MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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REFLEXIVITY IN WEIGHTED VECTOR-VALUED SEQUENCE SPACES

Issam Aboutaib and Lahbib Oubbi

Abstract. We deal with barrelledness, distinguishedness and reflexivity properties in various weighted vector-valued summable sequence spaces, with weights in the α-dual of a perfect scalar-valued sequence space Λ. A weaker notion of distinguishedness is introduced and characterized. A nice example showing the relevance of this notion is provided.

1. Introduction

The space $\Lambda(E)$ of all sequences $(x_n)_n$ in a Fréchet space E such that the series $\sum \alpha_n x_n$ converges for every weight $(\alpha_n)_n$ in the α-dual space of a perfect scalar sequence space Λ was studied in [6, 7]. The authors determined the topological dual $\Lambda(E)'$ whenever $\Lambda(E)$ has the property AK and established conditions under which such a space is reflexive. Since reflexivity is related to both barrelledness and distinguishedness, we want to investigate these three notions in $\Lambda(E)$ for a general locally convex Hausdorff space E.

Since the dual space of $\Lambda(E)$ is given by spaces of strongly Λ-summable sequence spaces, which in turn are defined by Λ-weakly summable sequence spaces, we also investigate these properties in the spaces $\Lambda\langle E\rangle$ and $\Lambda[E]$ of such sequences.

When is one of these spaces (quasi-)barrelled, distinguished or (semi-)reflexive is the question we are concerned with in this note. To answer this question, we first determine the topological dual space of each of these spaces and the equicontinuous sets therein, where E is an arbitrary locally convex Hausdorff space. We then give answers to our question. At the end, a weaker notion of distinguishedness is introduced and characterized in AK-spaces $\Lambda(E)$. An example is given to show the relevance of this notion.

The paper is presented as follows. After a section giving preliminary results and definitions, Section 3 is devoted to the (quasi-) barrelledness of the spaces $\Lambda(E)$ and

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$\Lambda\langle E\rangle$. Section 4 deals with reflexivity and semi-reflexivity in $\Lambda(E)$, while Section 5 focuses on the distinguishedness properties in $\Lambda(E)$.

2. Preliminaries

Throughout this paper, (E,τ) will be a Hausdorff locally convex space over the field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, E' will be its topological dual and $\mathcal{M}$ the collection of all $\sigma(E',E)$-closed equicontinuous discs of E'. Obviously τ is defined by the semi-norms

$$P_M(x)=\sup\{|a(x)|,\quad a\in M\},\quad x\in E,\quad M\in\mathcal{M}.$$

Denote by $\mathbb{N}$ the set of all positive integers, by $E^{\mathbb{N}}$ the set of all sequences from E and by $E^{(\mathbb{N})}$ the subspace of $E^{\mathbb{N}}$ consisting of those sequences with finite support.

The Köthe dual of a sequence space $\Omega\subset E^{\mathbb{N}}$ is defined by

$$\Omega^*:=\left\{(a_n)_n\subset E':\ \sum_{n=1}^{+\infty}|a_n(x_n)|<+\infty,\quad (x_n)_n\in\Omega\right\}.$$

If Ω is endowed with a linear topology $\mathcal{T}$ and contains $E^{(\mathbb{N})}$, then the closure of $E^{(\mathbb{N})}$ in Ω will be denoted by Ω_r. A sequences $(\alpha_n)_n\in\Omega_r$ turns out to be the limit of its finite sections $\alpha^{(k)}:=(\alpha_1,\dots,\alpha_k,0,0\dots)$. If for $t\in E$, te_n denotes the sequence whose only non-zero entree is the n^{th} one, which equals t, then $\alpha^{(k)}=\sum_{n=1}^{k}\alpha_n e_n$. The space $(\Omega,\mathcal{T})$ is AK, if $\Omega=\Omega_r$.

Now, for $\Lambda\subset\mathbb{K}^{\mathbb{N}}$ recall that a sequence $(x_n)_n\subset E$ is said to be

1. Λ-summable if the series $\sum\alpha_n x_n$ converges in E, for every $\alpha:=(\alpha_n)_n\in\Lambda^*$.

2. weakly Λ-summable if $\sum_{n=1}^{\infty}|\alpha_n a(x_n)|$ converges, for every $a\in E'$ and $\alpha:=(\alpha_n)_n\in\Lambda^*$. This is $(a(x_n))_n\in(\Lambda^*)^*$, for all $a\in E'$.

3. strongly Λ-summable, if for every $M\in\mathcal{M}$ and every $(a_n)_n\in\Lambda^*[E'_M]$, the sequence $(a_n(x_n))_n$ belongs to ℓ^1.

The sets $\Lambda(E)$, $\Lambda[E]$ and $\Lambda\langle E\rangle$ of all Λ-summable, weakly Λ-summable and strongly Λ-summable sequences from E, respectively, are linear spaces with respect to the coordinatewise operations, such that $\Lambda(E)\subset\Lambda[E]$.

From now on, $\Lambda\subset\mathbb{K}^{\mathbb{N}}$ will stand for a perfect space. This is $(\Lambda^*)^*=\Lambda$. Then necessarily $\mathbb{K}^{(\mathbb{N})}\subset\Lambda$. The space Λ will be equipped with the polar topology $\tau_{\mathcal{S}}$ defined by an upward directed cover $\mathcal{S}$ of Λ^*, consisting of normal $\sigma(\Lambda^*,\Lambda)$-closed and bounded discs. This topology is given by the semi-norms

$$P_S\big((\alpha_n)_n\big):=\sup\{\sum_{n=1}^{+\infty}|\alpha_n\beta_n|,\ (\beta_n)_n\in S\},\quad (\alpha_n)_n\in\Lambda,\quad S\in\mathcal{S}.$$

If A is a bounded disc in a Hausdorff locally convex space F, let F_A denote the linear span of A. When no topology is specified on F_A, it will be equipped with the gauge $\|\ \|_A$ of A as a norm. We will then consider without any further mention the spaces E_B, Λ_R, $E'_M:=(E')_M$, and $\Lambda^*_S:=(\Lambda^*)_S$, for any bounded discs $B\subset E$, $R\subset\Lambda$, $M\in\mathcal{M}$, and $S\in\mathcal{S}$, respectively.

