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ITERATIVE COMBINATIONS OF GENERALISED OPERATORS

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Abstract. In the present paper, We consider the generalised form of iterative combinations of positive linear operators with well-known Bernstein and Baskakov operators as its particular case. We have estimated the iterative operator's r^{th} moment and found a recurrence relation between the central moments and their derivatives. We deduce the Voronovskaya type asymptotic formula and the relation between the error of a continuous function and its norm with restrictions on its higher derivatives.

1. Introduction

In 1973 Micchelli [15] had given several results on the saturation class and studied properties of semigroups of operators of the Bernstein operator. In the last section of his dissertation, he defined the iterative operator as:

$$\mathrm{T}_{n,k}\left(f;x\right)=\left[I-\left(I-B_{n}\right)^{k}\right]\left(f;x\right)=-\sum_{v=0}^{k}\left(-1\right)^{v}\binom{k}{v}B_{n}^{v}\left(f;x\right), \tag{1}$$

where B_n is the Bernstein operator and B_n^r is the r^{th} iterate of the operator B_n with $k=1,2,\ldots$ and established the following result $|T_{n,k}(f;x)-f(x)|\leqslant\frac{3}{2}\left(2^k-1\right)\omega\left(f;\delta\right)$, where $\omega(f;\delta)$ is the modulus. The operators (1) proved to be a better approach to find the order for the Bernstein operator.

Inspired by Micchelli [12,15], Agarwal [3] et al. gave more intellectual and sharper results for the new Micchelli-type linear operators such as Voronovskaya type asymptotic approximation of a sufficiently smooth function for the Bernstein operators.

In 1998 Agarwal [3] extended his work on the Micchelli combination of Bernstein operators and gave simultaneous approximation results related to the inverse theorem. Deo [4, 5] had studied both Beta and Baskakov operators and estimated results for Baskakov operators based on Micchelli [15]. Some interesting approximation results are studied by several mathematicians (see [1, 2, 10, 11]).

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We now consider the sequence with the weighted function

$$q_{n.k}(x) = \frac{(-x)^k\,\phi_{n,c}^{(k)}(x)}{k!}; \quad \text{where} \quad \phi_{n,c}(x) = \begin{cases} (1+cx)^{-n/c}; & c=-1,\ x\in[0,1] \\ (1+cx)^{-n/c}; & c>0,\ \ x\in[0,\infty). \end{cases}$$

The generalised form of linear positive operators $L_{n,c}\colon C_2(\mathbb{R}_+)\to C(\mathbb{R}_+)$ is defined as:

$$L_{n,c}(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \tag{2}$$

where $C_2(\mathbb{R}_+) = \left\{ f \in C(\mathbb{R}_+) : \lim\limits_{x\to\infty} \frac{|f(x)|}{1+x^2} < 0 \right\}$. The space $C_2(\mathbb{R}_+)$ endowed with the norm $||f|| = \sup\left\{ \frac{|f(x)|}{1+x^2}; x \geq 0 \right\}$, such that $C_2(\mathbb{R}_+)$ is a Banach space. For $c=-1$, operators (2) represent Bernstein operators and for $c>0$, operators (2) represent Baskakov operators.

Now we consider Micchelli-type iterative combinations of generalized positive linear operators as:

$$T_{n,k} : C_2(\mathbb{R}_+) \to C(\mathbb{R}_+), \quad \text{and} \quad T_{n,k}\,(f;x) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} L_{n,c}^{r}\,(f;x)\,. \tag{3}$$

Only recently, Deo et al. [7] studied generalised positive linear operators based on the Pólya-Eggenberger distribution (PED) as well as the inverse Pólya-Eggenberger distribution (IPED), which are a generalised form of the classical Bernstein and Baskakov operators, and the authors established direct results for these operators, and a variant form of these generalised operators was studied by Dhamija et al. [8]. In 2011, Deo et al. [6, 8] established a generalised form of the Bernstein and Baskakov operators and gave a better approximation using [13].

