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THREE WEAK SOLUTIONS FOR A $p(x)$-LAPLACIAN EQUATION

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Abstract. We study the existence of three weak solutions to the Dirichlet boundary condition for a $p(x)$-Laplacian equation. Using a variational method and the three critical point theorem, we would show the existence and multiplicity of the solutions. For this purpose, we focus on a generalized variable exponent Lebesgue-Sobolev space.

1. Introduction

In this article we study the following problem:

$$\begin{cases} -div[O(x,|\nabla u|)\nabla u]+|u|^{p(x)-2}u=\lambda(a(x)|u|^{q(x)-2}-b(x)|u|^{r(x)-2})u & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with a sufficiently smooth boundary. Let λ be a positive real parameter and p, q and r be real continuous functions on $\bar{\Omega}$ with $1<q(x)<r(x)<p(x)<p^*(x)$, where $p^*(x)=\dfrac{Np(x)}{N-p(x)}$ and $p(x)<N$ for all $x\in\bar{\Omega}$, $O(x,\xi)$ is of type $|\xi|^{p(x)-2}$. $\Delta_{p(x)}u:=div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplacian operator (for details see [2, 8, 15]) . We consider the following conditions.

(H_1) $O:\bar{\Omega}\times[0,\infty)\to\mathbb{R}$ is a continuous function such that

$$C_1t^{p(x)-2}\le O(x,t)\le C_2t^{p(x)-2},$$

for all $t\ge 0$ and for all $x\in\bar{\Omega}$, where C_1, C_2 are positive constants and $p\in C(\bar{\Omega})$ such that $1<p(x)<p^*(x)$ for all $x\in\bar{\Omega}$.

(H_2) a and b are positive functions in $L^\infty(\bar{\Omega})$ and there exists $\varepsilon>0$ for all $x\in\bar{\Omega}$, such that $\varepsilon<a(x)$ and $\varepsilon<b(x)$.

(H_3) $\|a\|_\infty<\|b\|_\infty$.

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Many results have been obtained on this kind of problems. For literature in [11], the authors studied the existence of at least one positive radial solution for the problem:

$$\begin{cases} -\triangle_{p(x)}u + R(x)|u|^{p(x)-2}u = a(x)|u|^{q(x)-2}u - b(x)|u|^{r(x)-2}u & x \in B, \\ u > 0 & x \in B, \\ u = 0 & x \in \partial B, \end{cases}$$

where B is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$. In [15], V. F. Uta considered the existence of minimum action solutions and the concentration of the spectrum in a bounded interval for the following problem using the Mountain pass theorem and the Nehari manifold technique:

$$\begin{cases} -div[\Phi(x, |\nabla u|)\nabla u] = \lambda(g(x)|u|^{q(x)-2}u + |u|^{r(x)-2}u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

In [13], I. D. Stircu, studied the existence at least two weak solutions for the following problem using the Mountain pass theorem:

$$\begin{cases} -div[\Phi(x, |\nabla u|)\nabla u] + |u|^{p(x)-2}u = \lambda|u|^{r(x)-2}u - h(x)|u|^{s(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Ismail Aydin and Cihan Unal in [1] have found the existence of at least three weak solutions to the following Steklov problem using the three critical points theorem:

$$\begin{cases} div(a(x)|\nabla u|^{p(x)-2}\nabla u) = b(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = \lambda f(x, u) & \text{on } \partial\Omega. \end{cases}$$

In [14], S. Taarabti, Z. E. Allali and K. B. Haddouch studied the following $p(x)$-biharmonic problem using the three critical points theorem:

$$\begin{cases} \Delta^2_{p(x)} + a(x)|u|^{p(x)-2}u = \beta V(x)f(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = \dfrac{\partial}{\partial v}(|\Delta u|^{p(x)-2}\Delta u) = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, we study the existence and multiplicity of the solutions for the problem (1) by using the variational method and the three critical point theorem.

