

VOLUME OF A COMPLEX MATRIX AND ITS APPLICATIONS IN ORIENTED GRAPHS

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Abstract. For a complex matrix A of rank $r > 0$, the volume of A is the product of r nonzero singular values of A . We prove that the volume of A is the square root of the sum of squares of modulus of all $r \times r$ minors of A . Ben-Israel provided a formula for the Moore-Penrose inverse of A when A is a real matrix using the volumes of A and some of its submatrices. We generalize the result to complex matrices. As applications, we find the volume of a complex incidence matrix of an oriented graph with complex edge weights for oriented trees and unicyclic graphs. We also present a combinatorial formula for the complex Laplacian matrix of a weakly connected oriented graph with complex edge weights.

1. Introduction

For a complex matrix A of rank $r > 0$, the *volume* of A , denoted by $\text{vol}(A)$, is the product of r nonzero singular values of A . The following proposition shows for a nonsingular complex matrix A that $\text{vol}(A)$ is $|\det(A)|$, the modulus of the determinant of A . To be clear, the *modulus* of a complex number $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$.

PROPOSITION 1.1. *Let A be an $n \times n$ nonsingular complex matrix of rank $r > 0$ with singular values $\sigma_1, \sigma_2, \dots, \sigma_r$. Then*

$$\text{vol}(A) = \prod_{i=1}^r \sigma_i = |\det(A)|.$$

Proof. Suppose a SVD (singular value decomposition) of A is $A = UDV^*$ where U and V are complex unitary matrices and $D = [d_{ij}]$ is the $n \times n$ matrix whose only nonzero entries are $d_{ii} = \sigma_i$ for $i = 1, 2, \dots, r$. Since U and V are complex unitary matrices, $V^*V = I_n$ and $|\det(U)| = |\det(U^*)| = 1$. Since $AA^* = UDV^*VD^*U^* = UDD^TU^*$, we obtain

$$|\det(AA^*)| = |\det(UDD^TU^*)| = |\det(U)| \cdot |\det(DD^T)| \cdot |\det(U^*)| = (\sigma_1\sigma_2 \cdots \sigma_r)^2 = \text{vol}^2(A).$$

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On the other hand, $\det(AA^*) = \det(A)\overline{\det(A^T)} = \det(A)\overline{\det(A)} = |\det(A)|^2$. Hence, we conclude that $\text{vol}(A) = \prod_{i=1}^r \sigma_i = |\det(A)|$. $\square$

A natural question would be: what is $\text{vol}(A)$ in terms of determinant when (1) A is singular (i.e., $\det(A) = 0$) and (2) A is not even square in which case $\det(A)$ is not defined? We answer these questions in Section 2. The answers for the case of real matrix A were given in [6] which we extend to complex matrix A . As applications of the volume formulas, we present the formulas for the Moore-Penrose inverse of a complex matrix in Section 2 as well. We also investigate a combinatorial interpretation of volume via the complex incidence matrix of an oriented graph. In Section 3 we find the volume of a complex incidence matrix of an oriented graph with complex edge weights for oriented trees and unicyclic graphs which enables us to find the determinant of the complex Laplacian matrix of a weakly connected oriented graph with complex edge weights.

2. An equivalent volume formula with applications

Let A be an $m \times n$ matrix, $S \subseteq \{1, \dots, m\}$, and $T \subseteq \{1, \dots, n\}$. The submatrix of A obtained by retaining the rows indexed by S and columns indexed by T is denoted by $A[S; T]$. We use the notation $A(S; T)$ for the submatrix of A obtained by deleting the rows indexed by S and columns indexed by T . The submatrix of A obtained by retaining the rows indexed by S is denoted by $A[S;)$ where $A(; T)$ is defined similarly. Now we state the Cauchy-Binet formula which will be used later in this section.

THEOREM 2.1 (Cauchy-Binet, [8, Proposition 1.3.5]). *Let $m \leq n$. For $m \times n$ matrices A and B , we have*

$$\det(AB^T) = \sum_S \det(A(; S]) \det(B(; S]),$$

where the summation runs over $\binom{n}{m}$ m -subsets S of $\{1, 2, \dots, n\}$.

For an $m \times n$ complex matrix A of rank $r > 0$. Let $\Delta_r(A)$ denote the square root of the sum of squares of modulus of all $r \times r$ minors of A , i.e.,

$$\Delta_r^2(A) = \sum_{\substack{S \subseteq \{1, \dots, m\}, |S|=r \\ T \subseteq \{1, \dots, n\}, |T|=r}} |\det(A[S; T])|^2.$$

We prove that $\Delta_r(A) = \text{vol}(A)$ in Theorem 2.4. For the proof we need the following two lemmas.