The aim of this work is to provide generalised results for the operators (3) and to obtain better approximation and estimation results. In Section 2 we investigate auxiliary results and derive the value of the iterative operators (3) at the basic test functions $1, t, t^2$. Considering the auxiliary results, we obtain Voronovskaya-type results, direct estimates in terms of the modulus of continuity, and provide a computational approximation that graphically relates the convergence rate to the error estimation performed in the third section.

2. Auxiliary results

This section of the paper consists of some basic properties and definitions that are further used to prove the main theorems.

Definition 2.1 (Exponential operators). Let $S_\lambda(f;t)$ be a positive operator of the form

$$S_\lambda(f;t) = \int_{-\infty}^{\infty} W(\lambda,t,x) f(x)\,dx,$$

so that $S_\lambda(f;t)$ satisfies the following homogeneous partial differential equation:

$$\frac{\partial}{\partial t}W(\lambda,t,u)=\frac{\lambda}{p(t)}W(\lambda,t,u)(u-t), \tag{4}$$

where $p(t)$ is analytic and positive for $t\in(A,B)$ for some $A,B,-\infty\leqslant A<B\leqslant+\infty$, and the normalization condition

$$S_\lambda(1;t)=\int\limits_{-\infty}^{\infty}W(\lambda,t,u)du=1.$$

Then the operators $S_\lambda(f;t)$ are called exponential operators.

Let $m\in\mathbb{N}^\circ=\mathbb{N}\bigcup\{0\}$ (the set of all non-negative integers), $p\in\mathbb{N}$ (set of natural numbers) and $[\lambda]$ denote the integral part of λ.

Let the m^{th} order moment$a^{\{p\}}_{n,m}(x)$ can be defined as: $a^{\{p\}}_{n,m}(x)=L^{\{p\}}_{n,c}\left((t-x)^n;x\right)$. We will use $a_{n,m}(x)=a^{\{1\}}_{n,m}(x)$, $a^{(1)}_{n,m}(x)$ for the derivative of $a_{n,m}(x)$ with respect to x and $R(j,k;x)$ is the coefficient of $1/n^k$ in the $T_{n,k}\left((t-x)^j;x\right)$.

The following result can easily be proven by mathematical induction.

Lemma 2.2. *For the operator* $L^r_{n,c}(f;x)$*:*
(i) $L^r_{n,c}(1;x)$=1; *(ii)* $L^r_{n,c}(t;x)$=x; *(iii)* $L^r_{n,c}(t^2;x)=x^2\left(1+\frac{c}{n}\right)^r+\frac{x}{c}\left[\left(1+\frac{c}{n}\right)^r-1\right]$.

Lemma 2.3. *There is a recurrence holds for the function* $a_{n,m}(x)$*:*

$$n\,a_{n,m+1}(x)=x(1+cx)\left[a^{(1)}_{n,m}(x)+m\,a_{n,m-1}(x)\right];$$

with $a_{n,0}(x)=1$, $a_{n,1}(x)=0$ *and* $a_{n,2}(x)=\frac{x(1+cx)}{n}$.

Proof. The values corresponding to $a_{n,0}(x)$, $a_{n,1}(x)$ and $a_{n,2}(x)$ result directly from the definition. This proof is continued by proving

$$x(1+cx)\,a^{(1)}_{n,m}(x)=n\,a_{n,m+1}(x)-x(1+cx)m\,a_{n,m-1}(x).$$

The rest is basic computation which is left to the reader. □

3. Direct results

This section is dedicated to introducing approximation theorems and determining the asymptotic Voronovskaya-type formula to discuss the convergence properties.

