$ for each subfamily $\{ A_i : i \in J \}$ of I^X, $\bigwedge_{i \in J} A_i \in R(\tau)$, although the property $(R3)$ for each $A, B \in R(\tau)$, $A \vee B \in R(\tau)$ is not generally satisfied.

The main reason for establishing the topological concepts based on the family $P(\tau)$ is to exploite the properties and the results of CFTS because of the fact that various results in CFTS can be given without using the property $(P3)$, or equivalently $(R3)$. This idea allows us to find the topological concepts in a simple fashion, and the results presented in this paper will be better than given in [3]. This will be understandable whenever our results are introduced.

2. Fuzzy interior and fuzzy closure

Definition 2.1. Let (X,τ) be an *fts* and $A \in I^X$. Then $P(\tau)$-fuzzy interior of A ($P(\tau)$-fuzzy closure of A), denoted by $\overset{\circ}{A}$ ($\overline{A}$), is defined by

$$\overset{\circ}{A} = \bigvee\{\, K \in I^X : K \in P(\tau), K \subseteq A\,\} \quad (\overline{A} = \bigwedge\{\, K \in I^X : K^c \in P(\tau), A \subseteq K\,\}).$$

It can be easily observed that $\overline{A} = \bigwedge\{\, K \in I^X : K \in R(\tau), A \subseteq K\,\}$, $A \in P(\tau) \iff A = \overset{\circ}{A}$ and $A \in R(\tau) \iff A = \overline{A}$. Moreover, the following results are well-known properties of CFTS which are presented without using $(P3)$.

Proposition 2.2. *Let (X,τ) be an fts. Then for each $A, B \in I^X$ we have the following implications and equalities:*

(i) $A \subseteq B \implies \overset{\circ}{A} \subseteq \overset{\circ}{B}$, (ii) $A \subseteq B \implies \overline{A} \subseteq \overline{B}$, (iii) $(\overset{\circ}{A})^c = \overline{A^c}$, *i.e.* $\overset{\circ}{A} = (\overline{A^c})^c$, (iv) $(\overline{A})^c = (A^c)^\circ$, *i.e.* $\overline{A} = ((A^c)^\circ)^c$.

Proposition 2.3. *Let (X,τ) be an fts. Then for $A, B \in I^X$ we have the following:*

(i) $\overline{0} = 0$, $\overline{1} = 1$, $\overset{\circ}{0} = 0$, $\overset{\circ}{1} = 1$, (ii) $A \subseteq \overline{A}$, $\overset{\circ}{A} \subseteq A$, (iii) $\overline{(\overline{A})} = \overline{A}$, $(\overset{\circ}{A})^\circ = \overset{\circ}{A}$, (iv) $\overline{A} \vee \overline{B} = \overline{A \vee B}$, $(A \wedge B)^\circ \subseteq \overset{\circ}{A} \wedge \overset{\circ}{B}$.

Proposition 2.4. *Let (X,τ) be an fts and $A, B \in I^X$. Then:*

(i) $(\overset{\circ}{A})^c \leqslant \tau(\overset{\circ}{B})$ *and* $(\overset{\circ}{B})^c \leqslant \tau(\overset{\circ}{A}) \implies (A \wedge B)^\circ = \overset{\circ}{A} \wedge \overset{\circ}{B}$.

(ii) $\overline{A} \leqslant \tau^*(\overline{B})$ *and* $\overline{B} \leqslant \tau^*(\overline{A}) \implies \overline{A \vee B} = \overline{A} \vee \overline{B}$.

Proof. (i) Let $(\overset{\circ}{A})^c \leqslant \tau(\overset{\circ}{B})$ and $(\overset{\circ}{B})^c \leqslant \tau(\overset{\circ}{A})$. Since $(\overset{\circ}{A})^\circ \leqslant \tau(\overset{\circ}{A})$ for each $A \in I^X$, we may write $(\overset{\circ}{A})^c \leqslant \tau(\overset{\circ}{A}) \wedge \tau(\overset{\circ}{B})$ and $(\overset{\circ}{B})^c \leqslant \tau(\overset{\circ}{A}) \wedge \tau(\overset{\circ}{B})$. Then we observe that

$$(\overset{\circ}{A} \wedge \overset{\circ}{B})^c = (\overset{\circ}{A})^c \vee (\overset{\circ}{B})^c \leqslant \tau(\overset{\circ}{A}) \wedge \tau(\overset{\circ}{B}) \leqslant \tau(\overset{\circ}{A} \wedge \overset{\circ}{B}), \quad \text{i.e.} \quad \overset{\circ}{A} \wedge \overset{\circ}{B} \in P(\tau).$$