LEMMA 2.2. *Let C and R be complex matrices with full column and row rank r , respectively. Then $\Delta_r^2(C) = \det(C^*C)$ and $\Delta_r^2(R) = \det(RR^*)$.*

Proof. Let $C \in \mathbb{C}^{m \times r}$ and $R \in \mathbb{C}^{r \times n}$. By the Cauchy-Binet formula,

$$\det(C^*C) = \sum_{\substack{S \subseteq \{1, \dots, m\}, \\ |S|=r}} \det(C^*(; S]) \det(C[S;)) = \sum_{\substack{S \subseteq \{1, \dots, m\}, \\ |S|=r}} |\det(C[S;)]|^2 = \Delta_r^2(C)$$

and similarly

$$\det(RR^*) = \sum_{\substack{S \subseteq \{1, \dots, n\}, \\ |S|=r}} \det(R(; S]) \det(R^*[S;]) = \sum_{\substack{S \subseteq \{1, \dots, n\}, \\ |S|=r}} |\det(R(; S])|^2 = \Delta_r^2(R). \quad \square$$

The following lemma shows a nice decomposition of Δ_r^2 of a matrix with a full-rank factorization.

LEMMA 2.3. *Let A be an $m \times n$ complex matrix of rank $r > 0$ with a full-rank factorization $A = CR$. Then $\Delta_r^2(A) = \Delta_r^2(C)\Delta_r^2(R)$.*

Proof. Note that for all $S \subseteq \{1, \dots, m\}$ and $T \subseteq \{1, \dots, n\}$ such that $|S| = |T| = r$, we have $A[S; T] = C[S;)R(; T]$ and consequently $\det(A[S; T]) = \det(C[S;)R(; T]) = \det(C[S;)) \det(R(; T])$. Then

$$\begin{aligned} \Delta_r^2(A) &= \sum_{\substack{S \subseteq \{1, \dots, m\}, |S|=r \\ T \subseteq \{1, \dots, n\}, |T|=r}} |\det(A[S; T])|^2 = \sum_{\substack{S \subseteq \{1, \dots, m\}, |S|=r \\ T \subseteq \{1, \dots, n\}, |T|=r}} |\det(C[S;))|^2 |\det(R(; T])|^2 \\ &= \left(\sum_{S \subseteq \{1, \dots, m\}, |S|=r} |\det(C[S;))|^2 \right) \left(\sum_{T \subseteq \{1, \dots, n\}, |T|=r} |\det(R(; T])|^2 \right) \\ &= \Delta_r^2(C) \Delta_r^2(R). \quad \square \end{aligned}$$

Now we ready to state the main result of this section showing $\Delta_r(A)$ is equivalent to $\text{vol}(A)$.

THEOREM 2.4. *Let A be a complex matrix of rank $r > 0$. Then $\text{vol}(A) = \Delta_r(A)$.*

Proof. Consider a compact SVD of A : $A = UDV^*$ where $D = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$ and both U and V have r orthonormal columns. By Lemma 2.2, $\Delta_r^2(U) = \det(U^*U) = \det(I_r) = 1$ and $\Delta_r^2(V^*) = \det(V^*V) = \det(I_r) = 1$, and by Lemma 2.3,

$$\begin{aligned} \Delta_r^2(A) &= \Delta_r^2(UDV^*) = \Delta_r^2(U) \Delta_r^2(DV^*) \\ &= \Delta_r^2(U) \Delta_r^2(D) \Delta_r^2(V^*) = 1 \cdot \left(\prod_{i=1}^r \sigma_i \right)^2 \cdot 1 = \text{vol}^2(A). \quad \square \end{aligned}$$

EXAMPLE 2.5. Consider the following 3×3 complex matrix A of rank 2:

$$A = \begin{bmatrix} 1 & 3i & -4 \\ 2i & -1 & -3i \\ -3 & 1+2i & 1+i \end{bmatrix}$$

First we calculate $\text{vol}(A)$ as the product of its nonzero singular values:

$$\text{vol}(A) = \sigma_1 \sigma_2 = 6.44 \cdot 3.79 \approx 24.4$$

Now we calculate $\Delta_2(A)$ to verify that $\text{vol}(A) = \Delta_2(A)$.

$$|\det(A(3; 3))| = |\det(A(3; 2))| = |\det(A(3; 1))| = 5$$

$$|\det(A(2; 3))| = |\det(A(2; 2))| = |\det(A(2; 1))| = \sqrt{122}$$

$$|\det(A(1; 3))| = |\det(A(1; 2))| = |\det(A(1; 1))| = \sqrt{53}$$

$$\Delta_2(A) = \sqrt{\sum_{i=1}^3 \sum_{j=1}^3 |\det(A(i; j))|^2} = \sqrt{3(25 + 122 + 53)} = 10\sqrt{6} \approx 24.4$$

As an application of the volume of a complex matrix, we now present some volume formulas for the Moore-Penrose inverse of a complex matrix. The *Moore-Penrose inverse* of an $m \times n$ complex matrix A , denoted by A^+ , is the $n \times m$ complex matrix that satisfies the following equations (see [7]):

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)^* = AA^+, \quad (A^+A)^* = A^+A,$$

where $(AA^+)^*$ is the conjugate transpose of AA^+ . If a square matrix A is invertible, then $A^+ = A^{-1}$. In [6] Ben-Israel provided a formula for the Moore-Penrose inverse of A when A is a real matrix using the volumes of A and some of its submatrices. We extend Ben-Israel's formula to complex matrices which we present in Theorem 2.6. First we need some notations.