Theorem 3.1. *For every* $f\in C_2(\mathbb{R}_+)$ *with* $k\in\mathbb{N}^\circ$ *and* $n\in\mathbb{N}$,

$$|\,T_{n,k}(f;x)-f(x)|\leq\omega(f;\delta)\left[\left(2^k-1\right)+\frac{1}{\delta^2}\left\{\left(2+\frac{c}{n}\right)^k-2^k\right\}\left(x^2+\frac{x}{c}\right)\right].$$

Proof. We consider,

$$T_{n,k}\left(f;x\right)-f(x)=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}L^r_{n,c}\left(f(t)-f(x);x\right).$$

From this follows,

$$|T_{n,k}(f;x)-f(x)| \leqslant \sum_{r=1}^{k}\binom{k}{r}L_{n,c}^{r}\left(|f(t)-f(x)|;x\right). \tag{5}$$

Mond and et al. [16] give the result that for all $t,x\in[0,\infty]$ and $\delta>0$,

$$|f(t)-f(x)| \leq \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta). \tag{6}$$

From (5) and (6) and using Lemma 2.2 we obtain

$$\begin{aligned}
|\,T_{n,k}(f;x)-f(x)| &\leqslant \sum_{r=1}^{k}\binom{k}{r}L_{n,c}^{r}\left(\left(1+\frac{(t-\ x)^2}{\delta^2}\right);x\right)\omega(f;\delta)\\
&\leqslant \omega(f;\delta)\left[\sum_{r=1}^{k}\binom{k}{r}L_{n,c}^{r}(1;x)+\frac{1}{\delta^2}\sum_{r=1}^{k}\ L_{n,c}^{r}\left((t-x)^2;x\right)\right],
\end{aligned}$$

and with the help of the central moment $L_{n,c}^{r}((t-x)^2;x)=\left(x^2+\frac{x}{c}\right)\left[\left(1+\frac{c}{n}\right)^r-1\right]$, we get

$$\begin{aligned}
|\,T_{n,k}(f;x)-f(x)| &\leq \omega(f;\delta)\left[\left(2^k-1\right)+\frac{1}{\delta^2}\left\{\left(2+\frac{c}{n}\right)^k-2^k\right\}\left(x^2+\frac{x}{c}\right)\right]\\
&=\omega(f;\delta)\left(2^k-1\right)\left[1+\frac{1}{\delta^2}\left\{\frac{\left(2+\frac{c}{n}\right)^k-2^k}{2^k-1}\right\}\left(x^2+\frac{x}{c}\right)\right].
\end{aligned}$$

If we choose $\delta=n^{-1/2}$ as above, we finally obtain

$$||T_{n,k}(f)-f(x)|| \leq \omega(f;1/\sqrt{n})(2^k-1)\left[1+n\left\{\frac{\left(2+\frac{c}{n}\right)^k-2^k}{2^k-1}\right\}x\left(x+\frac{1}{c}\right)\right],$$

which is the desired answer. □

Remark 3.2. It is easy to verify that for $c>0$, $\frac{\left(2+\frac{c}{n}\right)^k-2^k}{2^k-1}\leqslant c^k\frac{k}{n}=\frac{c_1}{n}$, where $c_1=c^kk$. So we get

$$||T_{n,k}(f)-f|| \leqslant \omega\left(f;1/\sqrt{n}\right)\left(2^k-1\right)\left[1+kc^kx\left(x+\frac{1}{c}\right)\right],\quad x\in[0,\infty),$$

and for $c=-1$, we have

$$||T_{n,k}(f)-f|| \leqslant \omega\left(f;1/\sqrt{n}\right)\left(2^k-1\right)\left(1+\frac{k}{4}\right),\quad x\in[0,1].$$

Lemma 3.3 ([16]). *$a_{n,m}(x)$ is a polynomial in x and $1/n$ with degree of $a_{n,m}(x)$ in both is less than or equal to m with $a_{n,m}(x)=o\left(n^{-[(m+1)/2]}\right)$. The coefficient of $(1/n)^m$ in $a_{n,2m}(x)$ is also $(2m-1)!!\phi(x)^m$, where $a!!$= semifactorial of a and $\phi(x)=x(1+cx)$ and the coefficient of $(1/n)^m$ in $a_{n,2m+1}(x)$ $(2m+1)!!\,\phi^m(x)\phi^{'}(x)\left(\frac{m}{3}\right)$.*

Proof. According to the definition of exponential operators, (2) satisfies the partial differential equation (4). Furthermore, Bernstein polynomials, Szàsz, Post-Widder and Baskakov are all exponential types. The proof therefore follows from [14, Proposition 3.2]. □

LEMMA 3.4 ([3]). *There is the recurrence relation:*

$$a_{n,m}^{\{p+1\}}(x) = \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{m-j} \frac{1}{i!} \frac{d\left(a_{n,m}^{\{p\}}(x)\right)}{dx} a_{n,i+j}^{\{p\}}(x),$$

where $\frac{dy}{dx}$ *denotes the derivative of* y *with respect to* x.