2. Preliminaries

We recall some necessary definitions and propositions concerning the Lebesgue and Sobolev spaces. Let Ω be a bounded domain of $\mathbb{R}^N$. Set $C_+(\Omega) := \{s \in C(\bar{\Omega}); s(x)>1, \forall x \in \bar{\Omega}\}$. For any continuous function $s : \Omega \to (1, \infty)$, $s^- := \inf_{x\in\Omega} s(x)$ and $s^+ := \sup_{x\in\Omega} s(x)$. For $s \in C_+(\bar{\Omega})$, $L_{s(x)}(\Omega) := \{u : \Omega \to \mathbb{R}$ is a measurable function: $\int_\Omega |u|^{s(x)}dx<+\infty\}$, endowed with the norm $\|u\|_{s(x)}:=\inf\left\{\mu>0 : \int_\Omega \left|\dfrac{u(x)}{\mu}\right|^{s(x)} dx \leq 1\right\}$. $L_{s(x)}(\Omega)$ is well known that is a separable reflexive Banach space [3, 7, 9].

The modular of the $L_{s(x)}(\Omega)$ is defined by $\sigma_{s(x)}(u) := \int_\Omega |u(x)|^{s(x)}dx$.

Proposition 2.1 ([5,8]). *$(L_{s(x)}(\Omega), \|.\|_{s(x)})$ is separable, uniformly convex, reflexive Banach space and its conjugate space is $(L_{s'(x)}(\Omega), \|.\|_{s'(x)})$, where $\frac{1}{s(x)} + \frac{1}{s'(x)} = 1$, $\forall x \in \Omega$. For all $u \in L_{s(x)}(\Omega)$ and $w \in L_{s'(x)}(\Omega)$, we have*

$$|\int_\Omega uw\, dx| \leq (\frac{1}{s^-} + \frac{1}{s'^-})\|u\|_{s(x)}\|w\|_{s'(x)} \leq 2\|u\|_{s(x)}\|w\|_{s'(x)}.$$

Proposition 2.2 ([6,9]). *Suppose that $u, u_n \in L_{s(x)}(\Omega)$, we have*

$$\|u\|_{s(x)} < 1 \Rightarrow \|u\|_{s(x)}^{s^+} \leq \sigma_{s(x)}(u) \leq \|u\|_{s(x)}^{s^-}.$$
$$\|u\|_{s(x)} > 1 \Rightarrow \|u\|_{s(x)}^{s^-} \leq \sigma_{s(x)}(u) \leq \|u\|_{s(x)}^{s^+}.$$
$$\|u\|_{s(x)} < 1(resp, = 1; > 1) \Leftrightarrow \sigma_{s(x)}(u) < 1(resp, = 1; > 1).$$
$$\|u_n\|_{s(x)} \to 0(resp, \to +\infty) \Leftrightarrow \sigma_{s(x)}(u_n) \to 0(resp, \to +\infty).$$
$$\lim_{n\to\infty} \|u_n - u\|_{s(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \sigma_{s(x)}(u_n - u) = 0.$$

The Sobolev space $W^{1,s(x)}(\Omega)$, $W^{1,s(x)}(\Omega) := \left\{u \in L_{s(x)}(\Omega) : |\nabla u| \in L_{s(x)}(\Omega)\right\}$ is a separable and reflexive Banach spaces with norm $\|u\|_{1,s(x)} = \|u\|_{s(x)} + \|\nabla u\|_{s(x)}$. For more details, we refer to [4, 9].

On $W_0^{1,s(x)}(\Omega)$, we may consider the following equivalent norm $\|u\|_{s(x)} = \|\nabla u\|_{s(x)}$, where $W_0^{1,s(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{s(x)}\right) dx \leq 1\right\}.$$

It is known that $W_0^{1,s(x)}(\Omega) := \left\{u;\ u\Big|_{\partial\Omega} = 0,\ u \in L^{s(x)}(\Omega),\ |\nabla u| \in L^{s(x)}(\Omega)\right\}$. For more details, we refer to [2, 4, 15].

Proposition 2.3 ([5, Sobolev Embedding]). *For $s, s' \in C_+(\bar{\Omega})$ and $1 < s'(x) < s^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous compact embedding $W_0^{1,s(x)}(\Omega) \hookrightarrow L_{s'(x)}(\Omega)$, which is continuous and compact. Therefore, there is a constant $c_0 > 0$ such that $\|u\|_{s'(x)} \leq c_0\|u\|$.*

Proposition 2.4 ([10, Poincare Inequality]). *There is a constant $c > 0$ such that $\|u\|_{s(x)} \leq C\|\nabla u\|_{s(x)}$, for all $u \in W_0^{1,s(x)}(\Omega)$.*

Remark 2.5. From Proposition 2.4, $\|\nabla u\|_{s(x)}$ and $\|u\|_{1,s(x)}$ are equivalent norms on $W_0^{1,s(x)}(\Omega)$.