The spaces $\Lambda(E)$ and $\Lambda[E]$ will be equipped with the topology $\varepsilon_{\mathcal{S},\mathcal{M}}$ generated by the semi-norms

$$\varepsilon_{S,M}((x_n)_n) := \sup\left\{\sum_{n=1}^{\infty}|\alpha_n f(x_n)|, \quad \alpha\in S,\ f\in M\right\}, \quad S\in\mathcal{S},\ M\in\mathcal{M}.$$

With respect to this topology, $\Lambda(E)$, $\Lambda(E)_r$ and $\Lambda[E]_r$ are closed in $\Lambda[E]$. Therefore $\Lambda(E)_r = \Lambda[E]_r$. Moreover, it is easily seen that the following equalities hold algebraically $(\Lambda(E)_r)^* = \Lambda(E)^* = (\Lambda\,[E])^*$ (see [2,6] for more details).

Notice that, if B and R are closed bounded discs in E and in Λ, respectively, with R normal, then $\Lambda_R[E_B]\subset\Lambda[E]$ is a normed space, whose norm is $\varepsilon_{R^\circ,B^\circ}$, the polar of R being taken in $(\Lambda_R)^*$ and that of B in the Banach dual space $(E_B)'$.

The space $\Lambda\langle E\rangle$ will be equipped with the topology $\sigma_{\mathcal{S},\mathcal{M}}$, given by the semi-norms

$$\sigma_{S,M}((x_n)_n) := \sup\{\sum_{n=1}^{+\infty}|a_n(x_n)|,\ a=(a_n)_n\in\Lambda_S^*[E'_M], \varepsilon_{S^\circ,M^\circ}(a)\le 1\},$$

$(x_n)_n\in\Lambda\langle E\rangle$, where $S\in\mathcal{S}$ and $M\in\mathcal{M}$.

We refer to [5, Section 30] for details concerning Köthe theory of sequence spaces and to [1,4] for the terminology and notations concerning the general theory of locally convex spaces.

3. Barrelledness in $\Lambda(E)$ and $\Lambda\langle E\rangle$

In connection with barrelledness in the scalar sequence spaces, the property (B) of Pietsch [8] plays an important role. This property (B) extends naturally to the vector-valued sequence spaces as shown in [7]. To recall it, let $\mathcal{B}$ (resp. $\mathcal{B}'$, $\mathcal{R}$, $\mathcal{R}'$) denote the collection of all closed bounded discs of E (resp. all closed weak*-bounded discs of E', all closed bounded and normal discs of Λ, all closed $\sigma(\Lambda^*,\Lambda)$-bounded and normal discs of Λ^*).

Definition 3.1. We say that E is fundamentally (resp. weakly fundamentally) Λ-bounded, if for every bounded subset $\mathbb{B}$ of $\Lambda(E)$ (resp. of $\Lambda[E]$), there is $B\in\mathcal{B}$ and $R\in\mathcal{R}$, such that $\mathbb{B}\subset R(B)$ (resp. $\mathbb{B}\subset R[B]$), where $R[B] := \{(x_n)\in\Lambda[E] : (x'(x_n))_n\in R,\ x'\in B^\circ\}$ and $R(B) := \Lambda(E)\cap R[B]$. We thus say that E has property $(F\Lambda)$ (resp. $(WF\Lambda)$).

If E has $(WF\Lambda)$, then it also has $(F\Lambda)$. The sets $R(B)$ and $R[B]$ are easily seen to be bounded in $\Lambda[E]$. Such sets permit to give another expression of $\sigma_{S,M}(x)$ for every $S\in\mathcal{S}$ and $M\in\mathcal{M}$, as follows:

$$\sigma_{S,M}(x) := \sup\{\sum_{n=1}^{+\infty}|f_n(x_n)|,\ f=(f_n)_n\in S[M]\},\ x=(x_n)_n\in\Lambda\langle E\rangle\,.$$

Notice that, by [6, Theorem 4], if F is a continuous functional on $\Lambda(E)$, then the sequence $(a_n)_n$, given by $a_n(t) := F(te_n)$, belongs to $\Lambda_S^*\langle E'_M\rangle$ for some $S\in\mathcal{S}$ and some $M\in\mathcal{M}$. Moreover, [7, Theorem 2] asserts that if E and Λ are Fréchet spaces, then $(\Lambda(E)_r)' = \Lambda^*\langle E'_\beta\rangle$ and the identification $J : \left(\Lambda^*\langle E'_\beta\rangle, \sigma_{\mathcal{R},\mathcal{B}}\right) \to ((\Lambda(E)_r)')_\beta$ is continuous. It is open whenever E is reflexive. We get the following proposition improving [7, Theorem 2].