LEMMA 3.5 ([3]). *We have* $a_{n,m}^{\{p\}}(x) = o\left(n^{-[(m+1)/2]}\right)$.

LEMMA 3.6 ([3]). *For the* l*-th moment* ($l \in \mathbb{N}$) *of* $T_{n,k}$, *we have* $T_{n,k}\left((t-x)^l;x\right) = o\left(n^{-k}\right)$.

DEFINITION 3.7. $U_{n,s}(x) = \sum\limits_{k=0}^{\infty} \frac{(-x)^k \phi_{n,c}^{(k)}(x)}{k!} (k-nx)^s$.

The inequality $0 \le U_{n,s}(x) \le K\, n^{[s/2]}; \quad 0 \le x < \infty$, follows from Lemma 3.3.

THEOREM 3.8. *Let* $f \in C_2(\mathbb{R}_+)$ *and* $f^{(2k)}$ *exist at a fixed point* $x \in [0,\infty)$, *then*

$$\lim_{n\to\infty} n^k \left[T_{n,k}(f;x) - f(x)\right] = \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} R(j,k;x). \tag{7}$$

Thus, if $f^{(2k-1)} \in A.C.[0,c]$ *with* $f^{(2k)} \in L_\infty[0,c]$, *then for any proper interval* $[a,b]$ *of* $[0,c]$, *we have*

$$\|T_{n,k}(f;) - f\|_{C[a,b]} \leqslant \frac{C}{n^k}\left\{\|f\|_{C[a,b]} + \left\|f^{(2k)}\right\|_{L_\infty[0,c]}\right\}. \tag{8}$$

Proof. Since f is continuous in $[0,\infty)$, so it has a Taylors series expansion at $t = x$,

$$f(t) = f(x) + \sum_{j=1}^{2k} \frac{f^{(j)}(x)\,(t-x)^j}{j!} + \frac{f^{2k+1}(\xi)\,(t-x)^{2k+1}}{(2k+1)!}, 0 < \xi < \infty. \tag{9}$$

Now we consider

$$n^k\left[T_{n,k}(f;x) - f(t)\right] = n^k \sum_{j=1}^{2k} \frac{f^j(x)}{j!} T_{n,k}[(t-x)^j;x]$$
$$+ n^k \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} L_{n,c}^r\left(\epsilon(t;x)(t-x)^{2k};x\right) = G_1 + G_2,$$

where $\epsilon(t,x) \to 0$ as $t \to x$.

So for a given $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ such that $|\epsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. First we evaluate G_1,

$$G_1 = n^k \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} T_{n,k}\left((t-x)^j;x\right) + n^k\left[f'(x)\; T_{n,k}(t-x;x)\right].$$

From Lemma 3.6, we get $T_{n,k}((t-x)^j;x) = o(n^{-k})$. We can directly compute that,

$$G_1 = \sum_{j=2}^{2k} \frac{f^j(x)}{j!} R(j,k;x) + o(1). \tag{10}$$

Now, we estimate G_2,

Let $\Theta_\delta(t)$ be the characteristic function in $(t-\delta, t+\delta)$, we have

$$\begin{aligned} |\, G_2 \,| &\leqslant n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left(|\varepsilon(t,x)||t-x|^{2k} \Theta_\delta(t); x \right) \\ &\quad + n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left(|\varepsilon(t,x)||t-x| \right)^{2k} \left(1 - \Theta_\delta(t)\right); x \right) \\ &= G_{21} + G_{22}\,. \end{aligned}$$