Therefore, considering (2.1) and the inclusion $\overset{\circ}{A} \wedge \overset{\circ}{B} \subseteq A \wedge B$, we find that $\overset{\circ}{A} \wedge \overset{\circ}{B} \subseteq (A \wedge B)^\circ$, and so $(A \wedge B)^\circ = \overset{\circ}{A} \wedge \overset{\circ}{B}$ follows.

(ii) Let $\overline{A} \leqslant \tau^*(\overline{B})$ and $\overline{B} \leqslant \tau^*(\overline{A})$. Since $\tau^*(\overline{A}) \geqslant \overline{A}$ for each $A \in I^X$, we obviously have the inequalities $\overline{A} \leqslant \tau^*(\overline{B}) \wedge \tau^*(\overline{A})$ and $\overline{B} \leqslant \tau^*(\overline{A}) \wedge \tau^*(\overline{B})$. Thus $\overline{A} \vee \overline{B} \leqslant \tau^*(\overline{A}) \wedge \tau^*(\overline{B}) \leqslant \tau^*(\overline{A} \vee \overline{B})$, i.e. $\overline{A} \vee \overline{B} \in R(\tau)$. Then by taking into consideration (2.1) and the inclusion $A \vee B \subseteq \overline{A} \vee \overline{B}$, it is easily seen that $\overline{A \vee B} \subseteq \overline{A} \vee \overline{B}$, i.e. $\overline{A \vee B} = \overline{A} \vee \overline{B}$. ∎

In the following example, it is shown that the lefthand sides of the implications in (2.4) are not generally fulfilled, neither are the equalities $\overline{A \vee B} = \overline{A} \vee \overline{B}$ and $(A \wedge B)^\circ = \overset{\circ}{A} \wedge \overset{\circ}{B}$.

Example 2.5 For the crisp set $X=\{a,b\}$, let A,B,C,D be four fuzzy sets on X such that $A=a_{0,3}$, $B=b_{0,8}$, $C=A^c=a_{0,7}\vee b_1$, $D=B^c=a_1\vee b_{0,2}$. Defining the mapping $\tau^*\colon I^X\to I$ as $\tau^*(0)=\tau^*(1)=1$ and

$$\tau^*(U)=\begin{cases}0,3, & \text{if } U=A,\\ 0,8, & \text{if } U=B,\\ 0,5, & \text{if } U=A\vee B,\\ 0, & \text{if } U\notin\{0,1,A,B,A\vee B\},\end{cases}$$

one can easily observe that the mapping $\tau^*\colon I^X\to I$ satisfies the conditions $(C1)$–$(C3)$, i.e. the mapping $\tau\colon I^X\to I$, $\tau(U)=\tau^*(U^c)$ is an *ft* on X. It is clear that

$$\tau(U)=\begin{cases}0,3, & \text{if } U=C,\\ 0,8, & \text{if } U=D,\\ 0,5, & \text{if } U=C\wedge D,\\ 0, & \text{if } U\notin\{0_X,1_X,C,D,C\wedge D\}.\end{cases}$$

Now, let us evaluate $\mathring{A}$, $\mathring{B}$, $\mathring{A}\vee\mathring{B}$, $\mathring{C}$, $\mathring{D}$, $(C\wedge D)^\circ$ with respect to this *ft* τ on X. Since $A\leqslant 0,3=\tau^*(A)=0,3$ and $B\leqslant 0,8=\tau^*(B)=0,8$, it is obvious that $A=\overline{A}$, $B=\overline{B}$, i.e. $\overline{A}\vee\overline{B}=A\vee B=a_{0,3}\vee b_{0,8}$. On the other hand, $(A\vee B)(b)=0,8>\tau^*(A\vee B)=0,5$, i.e. $A\vee B\notin R(\tau)$, i.e. $A\vee B\neq\overline{A\vee B}$, i.e. $\overline{A\vee B}\neq\overline{A}\vee\overline{B}$.