For an $m \times n$ matrix A of rank $r > 0$, define

$$\mathcal{I}_A = \{S = \{i_1, i_2, \dots, i_r\} \mid S \subset \mathbb{Z}, 1 \leq i_1 < i_2 < \dots < i_r \leq m, \text{rank}(A[S;]) = r\}$$

$$\text{and } \mathcal{J}_A = \{T = \{j_1, j_2, \dots, j_r\} \mid T \subset \mathbb{Z}, 1 \leq j_1 < j_2 < \dots < j_r \leq n, \text{rank}(A(; T]) = r\}$$

where for any $I \in \mathcal{I}_A$, $\widehat{A[I;]}$ denotes the matrix obtained from A by replacing all the rows that are not in I by zero vectors and similarly for any $J \in \mathcal{J}_A$, $\widehat{A(; J]}$ denotes the matrix obtained from A by replacing all the columns of A that are not in J by zero vectors. For example, see Example 2.7.

THEOREM 2.6. *Let A be an $m \times n$ complex matrix of rank $r > 0$. Then*

$$(i) \quad A^+ = \frac{1}{\text{vol}^2(A)} \sum_{J \in \mathcal{J}_A} \text{vol}^2(A(; J]) \widehat{A(; J]}^+,$$

$$(ii) \quad A^+ = \frac{1}{\text{vol}^2(A)} \sum_{I \in \mathcal{I}_A} \text{vol}^2(A[I;]) \widehat{A[I;]}^+.$$

The proof is similar to that of the real case given in [6, Remark 6.3] and it follows from Theorem 2.4. The preceding theorem is illustrated by the following example.

EXAMPLE 2.7. Consider the following 3×3 complex matrix A of rank 2:

$$A = \begin{bmatrix} 1 & 3i & -4 \\ 2i & -1 & -3i \\ -3 & 1+2i & 1+i \end{bmatrix}$$

By the preceding theorem,

$$\begin{aligned} A^+ = \frac{1}{\text{vol}^2(A)} & \left(\text{vol}^2(A(; \{1, 2\}]) \widehat{A(; \{1, 2\}])}^+ + \text{vol}^2(A(; \{1, 3\}]) \widehat{A(; \{1, 3\}])}^+ \right. \\ & \left. + \text{vol}^2(A(; \{2, 3\}]) \widehat{A(; \{2, 3\}])}^+ \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{600} \left(200 \begin{bmatrix} 1 & 3i & 0 \\ 2i & -1 & 0 \\ -3 & 1+2i & 0 \end{bmatrix}^+ + 200 \begin{bmatrix} 1 & 0 & -4 \\ 2i & 0 & -3i \\ -3 & 0 & 1+i \end{bmatrix}^+ + 200 \begin{bmatrix} 0 & 3i & -4 \\ 0 & -1 & -3i \\ 0 & 1+2i & 1+i \end{bmatrix}^+ \right) \\
&= \frac{1}{600} \left(\begin{bmatrix} 18-9i & -3-31i & -40-5i \\ 3-43i & -16-6i & 5-25i \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -25-12i & -9-15i & -65-10i \\ 0 & 0 & 0 \\ -43-3i & -6+16i & -25-5i \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} 0 & 0 & 0 \\ -12-25i & -15-9i & 10-65i \\ -18+9i & 3+31i & 40+5i \end{bmatrix} \right) \\
&= \frac{1}{600} \begin{bmatrix} -7-21i & -12-46i & -105-15i \\ 15-68i & -1-15i & 15-90i \\ -61+6i & -3+47i & 15 \end{bmatrix}.
\end{aligned}$$

Bapat presented a volume formula for A^+ in [3] which was derived from the preceding theorem and was used in combinatorial setting in [1, 2]. Suppose $A(i_1, i_2, \dots, i_r)$, $1 \leq i_1 < i_2 < \dots < i_r \leq m$ denotes the matrix obtained from A by replacing all the columns of A that are different from columns $i_1, i_2, \dots, i_r$ by zero vectors. The following is the complex version of Bapat's result.

THEOREM 2.8. *Let A be an $m \times n$ complex matrix of rank $r > 0$. Then*

$$A^+ = \frac{1}{\text{vol}^2(A)} \sum_{(i_1, i_2, \dots, i_r) \in \mathcal{M}} \text{vol}^2(A(i_1, i_2, \dots, i_r)) A(i_1, i_2, \dots, i_r)^+,$$

where

$$\mathcal{M} = \{(i_1, i_2, \dots, i_r) \in \mathbb{Z}^r \mid 1 \leq i_1 < i_2 < \dots < i_r \leq n, \text{rank}(A(i_1, i_2, \dots, i_r)) = r\}.$$

3. Volume of a complex incidence matrix

In this section we find the volumes of complex incidence matrices of oriented trees and unicyclic graphs with complex edge weights. As an application, we obtain a combinatorial formula for the determinant of the complex Laplacian of a weakly connected oriented graph with complex edge weights. Although our results are self-contained, for additional background we refer [9] for spectra of digraphs and [17, 18] for the largest singular value of digraphs and mixed graphs.