Therefore,

$$G_{21} \leqslant \sup_{|t-x|<\delta} |\varepsilon(t,x)| n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left((t-x)^{2k}; x \right) \leqslant \varepsilon n^k \sum_{r=1}^{k} a_{n,2k}^{\{r\}} \binom{k}{r} = \varepsilon C_1.$$

Now

$$|G_{22}| \leqslant n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left(|\varepsilon(t,x)|(t- \ x)^{2k} \left(1 - \Theta_\delta(t)\right); x \right).$$

For an arbitrary $p > k$ and using lemma 3.5, we have

$$L_{n,c}^r(|\varepsilon(t,x)|\,|t-x|^{2k}; x) \leq \frac{M_2}{\delta^{2(p-k)}} L_{n,c}^r((t-x)^{2p}; x)$$

and

$$G_{22} \leqslant \frac{M_3}{n^{p-k}} = o(1).$$

Since $\varepsilon > 0$ is an arbitrary, $G_2 \to 0$ as $n \to \infty$. With G_1 and G_2, (7) follows immediately.

Consider

$$\begin{aligned} T_{n,k}(f(t);x) - f(x) &= T_{n,k}\left(\varphi(t)\left(f(t) - f(x)\right); x\right) + T_{n,k}\left(\left(1 - \varphi(t)\right)\left(f(t) - f(x)\right); x\right) \\ &= G_3 + G_4, \end{aligned}$$

where $\varphi(x)$ is the characteristics function of the closed interval on $[0, c]$.

For all $t \in [0, c]$ and $x \in [a, b]$, we have

$$f(t) - f(x) = \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!} (t-x)^l + \frac{1}{(2k-1)!} \int_x^t (t-s)^{2k-1} f^{(2k)}(s) ds$$

and

$$G_3 = \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!} T_{n,k}\left(\varphi(t)(t-x)^l; x\right) + \frac{1}{(2k-1)!} T_{n,k}\left(\int_x^t \varphi(t)(t-s)^{2k} f^{(2k)}(s) ds \right)$$

$$= \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!} \left[T_{n,k}\left((t-x)^l ; x \right) + T_{n,k}\left((\varphi(t)-1)\,(t-x)^l ; x \right) \right]$$

$$+ \frac{1}{(2k-1)!} \underset{n,k}{T} \left(\int_x^t \varphi(t)\,(t-s)^{2k} f^{(2k)}(s)ds \right)$$

$$= \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!} \left(G_{31}+G_{32}\right) + G_{33}.$$

Clearly, $G_{31} = o\left(\frac{1}{n^k}\right)$ by Lemma 3.6. Similarly by definition of characteristics function, $G_{31} = o\left(\frac{1}{n^k}\right)$. Convergence of G_{31} and G_{32} is uniform for all $x \in [a, b]$ & $t \in [0, b]$. Also,

$$G_{33} \leqslant \frac{\left\| f^{(2k)} \right\|_{L_\infty[0,c]}}{(2k-1)!} \leqslant \frac{M_{11} \left\| f^{(2k)} \right\|_{L_\infty[0,c]}}{n^k}, \text{ where } M_{11} \text{ is a constant.}$$

Combining the above results and by interpolation property of norms introduced by Goldberg and Meir [9], we have

$$G_3 \leqslant M_{12} \left\{ \|f\|_{C[a,b]} + \left\| f^{(2k)} \right\|_{L_\infty[0.c]} \right\}. \tag{11}$$

For G_4, we proceed on the same lines as we have done for (7) and for G_3.

$$G_4 \leqslant M_{21} \left\{ \sum_{l=0}^{2k-1} \left\| f^{(l)} \right\|_{C[a,b]} + \left\| f^{(2k)} \right\|_{L_\infty[0,c]} \right\}. \tag{12}$$