Proposition 3.2. *If E satisfies $(F\Lambda)$, then $(\Lambda(E)_r)'_\beta$ embeds continuously into $\Lambda^*\langle E'_\beta\rangle$. The embedding is an into topological isomorphism, provided E is semi-reflexive. Here Λ^* is endowed with the strong topology $\tau_\beta = \beta(\Lambda^*, \Lambda)$.*

Proof. Firstly, let us show that $(\Lambda(E)_r)' \subset \Lambda^*\langle E'_\beta\rangle$. According to [6, Theorem 7], it follows that $(\Lambda(E)_r)' = \bigcup_{S\in\mathcal{S},M\in\mathcal{M}} \Lambda^*_S\langle E'_M\rangle$. Then, we will demonstrate that $\bigcup_{S\in\mathcal{S},M\in\mathcal{M}} \Lambda^*_S\langle E'_M\rangle \subset \Lambda^*\langle E'_\beta\rangle$.

Let $S \in S$, $M \in M$ and $(a_n)_n \in \Lambda^*_S\langle E'_M\rangle$. For every H an equicontinuous subset of $(E'_\beta)'$ and $f = (f_n)_n \in \Lambda[(E'_\beta)'_H]$, let us show that $f \in (\Lambda^*_S)^*[(E'_M)']$.

The polar H° of H with respect to the duality $\langle E', (E'_\beta)'\rangle$ absorbs the equicontinuous (and then strongly bounded) subset M. Then there exists $r > 0$ such that $M \in rH^\circ$. But, for all $n \in \mathbb{N}$, let $\varepsilon_n > 0$ be such that $f_n \in \varepsilon_n H$. Then, for all $x' \in M$, one has $|f_n(x')| = r\varepsilon_n \left|\frac{f_n}{\varepsilon_n}(\frac{x'}{r})\right| \le r\varepsilon_n$.

Then for every $n \in \mathbb{N}$, $f_n \in (E'_M)'$. But, for every $x' \in E'_M$, the the mapping $\delta_{x'} : (E'_\beta)'_H \to \mathbb{K}, \delta_{x'}(x'') = x''(x')$ is linear and continuous. So, $(\delta_{x'}(f_n))_n = (f_n(x'))_n \in \Lambda \subset (\Lambda^*_S)^*$. Then $f = (f_n)_n \in (\Lambda^*_S)^*[(E'_M)']$. By Proposition 2 of [6] and since $(a_n)_n \in \Lambda^*_S\langle E'_M\rangle$, the series $\sum f_n(a_n)$ is absolutely convergent. Thus, $(a_n)_n \in \Lambda^*\langle E'_\beta\rangle$.

Now $(\Lambda(E)_r)'$ can be seen as a subspace of $\Lambda^*\langle E'_\beta\rangle$. Now, suppose that E satisfies $(F\Lambda)$ and let $\mathbb{B}$ be a bounded disc in $\Lambda(E)_r$. By $(F\Lambda)$ property, there exist $B \in \mathcal{B}$ and $R \in \mathcal{R}$, such that $\mathbb{B} \subset R(B)$. If H is the bipolar $B^{\circ\circ}$ of B in $(E'_\beta)'$, then $\mathbb{B}^\circ$ contains $V_{R,H}$, the unit ball of $\sigma_{R,H}$ restricted to $(\Lambda(E)_r)'$. Indeed, fix $a \in V_{R,H}$ and $x \in R(B)$. As $B^\circ = H^\circ$, we have $\varepsilon_{R^\circ,H^\circ}(x) \le 1$. But $\sum_{n=1}^{+\infty} |a_n(x_n)| \le 1$, by definition of $V_{R,H}$. Therefore $a \in R(B)^\circ \subset \mathbb{B}^\circ$.

Now, assume that E is semi-reflexive. Let H be an equicontinuous subset of $(E'_\beta)'$. Then there is a bounded disc B in E such that $H = J(B)$, where $J : E \to (E'_\beta)'$ is the canonical embedding. If R is a normal bounded subset of Λ, then $R(B)^\circ \subset V_{R,H}$. Indeed, let $a = (a_n)_n \in R(B)^\circ$ and $x = (x_n)_n \in \Lambda_R[E_B] = \Lambda_R[(E'_\beta)'_H]$, with $\varepsilon_{R^\circ,H^\circ}(x) \le 1$. Then, for every $b \in B^\circ$, one has $(b(x_n))_n \in R$. Let $\alpha \in R$, $b \in B^\circ$ and $(\gamma_n)_n \in c_0$ such that $\|(\gamma_n)_n\|_{c_0} \le 1$. Then

$$\sum_{n=1}^{+\infty} |\alpha_n b(\gamma_n x_n)| \le \sum_{n=1}^{+\infty} |\alpha_n b(x_n)| \le \varepsilon_{R^\circ,B^\circ}(x) \le 1.$$

Since $(\gamma_n x_n) \in \Lambda(E)_r$, then $(\gamma_n x_n) \in R(B)$. This shows that $\sum_{n=1}^{+\infty} |a_n(x_n)| \le 1$, consequently $(a_n)_n \in V_{R,H}$. □

Inspired by [3], we will say that E has property (ΛB) if the topology induced by $\Lambda^*\langle E'_\beta\rangle$ on $(\Lambda(E)_r)'$ coincides with the strong topology $\beta((\Lambda(E)_r)'$.