Let G be an oriented graph on n vertices $1, 2, \dots, n$ whose underlying undirected graph is denoted by G_u . To be clear, G is obtained from G_u by orienting each edge $\{u, v\}$ of G_u either from u to v with the directed edge (u, v) or from v to u with the directed edge (v, u) . For example, an oriented unicyclic graph is a directed graph obtained from an undirected unicyclic graph by orienting all of its edges. Note that an oriented cycle is not necessarily a directed cycle. In [4, 5] Barik et al. studied multidigraphs by associating them with certain oriented graphs with complex edge weights. We denote the nonzero complex weight of the edge (u, v) in G by w_{uv} .

We introduce complex adjacency, incidence, and Laplacian matrices for oriented graphs by following similar definitions for multidigraphs given by Barik et al. in [4, 5]. The *complex adjacency matrix* of G is the $n \times n$ complex matrix $A = [a_{ij}]$ with rows and columns indexed by the vertices of G and it is defined as follows:

$$a_{ij} = \begin{cases} w_{ij}, & \text{if } (i, j) \in E(G), \\ \overline{w_{ji}}, & \text{if } (j, i) \in E(G), \\ 0, & \text{otherwise} \end{cases}$$

Note that complex adjacency matrix is not quite same as Hermitian adjacency matrix in the literature [10, 15] because the entries of a Hermitian adjacency matrix are in $\{0, 1, i, -i\}$. When all arcs of an oriented graph have the weight i , its complex adjacency matrix and Hermitian adjacency matrix are the same. So complex adjacency matrix can be seen as one kind of weighted generalization of Hermitian adjacency matrix.

By definition, the complex adjacency matrix A is Hermitian and consequently it has real eigenvalues. The *complex Laplacian matrix* of G is the $n \times n$ complex matrix $L = D - A$ where $D = [d_{ii}]$ is the diagonal matrix of order n with $d_{ii} = \sum_{j \neq i} |a_{ij}|$.

Suppose $e_1, e_2, \dots, e_m$ are the edges of G . The *complex incidence matrix* of G is the $n \times m$ complex matrix $M = [m_{ij}]$ with rows and columns indexed by the vertices and columns of G , respectively, and it is defined as follows:

$$m_{ij} = \begin{cases} \sqrt{w_{uv}}, & \text{if } i = u, \\ -\sqrt{w_{uv}}, & \text{if } i = v, \\ 0, & \text{otherwise,} \end{cases}$$

where $e_j = (u, v)$ and $\sqrt{w_{uv}}$ denotes the principal square root of w_{uv} . It is easy to verify that $L = MM^*$.

EXAMPLE 3.1. Consider the oriented cycle C_H in Figure 1. Its complex Laplacian and incidence matrices are the following:

$$M = \begin{bmatrix} \sqrt{1+2i} & 0 & 0 & -\sqrt{1-i} \\ -\sqrt{1-2i} & \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & -\sqrt{1-i} & 0 \\ 0 & 0 & \sqrt{1+i} & \sqrt{1+i} \end{bmatrix}$$

$$L = D - A = \begin{bmatrix} 2 + \sqrt{2} & -(1+2i) & 0 & -(1-i) \\ -(1-2i) & 3 + \sqrt{5} & -3 & 0 \\ 0 & -3 & 3 + \sqrt{2} & -(1-i) \\ -(1+i) & 0 & -(1+i) & 2\sqrt{2} \end{bmatrix} = MM^*$$

Now we state some basic facts of complex numbers which will be used in this section. The principal square root of a complex number $z = re^{i\theta}$ where $r \geq 0$ and $\theta \in (-\pi, \pi]$ is $\sqrt{r}e^{i\theta/2}$ is denoted by $\sqrt{z}$ throughout this article. The following facts are repeatedly used in derivations later on:

1. $|\sqrt{z}| = |\sqrt{r}e^{i\theta/2}| = \sqrt{r} = \sqrt{|re^{i\theta}|} = \sqrt{|z|}$,

2. $|\operatorname{Im}(\sqrt{z})|^2 = r \sin^2(\theta/2) = \frac{r(1-\cos \theta)}{2} = \frac{1}{2}(|z| - \operatorname{Re}(z)) \geq 0$ and the equality holds if and only if $z \in \{0\} \cup \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of all positive real numbers.

The product of all edge weights of an oriented graph G is denoted by W_G . The following observation shows that $|W_G|$ is invariant under changing edge orientation.

OBSERVATION 3.2. Let G be an oriented graph with complex edge weights. Then the modulus $|W_G|$ of the product of all of its edge weights does not change if an edge (u, v) of G with weight w_{uv} is changed to the edge (v, u) of G with weight $w_{vu} = \overline{w_{uv}}$.

The above follows from the fact that $|zw_{vu}| = |z| \cdot |w_{vu}| = |z| \cdot |\overline{w_{uv}}| = |zw_{uv}|$.

Now we find a combinatorial expression of the volume of the complex incidence matrix of an oriented tree which will be useful later on.