Hence from (11) and (12), we obtain the required result 8. □

Theorem 3.9. *Suppose that* $k \in \mathbb{N}^\circ$. *If the function* $f, f', f'', \ldots, f^{(2k+1)}$ *are in the class of* $C[0,\infty)$ *and* $f^{(2k+1)} \in Lip_M\, 1$ *on* $[0,\infty)$, *then*

$$|\, T_{n.k}(f;x) - f(x)| = o\left(\frac{1}{n^{k+1}}\right), \quad \textit{uniformly as } n \to \infty \textit{ on } [0,\infty).$$

Proof. Let us define, $||f||_k = \max\limits_{0\le j\le 2k} \left\{ ||\, f^j\, ||, M \right\}$. It will be sufficient to prove that $|T_{n,k+1}(f;x) - f(x)| \le \frac{A_k\, ||f||_k}{n^{k+1}}$, where A_k is a constant independent of f and n. We will prove it by mathematical induction.

For the case $k = 0$, $|\, T_{n,0}(f;x) - f(x)| = |\, L_{n,c}(f;x) - f(x)|$. By Taylor series expansion, we obtain $|\, L_{n,c}(f(t) - f(x);x)| \le \frac{x(1+cx)}{2n} f''(x)$. Suppose the theorem is true for $j < k$ and f satisfies the hypothesis of the theorem then we have

$$f(x) - \sum_{i=0}^{2k+1} \frac{f^{(i)}(x)}{i!} (x-x_0)^i = \frac{1}{(2k+1)!} \int_{x_0}^{x} (x-\eta)^{2k+1} f^{(2k+1)}(\eta)d\eta$$

$$\le \frac{1}{(2k+1)!} \frac{\left[(x-\eta)^{2k+2}\right]_{x_0}^{x} ||f||_k}{(2k+2)}.$$

Since $L_{n,c}$ are positive linear operators,

$$\left| L_{n,c}f(x) - \sum_{i=0}^{2k+1} \frac{f^{(i)}(x)}{n^i i!} U_{n,i}(x) \right| \leqslant \frac{1}{(2k+1)! n^{2k+2}} U_{n,2k+2}(x) ||f||_k,$$

We can rewrite this inequality as:

$$(L_{n,c} - I)(f;x) = \sum_{i=0}^{2k+1} \frac{f^{(i)}(x)}{n^i \, i!} U_{n,i}(x) + e_n(x), \quad \text{where} \quad |\, e_n(x)| \leq \frac{||f||_k \, a_k}{n^{k+1}}.$$

Now, consider the function $J_{n,l}(x) = \frac{f^{(l)}(x)\, U_{n,l}(x)}{l!\, n^{[l/2]}}$, $l = 2, 3, \ldots, 2k$. This satisfies the hypothesis of the theorem for the integers $k_l = \left[k - \frac{l}{2}\right]$ as $k_l < k$.

By the definition of $J_{n,l}$ applying induction $\left|(L_{n,c} - I)^{k_i+1}(J_{n,i};x)\right| \leqslant \frac{b_{k_i}}{n^{k_i}} ||J_{n,i}||_{k_i}$. Also $\frac{J_{n,i}}{i!\, n^{[i/2]}}$ is a polynomial of degree i which is uniformly bounded in n:

$$0 \leq \frac{J_{n,i}}{i!\, n^{[i/2]}} \leq \frac{C}{i!}; \qquad \text{and} \qquad ||J_{n,i}||_{k_i} = ||\, f^{(i)} \,|| \leq ||f||_k \, .$$

We can easily estimate, $|\,(L_{n,c} - I)^k (J_{n,i};x)| \leq \frac{b_k}{n^{k+1}} \, ||f||_k$, $i = 2, 3 \ldots, 2k$. Moreover, $||\,(L_{n,c} - I)^k \,|| \leq 2^k \, m_k$, where m_k is independent of f and n. Therefore we have,

$$|\, T_{n,k+1}(f;x) - f(x)| = \left| (L_{n,c} - I)^k \left(\sum_{i=2}^{2k} \frac{J_{n,i} n^{[i/2]}}{n^i} + e_n(x); x \right) \right|$$