Proposition 3.3. *If $\Lambda(E)$ is an AK-space with property (ΛB), then E has $(F\Lambda)$.*

Proof. Let $\mathbb{B}$ be a bounded disc in $\Lambda(E) = \Lambda(E)_r$. Then the polar $\mathbb{B}^\circ$ of $\mathbb{B}$ is a 0-neighborhood in $\Lambda(E)'_\beta = (\Lambda(E)_r)'_\beta$. Since E has property (ΛB), there exist $R \in \mathbb{R}$

and $B \in \mathcal{B}$, such that $V_{R,B} := \{a \in (\Lambda(E)_r)' : \ \sigma_{R,B}(a) \leq 1\} \subset \mathbb{B}^\circ$. Let $a = (a_n)_n \in (R(B))^\circ$. Then for every $x \in R(B)$, $|\langle (a_n)_n, (x_n)_n \rangle| = |\sum_{n=1}^{\infty} a_n(x_n)| \leq 1$. In other words, for all $x = (x_n)_n \in \Lambda(E)$ such that $\varepsilon_{R^\circ, B^\circ}(x) \leq 1$, we have $|\sum a_n(x_n)| \leq 1$. It follows that $a \in V_{R,B}$, and then $(R(B))^\circ \subset \mathbb{B}^\circ$. Hence $\mathbb{B} \subset R(B)$ and E has $(F\Lambda)$. □

With the same proof, one gets that, if E has property (ΛB), then it also has $(F\Lambda_r)$, i.e., every bounded subset $\mathbb{B}$ of $\Lambda(E)_r$ is contained in $R(B)_r := R(B) \cap \Lambda(E)_r$ for some $B \in \mathcal{B}$ and $R \in \mathcal{R}$.

It is shown in [6] that E and Λ are closed subspaces of $\Lambda(E)$. The following lemma improves this result.

Lemma 3.4. *The space E is complemented in each of the spaces $\Lambda[E]$, $\Lambda(E)$, $\Lambda\langle E\rangle$, $\Lambda[E]_r$, $\Lambda(E)_r$ and $\Lambda\langle E\rangle_r$. Moreover, Λ is complemented in $\Lambda[E]$, $\Lambda(E)$ and $\Lambda\langle E\rangle$. If, in addition, $(\Lambda, \tau_{\mathcal{S}})$ is AK, then Λ is also complemented in $\Lambda[E]_r$, $\Lambda(E)_r$ and $\Lambda\langle E\rangle_r$.*

Proof. We give the proof for $\Lambda(E)$. The other cases, being similar, will be omitted.

Set $\Lambda(E)^1 := \{te_1 : t \in E\}$ and consider the mapping $p : \Lambda(E) \to \Lambda(E)^1$ defined for all $(x_n)_n \in \Lambda(E)$ by $p((x_n)_n) = x_1 e_1$. For all $(x_n)_n \in \Lambda(E)$, we have $p^2((x_n)_n) = p(x_1 e_1) = x_1 e_1 = p((x_n)_n)$. Since

$$\forall (x_n)_n \in \Lambda(E), \quad \forall (S, M) \in \mathcal{S} \times \mathcal{M}, \quad \varepsilon_{S,M}(p((x_n)_n)) \leq \varepsilon_{S,M}((x_n)_n),$$

the mapping p is a continuous linear projection. Therefore $\Lambda(E)^1$ is complemented in $\Lambda(E)$. Now, we define a linear mapping ϕ from E into $\Lambda(E)^1$ by $\phi(t) = te_1$. Then ϕ is a bicontinuous linear isomorphism, because for all $t \in E$ and all $(S, M) \in \mathcal{S} \times \mathcal{M}$, $\varepsilon_{S,M}(te_1) = P_S(e_1)P_M(t)$. Therefore the spaces $\Lambda(E)^1$ and E are topologically isomorphic. Identifying them, E is complemented in $\Lambda(E)$.

The same proof shows that E is complemented in $\Lambda(E)_r$.

Now, consider $f \in E'$ and $M_0 \in \mathcal{M}$, such that $f \in M_0$ and $f(t) = 1$ for some $t \in E$. Set $\Lambda t := \{(\alpha_n t)_n : (\alpha_n)_n \in \Lambda\}$ and define the map q_f from $\Lambda(E)$ into Λt by $q_f((x_n)_n) = (f(x_n)t)_n$. Since $(\alpha_n t)_n \in \Lambda(E)$ for every $(\alpha_n)_n \in \Lambda$, q_f is onto. Furthermore, it is clear that, for all $(x_n)_n \in \Lambda(E)$, it holds $q_f^2((x_n)_n) = q_f((f(x_n)t)_n) = (f(x_n)t)_n = q_f((x_n)_n)$. Moreover, for every $S \in \mathcal{S}$ and $M \in \mathcal{M}$, $\varepsilon_{S,M}\big((f(x_n)t)_n\big) = P_M(t)P_S((f(x_n))_n) \leq P_M(t)\varepsilon_{S,M_0}((x_n)_n)$. Then the mapping q_f is a continuous linear projection. Therefore Λt is complemented in $\Lambda(E)$. Now, define a linear isomorphism h from Λ onto Λt by $h((\alpha_n)_n) = (\alpha_n t)_n$. For all $\alpha \in \Lambda$ and all $(S, M) \in \mathcal{S} \times \mathcal{M}$ we have $P_S((\alpha_n)_n) = \frac{1}{P_M(t)}\varepsilon_{S,M}((\alpha_n t)_n)$. Therefore h is a bicontinuous linear isomorphism, and Λ is complemented in $\Lambda(E)$.

If $(\Lambda, \tau_{\mathcal{S}})$ is an AK-space, the map q_f restricted to $\Lambda(E)_r$ is still surjective. So Λt is complemented in $\Lambda(E)_r$, consequently Λ is also complemented in $\Lambda(E)_r$. □

We obtain a characterization of the barrelledness and the quasi-barrelledness properties in $\Lambda(E)_r$. First let us set $K_{R',B'} := \{(f_n)_n \in \Lambda[(E'_{B'})'] : \forall a \in B', (f_n(a)) \in (R')^\circ\}$ and $R'\langle B'\rangle := \big\{(a_n)_n \in (\Lambda(E)_r)' : \sum_{n=1}^{+\infty} |f_n(a_n)| \leq 1, \forall (f_n)_n \in K_{R',B'}\big\}$.