Similarly one can easily see that $C^c\leqslant 0,3=\tau(C)$, $D^c\leqslant 0,8=\tau(D)$. Thus $\mathring{C}=C=a_{0,7}\vee b_1$, $\mathring{D}=D=a_1\vee b_{0,2}$, i.e. $\mathring{C}\wedge\mathring{D}=C\wedge D=a_{0,7}\vee b_{0,2}$. On the other hand, $(C\wedge D)^c(b)=(A\vee B)(b)=0,8>\tau(C\wedge D)=0,5$, i.e. $C\wedge D\notin P(\tau)$. Thus $\mathring{C}\wedge\mathring{D}=C\wedge D\neq(C\wedge D)^\circ$.

Furthermore it can be easily checked that $\overline{B}\nleqslant\tau^*(\overline{A})$ and $(\mathring{D})^c\nleqslant\tau(\mathring{C})$.

The concepts of fuzzy continuous and fuzzy open maps are introduced in [5, 10–11]. In the following definition, weak form of these concepts are given.

Definition 2.6. Let (X,τ_X) and (Y,τ_Y) be two *fts*'s and $f\colon X\to Y$ a function.

(i) f is said to be P-fuzzy continuous if, for each $A\in P(\tau_Y)$, $f^{-1}(A)\in P(\tau_X)$.

(ii) f is called P-fuzzy open if, for each $A\in P(\tau_X)$, $f(A)\in P(\tau_Y)$.

Theorem 2.7. *Let (X,τ_X) and (Y,τ_Y) be two fts's and $f\colon X\to Y$ a function.*

(i) *If $f\colon(X,\tau_X)\to(Y,\tau_Y)$ is fuzzy continuous, then $f\colon(X,P(\tau_X))\to(Y,P(\tau_Y))$ is P-fuzzy continuous, and so $f\colon(X,T_{P(\tau_X)})\to(Y,T_{P(\tau_Y)})$ is fuzzy continuous (continuous in Chang's sense).*

(ii) *If $f\colon(X,\tau_X)\to(Y,\tau_Y)$ is fuzzy open, then $f\colon(X,P(\tau_X))\to(Y,P(\tau_Y))$ is P-fuzzy open. Thus $f\colon(X,T_{P(\tau_X)})\to(Y,T_{P(\tau_Y)})$ is fuzzy open (Chang's sense).*

Proof. (i) Let us suppose that $f\colon (X,\tau_X)\to(Y,\tau_Y)$ is fuzzy continuous and it is not P-fuzzy continuous. Then there exists a $V\in I^X$ such that $\tau_Y(V)\geqslant 1-V$ and $\tau_X(f^{-1}(V))\geqslant 1-f^{-1}(V)$. Then there exists $x\in X$ auch that $\tau_X(f^{-1}(V))<1-f^{-1}(V)(x)=1-V(f(x))\leqslant\tau_Y(V)$, a contradiction.

(ii) is obvious. ■

Remark 2.8. From (2.7) it can be concluded that a functor F from the category FT of fuzzy topological spaces to the category CFTS can be defined such that $F(X,\tau)=(X,T_{P(\tau)})$ and if $f\colon(X,\tau_X)\to(Y,\tau_Y)$ is fuzzy continuous, then $F(f)\colon(X,T_{P(\tau_X)})\to(Y,T_{P(\tau_Y)})$ is fuzzy continuous (Chang's sense) mapping.

Proposition 2.9. *Let τ_X and τ_Y be two ft's on X and Y, respectively. If a function $f\colon(X,\tau_X)\to(Y,\tau_Y)$ is fuzzy continuous, then we have:*

(i) *for every $A\in I^X$, $f(\overline{A})\subseteq\overline{f(A)}$,*

(ii) *for every $A\in I^Y$, $f^{-1}(\overline{A})\supseteq\overline{f^{-1}(A)}$,*

(iii) *for every $A\in I^Y$, $f^{-1}(\overset{\circ}{A})\subseteq(f^{-1}(A))^\circ$.*

Proof. If $f\colon(X,\tau_X)\to(Y,\tau_Y)$ is fuzzy continuous mapping, then by (2.7) (i), it is P-fuzzy continuous. In CFTS, noticing that the properties (i–iii) are derived without using ($P3$), the properties (i–iii) are nothing but well-known properties of CFTS. ■