THEOREM 3.3. Let T be an oriented tree on $n \geq 2$ vertices with complex edge weights and complex incidence matrix M . Then for all $i = 1, 2, \dots, n$, $|\det(M(i;))| = \sqrt{|W_T|}$ and consequently $\operatorname{vol}^2(M) = n|W_T|$.

Proof. Fix $i \in \{1, 2, \dots, n\}$. By permuting rows and columns of M (which may cause a sign change for a determinant), we can choose i to be a pendant vertex of T and by successive cofactor expansions across rows corresponding to pendant vertices of T , we get

$$\det(M(i;)) = \pm \sqrt{\prod_{\{u,v\} \in E(T_u)} a_{uv}}$$

where $a_{uv} \in \{w_{uv}, \overline{w_{uv}}\}$ for each edge $\{u, v\}$ in T_u . Then

$$|\det(M(i;))| = \left| \pm \sqrt{\prod_{\{u,v\} \in E(T_u)} a_{uv}} \right| = \sqrt{\prod_{\{u,v\} \in E(T_u)} |a_{uv}|} = \sqrt{\prod_{(u,v) \in E(T)} |w_{uv}|} = \sqrt{|W_T|}.$$

Thus, $\operatorname{vol}^2(M) = \sum_{i=1}^n |\det(M(i;))|^2 = n|W_T|$. $\square$

An oriented graph G is *weakly connected* if its underlying undirected graph G_u is connected. We now introduce weight of a walk in the underlying connected undirected graph G_u of a weakly connected oriented graph G . This definition has some similarity with that of the weight of a walk in a digraph given in [16].

DEFINITION 3.4. Let G be a weakly connected oriented graph on n vertices $1, 2, \dots, n$. Suppose the weight of a directed edge (u, v) in G is denoted by w_{uv} . For a G_u -walk $\gamma = (i_1, i_2, \dots, i_k)$ (i.e., a walk in the underlying connected undirected graph G_u of G), we define the *weight* of γ , denoted by $wt(\gamma)$, as follows $wt(\gamma) = b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{k-1} i_k}$, where

$$b_{ij} = \begin{cases} w_{ij}, & \text{if } (i, j) \in E(G), \\ \overline{w_{ji}}, & \text{if } (j, i) \in E(G). \end{cases}$$

The *weight* $wt(C)$ of an oriented cycle C is the weight $wt(\gamma)$ of a closed path γ in C_u specified in the context, if needed (the weights of two oppositely oriented closed paths are conjugate of each other). To be clear, $|wt(C)|$ and $\operatorname{Re}(wt(C))$ are the same for both oppositely oriented closed paths in C_u .

EXAMPLE 3.5. Consider the oriented unicyclic graph H in Figure 1.

$$\begin{aligned} W_H &= w_{12}w_{23}w_{43}w_{41}w_{54}w_{56} = (1+2i) \cdot 3 \cdot (1+i) \cdot (1+i) \cdot (1-i) \cdot (2+i) \\ &= 6(1+2i)(1+3i) = 30(-1+i) \end{aligned}$$

Using $\gamma(C_H) = (1, 2, 3, 4, 1)$,

$$wt(C_H) = w_{12}w_{23}\overline{w_{43}}w_{41} = (1+2i) \cdot 3 \cdot (1-i) \cdot (1+i) = 6+12i \notin \mathbb{R}^+.$$

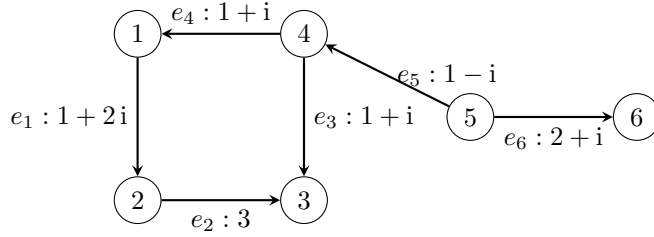


Figure 1: An oriented unicyclic graph H on 6 vertices with $wt(C_H) \notin \mathbb{R}^+$

Now we find the volume of the complex incidence matrix of an oriented cycle.

THEOREM 3.6. Let H be an oriented cycle on n vertices with complex edge weights and complex incidence matrix M . Then $|\det(M)| = \sqrt{2(|wt(H)| - \operatorname{Re}(wt(H)))}$.

(a) If $wt(H) \notin \mathbb{R}^+$, then

$$\operatorname{vol}^2(M) = |\det(M)|^2 = 2(|wt(H)| - \operatorname{Re}(wt(H))) \neq 0.$$