Proposition 3.5. *The space $\Lambda(E)_r$ is barrelled (resp. quasi-barrelled), whenever*
(i) E and Λ are barrelled (resp. quasi-barrelled), and

(ii) for each weak bounded (resp. strongly bounded) subset* $\mathbb{B}$ *of* $(\Lambda(E)_r)'$*, there exist* $B' \in \mathcal{B}'$ *and* $R' \in \mathcal{R}'$*, such that* $\mathbb{B} \subset R' \langle B' \rangle$.
The converse holds whenever $(\Lambda, \tau_{\mathcal{S}})$ *is AK.*

Proof. Let T be a barrel (resp. bornivorous barrel) in $\Lambda(E)_r$. Then T° is a weakly bounded (resp. strongly bounded) subset of $(\Lambda(E)_r)'$. By (ii) there exists $R' \in \mathcal{R}'$ and $B' \in \mathcal{B}'$ such that $T^\circ \subset R'\langle B' \rangle$. Since E is barrelled (resp. quasi-barrelled), B' is equicontinuous, hence contained in some $M \in \mathcal{M}$. Similarly, since Λ is barrelled (resp. quasi-barrelled), there exists $S \in \mathcal{S}$ such that $R' \subset S$. Consequently $T^\circ \subset R'\langle B' \rangle \subset S\langle M \rangle$. Therefore T° is equicontinuous and then T is a neighborhood of 0 in $\Lambda(E)_r$.

Now, assume that $\Lambda(E)_r$ is barrelled. Then both spaces Λ and E, being complemented in $\Lambda(E)_r$ by Lemma 3.4, are barrelled (resp. quasi-barrelled). Moreover, if $\mathbb{B}$ is a weakly bounded (resp. strongly bounded) subset of $(\Lambda(E)_r)'$, then $\mathbb{B}$ is an equicontinuous subset of $(\Lambda(E)_r)'$. By [6, Theorem 8], there exist $S \in \mathcal{S}$ and $M \in \mathcal{M}$ such that $\mathbb{B} \subset S\langle M \rangle$. Whence the result. □

Remark 3.6. The condition "$(\Lambda, \tau_{\mathcal{S}})$ is an AK-space" in Proposition 3.5 is not involved in the conclusion (ii). It is used only to conclude that Λ is barrelled (resp. quasi-barrelled).

With a similar proof, we obtain the following proposition.

Proposition 3.7. *The space* $\Lambda\langle E \rangle_r$ *is barrelled (resp. quasi-barrelled) provided*
(i) E *and* Λ *are barrelled (resp. quasi-barrelled), and*

(ii) for each weak-bounded (resp. strongly bounded) subset* $\mathbb{B}$ *of* $(\Lambda\langle E \rangle_r)'$ *there exists* $B' \in \mathcal{B}'$ *and* $R' \in \mathcal{R}'$ *such that* $\mathbb{B} \subset R'[B']$.
The converse holds whenever $(\Lambda, \tau_{\mathcal{S}})$ *is AK.*

4. Reflexivity of $\Lambda(E)$

Recall that a locally convex space is said to be C-barrelled if every weak*-Cauchy sequence in its topological dual is equicontinuous (see [12]).

The following proposition gives a relation between the topological and the Köthe dual of $\Lambda(E)_r$.

Proposition 4.1. *(i) The inclusion* $(\Lambda(E)_r)' \subset \Lambda(E)^*$ *always holds. Moreover,* $\Lambda(E)$ *is an AK-space if and only if* $\Lambda(E)' \subset \Lambda(E)^*$.

(ii) If $\Lambda(E)_r$ *is C-barrelled, then the equality* $(\Lambda(E)_r)' = \Lambda(E)^*$ *holds algebraically.*

Proof. (i) Let $F \in \Lambda(E)_r'$ be arbitrary. Define $a_n(t) := F(te_n)$, $n \in \mathbb{N}$ and $t \in E$. Then $a_n \in E'$ and, for every $x = (x_n)_n \in \Lambda(E)_r$, $\sum_{n=1}^{+\infty} a_n(x_n) = F(x)$. Therefore the sequence $(a_n)_n$ belongs to $\Lambda(E)^*$. Since the sequence (a_n) is uniquely determined, $\Lambda(E)_r' \subset \Lambda(E)^*$. Now, if $\Lambda(E)$ is AK, then $\Lambda(E)' \subset \Lambda(E)^*$.

Conversely, assume that the latter inclusion holds and that some $(x_n)_n \in \Lambda(E)$ does not belong to $\Lambda(E)_r$. Then, by Hahn-Banach theorem, we can find some $F \in \Lambda(E)'$ such that $F((x_n)_n) = 1$ and F vanishes identically on $\Lambda(E)_r$. Since $\Lambda(E)' \subset \Lambda(E)^*$, F belongs to $\Lambda(E)^*$. Therefore there exists a sequence $(b_n)_n \subset E'$ such that $F(y) = \sum_{n=1}^{+\infty} b_n(y_n)$ for every $y := (y_n)_n \in \Lambda(E)$. In particular $F((x_n)_n) = \sum_{n=1}^{+\infty} b_n(x_n) = 1$. But $1 = \sum_{n=1}^{\infty} b_n(x_n) = \lim_{p\to\infty} F((x_n)_n^{(p)}) = 0$, for $(x_n)^{(p)} \in \Lambda(E)_r$. This contradiction shows that $(x_n)_n \in \Lambda(E)_r$ and then that $\Lambda(E) = \Lambda(E)_r$.