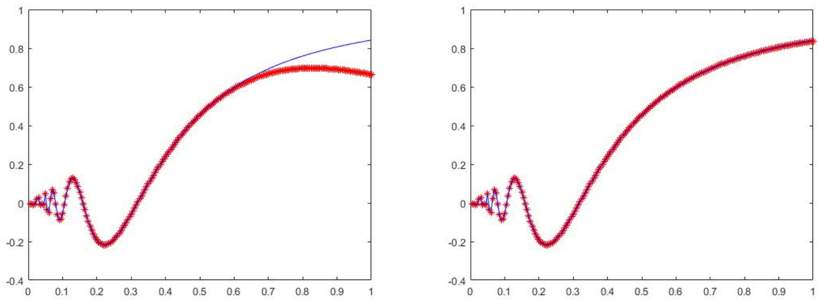
$$\leq \left\{ A_k \sum_{i=2}^{2k} \frac{n^{[i/2]}}{n^{k_i+1}\, n^i} + \frac{2^k \, a_k \, m_k}{n^{k+1}} \right\} ||f||_k = \frac{2^k \, a_k \, m_k + b_k(2k-1)}{n^{k+1}} \, ||f||_k \, .$$

So, we obtain the required result by mathematical induction. □

x	$\|\, T_{n,10}(f;x) - f(x)\|$	$\|\, T_{n,30}(f;x) - f(x)\|$
0.88	0.10510	0.0120
0.89	0.11070	0.00140
0.895	0.11650	0.00160
0.9050	0.12240	0.00180
0.9150	0.12830	0.00210
0.9300	0.13440	0.00240
0.940	0.14040	0.00280
0.950	0.14660	0.00310
0.960	0.15280	0.00360
0.970	0.15900	0.00400
0.980	0.16530	0.00450
0.990	0.17160	0.00510
1.00	0.17800	0.00560

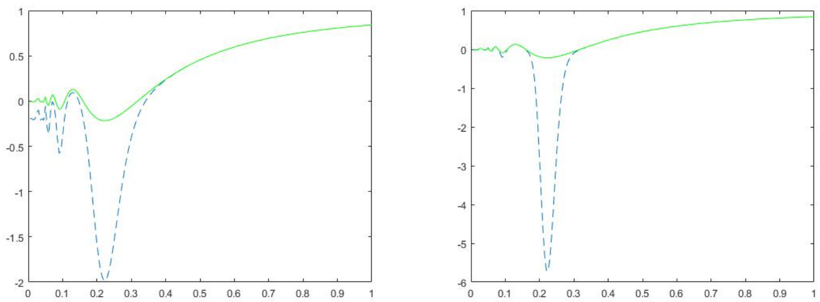
Table 1: Comparing convergence of operators for different values of k for $f(x) = x \sin(1/x)$ and $c > 0$

EXAMPLE 3.10. We have approximated the rate of convergence of the operators $T_{n,k}(f)$ to the function $f(x) = x\sin(1/x)$ for different values of k while keeping $c > 0$ (see Table 1). As the conclusion comes from the table and using the graphical technique, for the value of $k = 10$ the error estimation is less than 0.1 and for $k = 30$ the error is less than 0.001.



(a) k=10, c =23.6 (b) k=30, c =23.6

Figure 1: $T_{n,k}(f;x) * * **$, $f(x) = x\sin(1/x) --$, $n = 100$



(a) k=10, c = 1 (b) k=30, c = 1

Figure 2: $T_{n,k}(f;x) - - -$, $f(x) = x\sin(1/x) -$, $n = 100$

4. Conclusion

We take the generalised form of iterative combinations of positive linear operators with known Bernstein and Baskakov operators to obtain better approximate as well as estimated results. The iterative operators rth moment has been computed at the basic test functions $1, t, t^2$. We establish a relationship between the central moments and their derivatives.

We obtain the asymptotic formula and define the connection between the error of the continuous function in terms of the modulus of continuity and its norm with limits on its higher derivatives.

The effectiveness of the modified operators is illustrated by the inclusion of graphs and corresponding results.

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