(b) If $wt(H) \in \mathbb{R}^+$, then $\det(M) = 0$ and

$$\operatorname{vol}^2(M) = n|W_H| \sum_{(u,v) \in E(H)} \frac{1}{|w_{uv}|}.$$

Proof. For all $i = 1, 2, \dots, n$, suppose $e_i = (u_i, v_i)$ and choose $a_i \in \{\sqrt{w_{u_i v_i}}, \sqrt{\overline{w_{u_i v_i}}}\}$ such that $a_1 a_2 \cdots a_n = \sqrt{wt(H)}$. Then

$$M = PM'QS = P \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & -\overline{a_n} \\ -\overline{a_1} & a_2 & 0 & \cdots & 0 & 0 \\ 0 & -\overline{a_2} & a_3 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & -\overline{a_{n-1}} & a_n \end{bmatrix} QS,$$

for some permutation matrices P and Q and a signature matrix S (which multiplies the columns of M' with appropriate signs). By a cofactor expansion across the first row,

$$\begin{aligned} \det(M) &= \det(P) \det(M') \det(Q) \det(S) \\ &= (\pm 1)[a_1 a_2 \cdots a_n + (-1)^{n+1}(-\overline{a_n})(-\overline{a_1})(-\overline{a_2}) \cdots (-\overline{a_{n-1}})](\pm 1) \end{aligned}$$

$$= \pm(a_1 a_2 \cdots a_n - \overline{a_1 a_2 \cdots a_n}) = \pm(\sqrt{wt(H)} - \sqrt{\overline{wt(H)}}) = \pm 2i \operatorname{Im} \sqrt{wt(H)}.$$

Then $|\det(M)|^2 = |2i \operatorname{Im} \sqrt{wt(H)}|^2 = 2(|wt(H)| - \operatorname{Re}(wt(H)))$.

(a) Suppose $wt(H) \notin \mathbb{R}^+$. Then $\operatorname{vol}^2(M) = |\det(M)|^2 = 2(|wt(H)| - \operatorname{Re}(wt(H))) \neq 0$.

(b) Suppose $wt(H) \in \mathbb{R}^+$. Then $|\det(M)|^2 = 2(|wt(H)| - \operatorname{Re}(wt(H))) = 0$ and M does not have the full rank. We proceed to investigate all minors of order $n-1$. We delete column j of M (i.e., delete edge $e_j = (u, v)$) and consider the corresponding incidence matrix $M(:, j)$. Note that $H \setminus e_j$ is a tree and for all $i = 1, 2, \dots, n$, $|\det(M(i; j))| = \sqrt{|W_{H \setminus e_j}|} = \sqrt{\frac{|W_H|}{|w_{uv}|}}$ and consequently $\operatorname{vol}^2(M(:, j)) = n \frac{|W_H|}{|w_{uv}|}$. Therefore, $\operatorname{vol}^2(M) = n |W_H| \sum_{(u,v) \in E(H)} \frac{1}{|w_{uv}|}$. $\square$

COROLLARY 3.7. *Let H be an oriented cycle on n vertices with complex edge weights and complex incidence matrix M . Then $\det(M) = \pm 2i \operatorname{Im} \sqrt{wt(H)}$.*

The next result provides a combinatorial expression of the volume of the complex incidence matrix of an oriented unicyclic graph.

THEOREM 3.8. *Let H be an oriented unicyclic graph with complex edge weights and complex incidence matrix M . Suppose C_H is the oriented cycle in H and $H \setminus C_H$ is the oriented forest obtained from H by deleting edges of C_H . Then*

$$|\det(M)| = \sqrt{2|W_{H \setminus C_H}|(|wt(C_H)| - \operatorname{Re}(wt(C_H)))}.$$

(a) If $wt(C_H) \notin \mathbb{R}^+$, then

$$\operatorname{vol}^2(M) = |\det(M)|^2 = 2|W_{H \setminus C_H}|(|wt(C_H)| - \operatorname{Re}(wt(C_H))).$$

(b) If $wt(C_H) \in \mathbb{R}^+$, then $\det(M) = 0$ and

$$\operatorname{vol}^2(M) = n |W_H| \sum_{(u,v) \in E(C_H)} \frac{1}{|w_{uv}|}.$$

Proof. For any pendant edge $e_j = (u, v)$, we get $\det(M)$ by a cofactor expansion along the row corresponding to the pendant vertex between u and v and get $|\det(M)| = \sqrt{|w_{uv}|} |\det(M(u; j))|$ or $|\det(M)| = \sqrt{|w_{uv}|} |\det(M(v; j))|$. Using this process successively, we get $|\det(M)| = \sqrt{|W_{H \setminus C_H}|} |\det(M_{C_H})|$, where M_{C_H} is the incidence matrix of C_H . By Theorem 3.6,

$$\begin{aligned} |\det(M)| &= \sqrt{|W_{H \setminus C_H}|} \sqrt{2(|wt(C_H)| - \operatorname{Re}(wt(C_H)))} \\ &= \sqrt{2|W_{H \setminus C_H}|(|wt(C_H)| - \operatorname{Re}(wt(C_H)))}. \end{aligned}$$

(a) If $wt(C_H) \notin \mathbb{R}^+$, then

$$\operatorname{vol}^2(M) = |\det(M)|^2 = 2|W_{H \setminus C_H}|(|wt(C_H)| - \operatorname{Re}(wt(C_H))) \neq 0.$$

(b) Suppose $wt(C_H) \in \mathbb{R}^+$. Then

$$|\det(M)|^2 = 2|W_{H \setminus C_H}|(|wt(C_H)| - \operatorname{Re}(wt(C_H))) = 0.$$

We proceed to investigate all minors of order $n-1$. We delete column j of M (i.e., delete edge $e_j = (u, v)$) and consider the corresponding incidence matrix $M(:, j)$.