3. Main results

Before to the proceed the results, we need some notions.

Definition 3.1. $u \in W_0^{1,p(x)}(\Omega)$ is called a *weak solution* for (1) if

$$\int_\Omega O(x, |\nabla u(x)|)\nabla u(x)\nabla h(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x)h(x)dx$$

$$= \lambda \int_\Omega [a(x)|u(x)|^{q(x)-2}u(x)h(x) - b(x)|u(x)|^{r(x)-2}u(x)h(x)]dx,$$

for all $h \in W_0^{1,p(x)}(\Omega)$. In what follows

$$A_0(x,z) := \int_0^z O(x,t)t\,dt,$$

and $$A : W_0^{1,p(x)}(\Omega) \to \mathbb{R} \quad \text{by} \quad A(u) := \int_\Omega A_0(x, |\nabla u(x)|)dx.$$

The energy functional associated to problem (1) can obtained by

$$J(u) = \int_\Omega A_0(x,|\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \lambda \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}dx + \lambda \int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)}dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is well defined, C^1 functional and for all $u, h \in W_0^{1,p(x)}(\Omega)$,

$$\langle J'(u), h\rangle = \int_\Omega O(x,|\nabla u|)\nabla u.\nabla h dx + \int_\Omega |u|^{p(x)-2}uhdx$$
$$- \lambda \int_\Omega a(x)|u|^{q(x)-2}uhdx + \lambda \int_\Omega b(x)|u|^{r(x)-2}uhdx.$$

Therefore, critical points of this energy functional are week solutions for the problem (1). We consider $\Omega \subset \mathbb{R}^N (N > 3)$ as a bounded domain with smooth boundary and $p \in C_+(\Omega)$ such that

$$1 < q^- \le q(x) \le q^+ < r^- \le r(x) \le r^+ < p^- \le p(x) \le p^+ < p^*(x) \tag{2}$$

THEOREM 3.2 ([12]). *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a continuous Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X', $\Psi : X \to \mathbb{R}$ is a continuous Gateaux differentiable functional whose Gateaux derivative is compact. Suppose the following assertions:*
(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \pm\infty$, *for all* $\lambda > 0$,
(ii) There exist $e \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < e < \Phi(u_1)$,
(iii) $\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) > \dfrac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}$.
Then there exist an open interval $\Lambda \subset (0, +\infty)$ and a positive real number γ such that the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ admits at least three solutions in X whose norms are less than γ, for all $\lambda \in \Lambda$.

THEOREM 3.3. *If* (2) *and (H_1)-(H_3) hold. Then, there exist an open interval $\Lambda \subset (0, +\infty)$ and a positive real number γ such that for any $\lambda \in \Lambda$, the problem* (1) *has at least three solutions in $W_0^{1,p(x)}(\Omega)$ whose norms are less than γ.*

PROPOSITION 3.4 ([1]). *Let us define the functional $\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by*

$$\Phi(u) = \int_\Omega A_0(x,|\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. *Then, we have*

(i) $\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *is sequentially weakly lower semicontinuous and* $\Phi \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. *Moreover, the derivative operator* Φ' *of* Φ *define as*

$$\langle \Phi'(u), h\rangle = \int_\Omega O(x, |\nabla u|)\nabla u \nabla h dx + \int_\Omega |u|^{p(x)-2} uh dx.$$

for all $u, h \in W_0^{1,p(x)}(\Omega)$.

(ii) $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ *is a continuous, bounded and strictly monotone operator.*

(iii) The mapping $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ *is of* (S_+) *type, i.e., if* $u_n \rightharpoonup u$ *as* $n \to \infty$ *and* $\limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \le 0$ *implies* $u_n \to u$.

(iv) $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ *is a homeomorphism.*

Let

$$\Psi(u) = \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)} dx - \int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)} dx.$$

We have
$$\langle \Psi'(u), h\rangle = \int_\Omega a(x)|u|^{q(x)-2} uh\, dx - \int_\Omega b(x)|u|^{r(x)-2} uh\, dx.$$

Therefore, Ψ is a C^1- function on $W_0^{1,p(x)}(\Omega)$ and by [3], Ψ' satisfied the condition (S_+). By using H_2 and the compact Sobolev embedding $W_0^{1,s(x)}(\Omega) \hookrightarrow L_{q(x)}(\Omega)$ and $W_0^{1,s(x)}(\Omega) \hookrightarrow L_{r(x)}(\Omega)$. It is direct to see that Ψ' is compact.