Proposition 2.10. *Let $f\colon(X,\tau_X)\to(Y,\tau_Y)$ be a surjective fuzzy open map. Then, for each $A\in I^X$, $f(\overset{\circ}{A})\subseteq f(A)^\circ$.*

Proof. From (2.7) f is a surjective and P-fuzzy open map. Thus, the required inclusion is a well-known result of CFTS which is established in CFTS without needing ($P3$). ■

3. Various types of fuzzy compactness and concerned relations

Smooth compact, smooth almost compact, smooth regular and smooth nearly compact *fts*'s are defined in [3], and more details can be supplied from [3]. In this section, we introduce some new types of fuzzy compactness weaker than the corresponding concepts presented in [3]. The properties of such fuzzy compactness and the connection between such fuzzy compactness and the corresponding concepts given in [3] are investigated.

Proposition 3.1. *Let (X,τ) be an fts. Then we have $A\in P(\tau)\implies\tau(A)>0$, for all $A\in I^X$.*

Proof. For $A\in I^X$, let us assume that $A\in P(\tau)$, i.e. $A^c\leqslant\tau(A)$. If $A=1$, then $\tau(A)=\tau(1)=1>A^c=0$. For the case of $A\neq 1$ we have $A^c\neq 0$ and so there exists $x_0\in X$ such that $A^c(x_0)>0$, i.e. $\tau(A)\geqslant A^c(x_0)>0$ which is the desired result. ■

Definition 3.2. An *fts* (X, τ) is called P-fuzzy compact (*pfc* for short) if every cover of X in $P(\tau)$ has a finite subcover.

Definition 3.3. An *fts* (X, τ) is called P-fuzzy almost compact (*pfac* for short) if for every subfamily $\{ A_j : j \in J \}$ of $P(\tau)$-covering of X, there exists a finite subset J_0 of J such that $\bigvee_{j \in J_0} \overline{A}_j = 1$.

Definition 3.4. An *fts* (X, τ) is said to be P-fuzzy nearly compact (*pfnc* for short) if for every subfamily $\{ A_j : j \in J \}$ of $P(\tau)$-covering of X, there exists a finite subset J_0 of J such that $\bigvee_{j \in J_0} (\overline{A}_j)^{\circ} = 1$.

Theorem 3.5. (i) *smooth compact* $\Longrightarrow$ *pfc,* (ii) *smooth nearly compact* $\Longrightarrow$ *pfnc,* (iii) *smooth almost compact* $\Longrightarrow$ *pfac.*

Proof. Considering (3.2)–(3.4) and using (3.1), the required implications are easily seen. ∎

Proposition 3.6. *pfc* $\Longrightarrow$ *pfnc* $\Longrightarrow$ *pfac.*

Proof. These implications were proven in CFTS without needing $(P3)$ [2, 4], and so the proof is obvious. ∎

It has to be noticed that if an *ft* τ is a CFT, then $P(\tau) = \tau$, and so the *pfc*, *pfnc* and *pfac* are precisely fuzzy compactness, fuzzy nearly compactness and fuzzy almost compactness in Chang's sense, respectively. In CFTS, it was demonstrated that the converse implications in (3.6) are not satisfied in general [2, 4]. Thus the converse implications in (3.6) are not generally true.

Proposition 3.7. *Let τ and σ be two fuzzy topologies on X and Y, respectively, and $f \colon X \to Y$ a surjective P-fuzzy continuous map.*

(i) *If the fts (X, τ) is pfac, then so is (Y, σ).*

(ii) *If the fts (X, τ) is pfnc, then the fts (Y, σ) is pfac.*

Proposition 3.8. *Let $f \colon (X, \tau) \to (Y, \sigma)$ be a surjective and fuzzy continuous function. Then the following hold:*

(i) *If the fts (X, τ) is pfac, then so is (Y, σ).*

(ii) *If the fts (X, τ) is pfnc, then (Y, σ) is pfac.*

Definition 3.9. An *fts* (X, τ) is P-fuzzy regular if each fuzzy set $A \in P(\tau)$ can be written in the form of $A = \bigvee \{ K \in I^X : \tau(K) \geqslant \tau(A), \overline{K} \subseteq A \}$.

One can easily see that a smooth regular *fts* [3] is P-fuzzy regular.

Proposition 3.10. *A pfac and P-fuzzy regular fts is pfc.*

Proposition 3.11. *A pfnc and P-fuzzy regular fts is pfc.*

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