Suppose e_j is not on the cycle C_H in H . Then $H \setminus e_j$ has a tree component T_H and a unicyclic component U_H with corresponding incidence matrices M_T and M_U respectively. Since $wt(U_H) \in \mathbb{R}^+$, then $\det(M_U) = 0$. For each vertex i in T_H , $\det(M(i; j)) = \det(M_T(i;)) \det(M_U) = 0$. For each vertex i in U_H , $\det(M(i; j)) = 0$ because the columns of $M(i; j)$ corresponding to the edges of T_H are linearly dependent.

Suppose e_j is on the cycle C_H in H . Then $H \setminus e_j$ is a tree on n vertices and for all $i = 1, 2, \dots, n$, $|\det(M(i; j))| = \sqrt{|W_{H \setminus e_j}|} = \sqrt{\frac{|W_H|}{|w_{uv}|}}$ and consequently $\text{vol}^2(M(i; j)) = n \frac{|W_H|}{|w_{uv}|}$. Therefore, $\text{vol}^2(M) = n|W_H| \sum_{(u,v) \in E(C_H)} \frac{1}{|w_{uv}|}$. $\square$

COROLLARY 3.9. *Let H be an oriented unicyclic graph with complex edge weights and complex incidence matrix M . Suppose C_H is the oriented cycle in H and $H \setminus C_H$ is the oriented forest obtained from H by deleting edges of C_H . If all the edges of $H \setminus C_H$ are oriented towards the cycle C_H , then $\det(M) = \pm 2i (\sqrt{|W_{H \setminus C_H}|}) \text{Im} \sqrt{wt(C_H)}$.*

EXAMPLE 3.10. (a) Consider the oriented unicyclic graph H in Figure 1, where $wt(C_H) = 6(1 + 2i) \notin \mathbb{R}^+$ and

$$\text{vol}^2(M) = 2|W_{H \setminus C_H}|(|wt(C_H)| - \text{Re}(wt(C_H))) = 2|3 - i|(6\sqrt{5} - 6) = 12\sqrt{10}(\sqrt{5} - 1).$$

(b) Consider the oriented unicyclic graph H' in Figure 2. Note that $|W_{H'}| = |12(3 - i)| = 12\sqrt{10}$ and $wt(C_{H'}) = 2 \cdot 3 \cdot (1 - i) \cdot (1 + i) = 12 \in \mathbb{R}^+$. Then $\det(M) = 0$ and

$$\begin{aligned} \text{vol}^2(M) &= n|W_{H'}| \sum_{(u,v) \in E(C_{H'})} \frac{1}{|w_{uv}|} \\ &= 6 \cdot 12\sqrt{10} \left(\frac{1}{|2|} + \frac{1}{|3|} + \frac{1}{|1+i|} + \frac{1}{|1-i|} \right) = 12\sqrt{10}(5 + 6\sqrt{2}). \end{aligned}$$

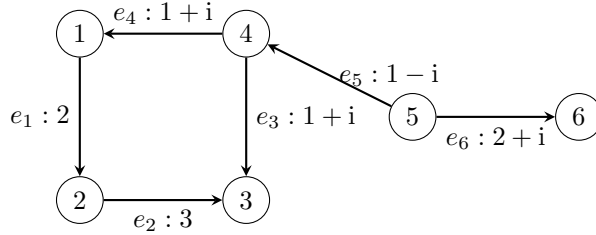


Figure 2: An oriented unicyclic graph H' on 6 vertices with $wt(C_{H'}) \in \mathbb{R}^+$

As a consequence of the volume formula of the complex incidence matrix of an oriented unicyclic graph, we find the determinant of the complex Laplacian matrix of a weakly connected oriented graph as follows.

THEOREM 3.11. *Let G be a weakly connected oriented graph on $n \geq 2$ vertices with complex edge weights and complex Laplacian matrix L . Then*

$$\det(L) = \sum_H \prod_{i=1}^{c(H)} 2|W_{U_i \setminus C_i}|(|wt(C_i)| - \text{Re}(wt(C_i))),$$

where the summation runs over all weakly connected spanning subdigraphs H of G with n edges consisting of $c(H)$ oriented unicyclic components $U_1, U_2, \dots, U_{c(H)}$ whose corresponding cycles' $(C_1, C_2, \dots, C_{c(H)})$, respectively weights are not positive real numbers.