(ii) By (i), we have $(\Lambda(E)_r)' \subset \Lambda(E)^*$. Let us show that $\Lambda(E)^* \subset (\Lambda(E)_r)'$. Choose an arbitrary $(a_n)_n \in \Lambda(E)^*$. Then the series $\sum_{n=1}^{+\infty} a_n(x_n)$ converges for every $(x_n)_n \in \Lambda(E)_r$. Next, consider on $(\Lambda(E)_r)$ the linear functionals F_a and $F_{a,n}$, defined by $F_a : (x_k)_k \longmapsto \sum_{k=1}^{+\infty} a_k(x_k)$ and $F_{a,n} : (x_k)_k \longmapsto \sum_{k=1}^{n} a_k(x_k)$, $n \in \mathbb{N}$. It is clear that, for every $n \in \mathbb{N}$, $F_{a,n}$ is a continuous linear functional on $(\Lambda(E)_r)$ and that for every $x := (x_k)_k \in \Lambda(E)_r$, $(F_{a,n}(x))_n$ converges to $F_a(x)$. Hence the sequence $(F_{a,n})_n$ is weak*-Cauchy in $(\Lambda(E)_r)'$. Since $\Lambda(E)_r$ is C-barrelled, $(F_{a,n})_n$ is equicontinuous. Therefore F_a is continuous. Hence $(\Lambda(E)_r)' = \Lambda(E)^*$. □

It follows from Proposition 4.1 that $(\Lambda, \tau_{\mathcal{S}})$ is AK if and only if $(\Lambda, \tau_{\mathcal{S}})' = \Lambda^*$, and if and only if every $S \in \mathcal{S}$ is $\sigma(\Lambda^*, \Lambda)$-compact.

The following lemma yields a condition under which the dual of $\Lambda(E)_r$ coincides with the whole $\Lambda^*\langle E'_\beta\rangle$.

Lemma 4.2. *If $\Lambda(E)_r$ is C-barrelled and if E satisfies $(F\Lambda)$, then the equality $(\Lambda(E)_r)' = \Lambda^*\langle E'_\beta\rangle$ holds algebraically. If in addition E has property (ΛB), then the equality holds also topologically, when Λ^* is endowed with the strong topology $\beta(\Lambda^*, \Lambda)$.*

Proof. Suppose $\Lambda(E)_r$ C-barrelled and E fundamentally Λ-bounded. Then the equality $(\Lambda(E)_r)' = \Lambda(E)^*$ holds algebraically by Proposition 4.1 *ii*), and $\Lambda^*\langle E'_\beta\rangle \subset \Lambda(E)^*$ by Proposition 3.2. Therefore $(\Lambda(E)_r)' \subset \Lambda^*\langle E'_\beta\rangle \subset \Lambda(E)^* = (\Lambda(E)_r)'$, whereby $(\Lambda(E)_r)' = \Lambda^*\langle E'_\beta\rangle$. Again by Proposition 3.2, the strong topology on $(\Lambda(E)_r)'$ is coarser than that of $\Lambda^*\left\langle E'_\beta\right\rangle$. Now, if E has property (ΛB), then the two topologies coincide. □

In [7] it is given conditions under which $\Lambda(E)$ is reflexive for a Fréchet space E. We improve this result by characterizing the semi-reflexivity and reflexivity of $\Lambda(E)$, for general E, assuming $\Lambda(E)$ C-barrelled. This condition is obviously enjoyed in the Fréchet case.

Theorem 4.3. *If $\Lambda(E)$ is C-barrelled and E satisfies $(F\Lambda)$, then $\Lambda(E)$ is semi-reflexive if, and only if, the following assertions hold:*
(i) E and Λ are semi-reflexive.

(ii) $\Lambda(E)$ and $\Lambda^*\langle E'_\beta\rangle$ are both AK-spaces.

Proof. Necessity: Suppose that $\Lambda(E)$ is semi-reflexive. Since E and Λ and $\Lambda(E)_r$ are closed subspaces of $\Lambda(E)$, they are all semi-reflexive. In particular, $\Lambda(E)_r$ is weakly quasi-complete [4], then also weakly sequentially complete. Now, for every $x = (x_n)_n \in \Lambda(E)$, the sequence $(x^{(p)})_p$ is weakly Cauchy in $\Lambda(E)_r$. Indeed, if

$F \in (\Lambda(E)_r)'$, then there is a sequence $(a_n)_n$ in E' as in the proof of Proposition 4.1, such that the series $\sum a_n(x_n)$ converges. Therefore $\left(F(x^{(p)})\right)_p = \left(\sum_{k=1}^{p} a_n(x_n)\right)_p$ is a Cauchy sequence. Hence $(x^{(p)})_p$ converges weakly to a limit $y = (y_n)_n \in \Lambda(E)_r$, which is nothing but $(x_n)_n$. Consequently $\Lambda(E)$ is a AK-space. As $\Lambda(E) = \Lambda(E)_r$ is C-barrelled and E satisfies $(F\Lambda)$, $(\Lambda(E)_r)' = \Lambda^*\langle E'_\beta\rangle$ is sequentially complete with respect to the topology $\sigma(\Lambda^*\langle E'_\beta\rangle, \Lambda(E)_r)$. Let $a = (a_n)_n \in \Lambda^*\langle E'_\beta\rangle$. Then $(a^{(k)})_k \in \Lambda^*\langle E'_\beta\rangle_r$ and it is $\sigma(\Lambda^*\langle E'_\beta\rangle, \Lambda(E)_r)$-Cauchy in it. Indeed, consider $x = (x_n)_n \in \Lambda(E)_r$. Then $\langle (a_n)_n, (x_n)_n\rangle = \sum_{n=1}^{+\infty} a_n(x_n)$. Therefore $\left(\sum_{n=1}^{k} a_n(x_n)\right)_k$ is a Cauchy sequence, hence $(a^{(k)})_k$ converges to $b = (b_n)_n$ in $(\Lambda^* \left\langle E'_\beta \right\rangle_r, \sigma(\Lambda^* \left\langle E'_\beta \right\rangle_r, \Lambda(E)_r))$. Fix $n \in \mathbb{N}$ and $t \in E$. Applying the functional $l_{n,t}$ defined on $\Lambda^* \left\langle E'_\beta \right\rangle_r$ by $l_{n,t}((a_i)_i) = a_n(t)$, we get $(a_n)_n = (b_n)_n$. Hence $\Lambda^* \left\langle E'_\beta \right\rangle$ is an AK-space.