Proof (Proof of Theorem 3.3). To prove this theorem, we first verify the condition (i) of Theorem 3.2

$$\Phi(u) = \int A_0(x, |\nabla u|)dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx = \int_\Omega \left[\int_0^{|\nabla u|} O(x,t)t\, dt\right] + \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx$$

$$\ge \int_\Omega \left[C_1 \int_0^{|\nabla u|} t^{p(x)-1} dt\right] dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx \ge \frac{C_1}{p^+}\int_\Omega |\nabla u|^{p(x)} dx + \frac{1}{p^+}\int |u|^{p(x)} dx.$$

Set $C_2 = \min\{\frac{C_1}{p^+}, \frac{1}{p^+}\}$. If $\sigma_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)} dx$ and $\sigma_{s(x)}(u) > 1$, by Proposition 2.4, Proposition 2.2 and (2)

$$\Phi(u) \ge C_2 \|u\|^{p^-}. \tag{3}$$

On the other hand,

$$\Psi(u) = \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)} dx - \int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)} dx$$
$$\ge \frac{1}{q^+}\int_\Omega a(x)|u|^{q(x)} dx - \frac{\|b\|_\infty}{r^-}\int_\Omega |u|^{r(x)} dx \ge -\frac{\|b\|_\infty}{r^-}\int_\Omega |u|^{r(x)} dx.$$

If $\sigma_p(u) > 1$, by Proposition 2.4, Proposition 2.2 and (2)

$$\Psi(u) \ge -\frac{\|b\|_\infty}{r^-}\|u\|^{r^+}. \tag{4}$$

By (3), (4) and for any $\lambda > 0$, we obtain $\Phi(u) + \lambda\Psi(u) \geq C_2\|u\|^{p^-} - \lambda\frac{\|b\|_\infty}{r^-}\|u\|^{r^+}$. Since (2), then $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$, for all $\lambda > 0$ and (i) of Theorem 3.2 is verified.

Choosing $k < d^{p^-}|\Omega|$, $0 < e < \frac{k}{p^+}$, $u_0(x) = 0$ and $u_1(x) = d$ such that $d > 1$, then

$$\Phi(u_0) = \Psi(u_0) = 0 \quad \text{and} \quad \Phi(u_1) = \int_\Omega \frac{1}{p(x)} d^{p(x)} dx \geq \frac{d^{p^-}}{p^+}|\Omega| > e.$$

Thus $\Phi(u_0) < e < \Phi(u_1)$. Then (ii) of Theorem 3.2 is verified.

On the other hand, by (H_3), (2), $d > 1$ and choosing $\frac{d^{q^+}}{q^-} < \frac{d^{r^-}}{r^+}$,

$$\begin{aligned}
&-\frac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} = -e\frac{\Psi(u_1)}{\Phi(u_1)} \\
&= -e\frac{\int_\Omega \frac{a(x)}{q(x)} d^{q(x)} dx - \int_\Omega \frac{b(x)}{r(x)} d^{r(x)} dx}{\int_\Omega \frac{1}{p(x)} d^{p(x)} dx} > -e\frac{\frac{\|a\|_\infty}{q^-} d^{q^+}|\Omega| - \frac{\|a\|_\infty}{r^+} d^{r^-}|\Omega|}{\frac{d^{p^+}}{p^-}|\Omega|} \\
&= -e\frac{(\frac{d^{q^+}}{q^-} - \frac{d^{r^-}}{r^+})\|a\|_\infty}{\frac{d^{p^+}}{p^-}} > 0.
\end{aligned} \tag{5}$$

Let $u \in W_0^{1,p(x)}(\Omega)$ such that $\Phi(u) \leq e$ and $e < C_2$. By (3) and Proposition 2.2, we have $C_2\|u\|^{p^-} \leq \Phi(u) \leq e$. So

$$\|u\| \leq (\frac{e}{C_2})^{\frac{1}{p^-}} < 1. \tag{6}$$

From (H_2), (2), (4) and (6)

$$-\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,e]} -\Psi(u) \leq \sup[\frac{\|b\|_\infty}{r^-}\|u\|^{r^+} - \frac{\varepsilon}{q^+}\|u\|^{q^-}] \leq 0. \tag{7}$$

Then by (5) and (7)

$$-\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) < -\frac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)},$$

and
$$\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) > \frac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

This completes the proof. □

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