Proof. Suppose M is the complex incidence matrix of G . Then $L = MM^*$. Since G is weakly connected, the number m of edges of G is at least $n - 1$. If $m = n - 1$, $\text{rank}(L) = \text{rank}(M) = n - 1$ and $\det(L) = 0$ which equals to the empty summation. Suppose $m = n - 1$. By Cauchy-Binet Theorem,

$$\det(L) = \det(MM^*) = \sum_S \det(M(:, S]) \det(\overline{M(:, S)}) = \sum_S |\det(M(:, S])|^2,$$

where the summation runs over n -subsets S of $\{1, 2, \dots, m\}$ which corresponds to a subdigraph H of G consisting of n edges. We denote the incidence matrix of H by M_H . Note that $\det(M_H) = 0$ when H is not a spanning subdigraph of G . So we consider all spanning subdigraphs H of G consisting of n edges. If any weakly connected component of H has more edges than vertices, then there is an oriented tree as a weakly connected component of H and corresponding columns of M_H will be linearly dependent making $\det(M_H) = 0$. Then by Theorem 3.8,

$$\det(L) = \sum_H |\det(M_H)|^2 = \sum_H \prod_{i=1}^{c(H)} 2|W_{U_i \setminus C_i}|(|\text{wt}(C_i)| - \text{Re}(\text{wt}(C_i))),$$

where the summation runs over all weakly connected spanning subdigraphs H of G with n edges consisting of $c(H)$ oriented unicyclic components $U_1, U_2, \dots, U_{c(H)}$ whose corresponding cycles' $(C_1, C_2, \dots, C_{c(H)})$, respectively weights are not positive real numbers. $\square$

Note that a cycle C of a weakly connected oriented graph G with $\text{wt}(C) \notin \mathbb{R}^+$ can be extended to a spanning unicyclic graph H of G with incidence matrix M_H such that $\det(M_H) \neq 0$. This observation together with the preceding theorem, we have the following result.

COROLLARY 3.12. *Let G be a weakly connected oriented graph on $n \geq 2$ vertices with complex edge weights and complex incidence matrix M . If G contains a cycle C for which $\text{wt}(C) \notin \mathbb{R}^+$, then*

$$\text{vol}^2(M) = \det(MM^*) = \sum_H \prod_{i=1}^{c(H)} 2|W_{U_i \setminus C_i}|(|\text{wt}(C_i)| - \text{Re}(\text{wt}(C_i))),$$

where the summation runs over all weakly connected spanning subdigraphs H of G with n edges consisting of $c(H)$ oriented unicyclic components $U_1, U_2, \dots, U_{c(H)}$ whose corresponding cycles' $(C_1, C_2, \dots, C_{c(H)})$ respectively weights are not positive real numbers.

EXAMPLE 3.13. Consider the oriented graph G in Figure 3 which has 3 unicyclic spanning subgraphs whose cycle weight $\text{wt}(C_H) \notin \mathbb{R}^+$: H in Figure 1 and H_2 and H_3 in Figure 4. Note that

$$2|W_{H \setminus C_H}|(|\text{wt}(C_H)| - \text{Re}(\text{wt}(C_H))) = 12\sqrt{10}(\sqrt{5} - 1),$$

$$2|W_{H_2 \setminus C_{H_2}}|(|wt(C_{H_2})| - \operatorname{Re}(wt(C_{H_2}))) = 2|4 + 2i|(6\sqrt{5} - 6) = 24\sqrt{5}(\sqrt{5} - 1),$$

$$2|W_{H_3 \setminus C_{H_3}}|(|wt(C_{H_3})| - \operatorname{Re}(wt(C_{H_3}))) = 2|2 + i|(12\sqrt{5} - 12) = 24\sqrt{5}(\sqrt{5} - 1).$$

Then $\det(L) = \left[12\sqrt{10} + 24\sqrt{5} + 24\sqrt{5}\right](\sqrt{5} - 1) \approx 179.57$.

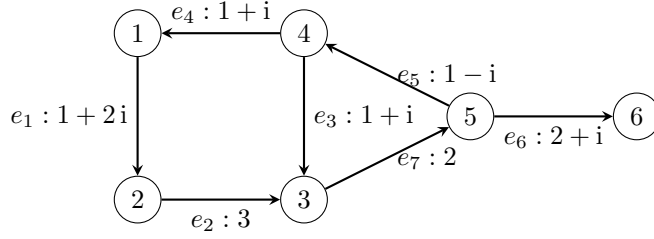


Figure 3: An oriented graph G on 6 vertices

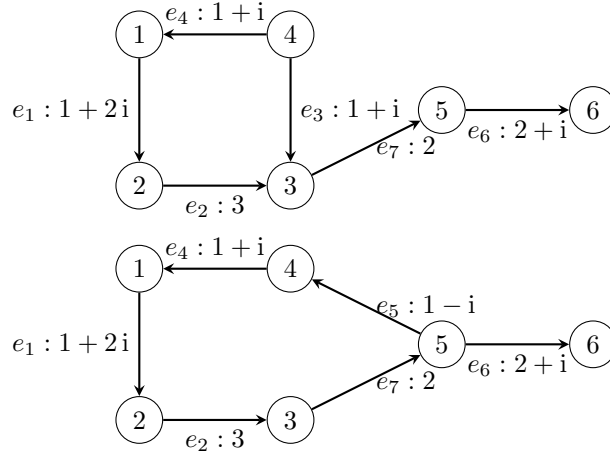


Figure 4: H_2 and H_3

A natural next step would be to answer the following open problem:

QUESTION 3.14. Find M^+ and L^+ for a weakly connected oriented graph with complex edge weights.

Similar questions were studied for graphs and signed graphs in the literature [1, 2, 11–14].

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