Conversely, assume that (i) and (ii) are satisfied. Then :

$$\begin{aligned}\Lambda(E)'' &\overset{(ii)}{=} (\Lambda(E)_r)'' = (\Lambda^* \left\langle E'_\beta\right\rangle)' \overset{(ii)}{=} (\Lambda^* \left\langle E'_\beta\right\rangle_r)' \\ &\overset{[7]}{=} \bigcup_{\mathcal{R},\mathcal{B}} \Lambda_R[(E'_\beta)'_{B^{\circ\circ}}] \overset{(i)}{=} \bigcup_{\mathcal{R},\mathcal{B}} \Lambda_R[E_B] \subset \Lambda[E].\end{aligned}$$

Let us prove that $\Lambda[E] = \Lambda(E)$. Let $\alpha = (\alpha_n)_n \in \Lambda^*$, $(R, B) \in \mathcal{R}\times\mathcal{B}$, $\beta = (\beta_n)_n \in R$ and $x \in B$. Put for every $n \in \mathbb{N}$, $f_n = \beta_n x$, then $f = (f_n)_n \in \Lambda_R\left[(E'_\beta)'_{B^{\circ\circ}}\right] = \Lambda_R[E_B]$ and $\varepsilon_{R^\circ,B^\circ}(f) \leq 1$. So for every $a \in E'$,

$$\sum_{n=1}^{+\infty} |\alpha_n\beta_n a(x)| = \sum_{n=1}^{+\infty} |\alpha_n f_n(a)| \leq \sigma_{R,B}((\alpha_n a)_n).$$

Then $P_R((\alpha_n)_n)|a(x)| \leq \sigma_{R,B}((\alpha_n a)_n)$. Since $\left(\Lambda^* \left\langle E'_\beta \right\rangle, \sigma_{\mathcal{R},\mathcal{B}}\right)$ is an AK-space, $(\Lambda^*, \tau_{\mathcal{R}}) = (\Lambda^*, \beta(\Lambda^*, \Lambda))$ is an AK-space. Therefore by [5, 30.7.(4)], $(\Lambda, \tau_{\mathcal{N}})$ is semi-reflexive. Due to [2, Corollary 1.4] we have $\Lambda[E] = \Lambda(E)$, whereby $\Lambda(E)$ is semi-reflexive. □

The main result of [7] states that if E and Λ are Féchet spaces, then $\Lambda(E)$ is reflexive if, and only if, E and Λ are reflexive, and $\Lambda(E)$ and $\Lambda^*\langle E'_\beta\rangle$ are both AK-spaces. As a direct corollary of Theorem 4.3, we get an improvement of this result.

THEOREM 4.4. *If $\Lambda(E)$ is barrelled and E is fundamentally Λ-bounded, then $\Lambda(E)$ is reflexive if, and only if, the following assertions hold*
(i) E and Λ are reflexive.
(ii) $\Lambda(E)$ and $\Lambda^*\langle E'_\beta\rangle$ are both AK-spaces.

A further generalization of the same result is the following.

THEOREM 4.5. *If E is fundamentally Λ-bounded, then $\Lambda(E)$ is reflexive if, and only if, the following assertions hold :*

(i) E and Λ are reflexive.

(ii) $\Lambda(E)$ and $\Lambda^*\langle E'_\beta\rangle$ are both AK-spaces.

(iii) for every bounded subset $\mathbb{B}$ of $\Lambda^*\langle E'_\beta\rangle$, there are bounded sets $B' \in \mathcal{B}'$, $R' \in \mathcal{R}'$ such that $\mathbb{B} \subset R'\langle B'\rangle$.

Proof. If $\Lambda(E)$ is reflexive, then it is semi-reflexive and barrelled, then it is semi-reflexive and C-barrelled. By Theorem 4.3 (i), E and Λ are semi-reflexive, then reflexive since E and Λ are barrelled, as complemented subspaces of $\Lambda(E)$. Now, by (ii) of the same theorem, $\Lambda(E)$ and $\Lambda^*\langle E'_\beta\rangle$ are both AK-spaces. Since $\Lambda(E)_r = \Lambda(E)$ is barrelled, by Proposition 3.5 (i) and Remark 3.6, for each weakly bounded (resp. strongly bounded) subset $\mathbb{B}$ of $(\Lambda(E)_r)' = \Lambda^*\langle E'_\beta\rangle$, there exist $B' \in \mathcal{B}'$ and $R' \in \mathcal{R}'$, such that $\mathbb{B} \subset R'\langle B'\rangle$.

Conversely, by (i) and (iii) the space $\Lambda(E)_r$ is barrelled (resp. quasi-barrelled) and by (i) and (ii) $\Lambda(E)$ is semi-reflexive. Then $\Lambda(E)$ is reflexive. □

5. Distinguishedness in $\Lambda(E)$

A locally convex space is said to be distinguished if its strong dual is barrelled. A bornological (in particular metrizable) space is distinguished if and only it its strong dual is quasi-barrelled, for it is complete. This raises the natural question whether the strong dual of a locally convex space is barrelled, whenever it is quasi-barrelled. As we will see later on, this is not the case. We then introduce the following definition.

Definition 5.1. A Hausdorff locally convex space is said to be quasi-distinguished if its strong dual is quasi-barrelled.

Actually, the appellation “quasi-distinguished space” has already been used in the literature, but with a different meaning [11].

Every semi-reflexive space and every normed space are quasi-distinguished. With standard arguments, one gets the following characterization of quasi-distinguished spaces.

Proposition 5.2. *A locally convex space E is quasi-distinguished if and only if, for every bounded subset B of $(E'', \beta(E'', E'))$, there exists a bounded subset A of E, such that B is contained in the $\sigma(E'', E')$-closure $\overline{A}^{\sigma(E'',E')}$ of A in E''.*

A Féchet space need not be distinguished (neither quasi-distinguished) as shown by Taskinen in [10]. Now, following an idea of Vladimir Kadets, answering a question of the second named author (see https://www.researchgate.net/profile/Lahbib_Oubbi), we provide an example of a quasi-distinguished space which fails to be distinguished.

Example 5.3. Let $\mathcal{P}$ be the set of all (positive) prime numbers and for every $p \in \mathcal{P}$, $A_p = \{p^n : n \in \mathbb{N}\}$. Define for each $p \in \mathcal{P}$ the sequence $(x_{p,n})_n \in \ell^\infty$ by $x_{p,n} = 1$ if $n \in A_p$ and $x_{p,n} = 0$ otherwise. Let D be the linear span of $\{(x_{p,n})_n : p \in \mathcal{P}\} \oplus c_0$.

Then by [9, Example 2.4.], $(D, \| \|_\infty)$ is a non barrelled subspace of $(\ell^\infty, \| \|_\infty)$. Now, consider $E = (\ell^1, \sigma(\ell^1, D))$. Then $E' = D$. Moreover $c_0 \subset D \subset \ell^\infty$. Then, by Banach-Steinhaus theorem, a disc $A \subset E$ is $\sigma(\ell^1, D)$-bounded if and only if A is bounded in the norm topology $\| \|_1$ of ℓ^1. Therefore the strong topology $\beta(D, \ell^1)$ coincides with that of the relative norm induced by $\| \|_\infty$ of ℓ^∞. Hence $(E', \beta(E', E))$ being equal to $(D, \beta(D, \ell^1))$ is a non-barrelled quasi-barrelled space. Consequently E is a non-distinguished quasi-distinguished space.

THEOREM 5.4. *Assume that $\Lambda(E)_r$ is C-barrelled and that E satisfies both $(F\Lambda)$ and (ΛB). Then $\Lambda(E)_r$ is distinguished (resp. quasi-distinguished) if and only if the following three conditions hold.*
(i) E and Λ are distinguished (resp. quasi-distinguished).

(ii) $\Lambda^*\langle E'_\beta \rangle$ is an AK-space.

(iii) for every weak-bounded (resp. strongly-bounded) subset $\mathbb{B}$ of $(\Lambda^*\langle E'_\beta \rangle)'$, there exist $B \in \mathcal{B}$ and $R \in \mathcal{R}$, such that $\mathbb{B} \subset R[B^{\circ\circ}]$.*

Proof. We give the proof in the distinguished case. The other case is similar. Suppose that $\Lambda(E)_r$ is distinguished. Then $(\Lambda(E)_r)'_\beta$ is barrelled. Since $\Lambda(E)_r$ is C-barrelled, by Lemma 4.2, the equality $(\Lambda(E)_r)'_\beta = (\Lambda^*\langle E'_\beta \rangle, \sigma_{\mathcal{R},\mathcal{B}})$ holds algebraically and topologically. Therefore $(\Lambda^* \left\langle E'_\beta \right\rangle, \sigma_{\mathcal{R},\mathcal{B}})$ is barrelled. By Proposition 3.7 (i), both spaces E'_β and $(\Lambda^*, \tau_{\mathcal{R}}) = \Lambda'_\beta$ are barrelled. Hence E and Λ are distinguished. Since $\Lambda(E)_r$ is C-barrelled, $(\Lambda(E)_r)' = \Lambda^*\langle E'_\beta \rangle$ is weakly sequentially complete and so the arguments of the proof of Theorem 4.3 still work to prove that the space $\Lambda^*\langle E'_\beta \rangle$ is an AK-space. Therefore, by Proposition 3.7 (ii), for each weakly bounded subset $\mathbb{B}$ of $(\Lambda^*\langle E'_\beta \rangle_r)'$, there exist $B \in \mathcal{B}$ and $R \in \mathcal{R}$ such that $\mathbb{B} \subset R[B^{\circ\circ}]$.

Conversely, assume that (i), (ii) and (iii) are satisfied. Then, an application of Proposition 3.7 gives that $(\Lambda^*\langle E'_\beta \rangle, \sigma_{\mathcal{R},\mathcal{B}})$ is barrelled, so $(\Lambda(E)_r)'_\beta$ is barrelled. Consequently $\Lambda(E)_r$ is distinguished. □

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Department of Mathematics, Faculty of Sciences, Mohammed V University in Rabat, Avenue Ibn Battouta 4, Rabat, 10108, Morocco

E-mail: issamaboutaib@gmail.com

ORCID iD: https://orcid.org/0000-0003-4398-7717

Department of Mathematics, Ecole Normale Superieure, Mohammed V University in Rabat, Avenue Mohamed Belhassan El Ouzzane, Rabat, 10105, Morocco

E-mail: lahbib.oubbi@ens.um5.ac.ma

ORCID iD: https://orcid.org/0000-0003-2119-9293