

ON CERTAIN GENERALIZATION OF SPLIT CONTINUITY AND ITS TOPOLOGICAL CONSEQUENCES

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Abstract. In the context of functions between topological spaces, we introduce a weaker form of split continuity [G. Beer, C. Bloomfield, G. Robinson, *Split continuity*, J. Math. Anal. Appl., **474** (2019), 833–851], namely finitely split continuity. We examine its association with upper semicontinuous multifunctions from a wider perspective. An interesting aspect of this investigation is the significant involvement of the closure of the function's graph. In the final section, we analyze the concept of finitely split continuity more thoroughly within metric spaces.

1. Introduction and preliminaries

Continuity has played a significant role in the development of analysis. Consequently, many mathematicians have been motivated to investigate weaker notions of continuity. In 2019, a weaker form of continuity, known as split continuity, was introduced by Beer et al. in [5]. The authors introduced this concept in the context of metric spaces. More recently, split continuity has been generalized to topological spaces in [9, Definition 3.1].

Before proceeding, it is essential to present the precise definition of split continuity. Throughout this work, all spaces are assumed to be Hausdorff unless otherwise stated.

DEFINITION 1.1 ([9]). Let X, Y be topological spaces. Then a function $f : X \rightarrow Y$ is said to be split continuous at $x \in X$ if there exist some $y_x \in Y$ such that

- (i) For every pair open neighborhoods U of x and V of y_x , $f(U) \cap V \neq \emptyset$;
- (ii) For every pair of open neighborhoods V_1 of $f(x)$ and V_2 of y_x there exists an open set W containing x such that $f(W) \subset V_1 \cup V_2$.

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Whenever f is split continuous at every point $x \in X$, it is referred to as split continuous on X . In the case of split continuity for a function f , the point y_x is unique for each point $x \in X$ [9]. Consequently, for every split continuous function $f : X \rightarrow Y$, there exists a corresponding function $f^* : X \rightarrow Y$ such that $f^*(x) = y_x$ for all $x \in X$. For a function f to be continuous at $x \in X$, it is necessary and sufficient that $f(x)$ equals $f^*(x)$. In cases where $f(x)$ and $f^*(x)$ are not equal, f is classified as strictly split continuous at $x \in X$ (for more details, see [1, 2, 5, 6, 9, 10]).

A multifunction Γ from X to Y assigns each $x \in X$ to a non-empty subset $\Gamma(x) \subset Y$. In [5], the authors had established a strong connection between split continuous functions and multifunctions in the context of metric spaces. It has been shown that for a function $f : (X, d) \rightarrow (Y, \rho)$ between two metric spaces, there exists a function $g : (X, d) \rightarrow (Y, \rho)$ such that the multifunction $\Gamma : (X, d) \rightrightarrows (Y, \rho)$, defined by $\Gamma(x) = \{f(x), g(x)\}$, is upper semicontinuous on X if and only if f is split continuous on X .

In Section 2 we introduce a weaker form of split continuity, named as finitely split continuity, within the context of topological spaces. Naturally, our definition provides a more general form of split continuity, which covers a large class of discontinuous maps. We investigate the relation between finitely split continuous maps and the class of all upper semicontinuous multifunctions with compact values. An interesting point of this investigation is the significant involvement of the closure of the function's graph. There is a rich literature exploringusco maps with regard to the properties of their selections. The important property there is called subcontinuity [12]. We study the connection of subcontinuous functions with finitely split continuous functions. At the end of Section 2, we prove that the composition of two finitely split continuous functions on their respective domains is finitely split continuous, whereas this is not generally true for split continuous functions. In the final section, we focus on the study of finitely split continuous functions within metric spaces. Throughout, $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of all real numbers and natural numbers, respectively, and our topological terminologies and notations are as in the book [7], from which the notions (undefined within the article) can be found.

2. Finitely split continuous function

An elementary question of analysis is how far one can characterize discontinuous maps. The concept of split continuity assists us to bind large numbers of discontinuous maps in a certain class. But still, there are lots of well-known functions in analysis those are not split continuous. For example if we consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0, \infty) \end{cases}$$

then f is not split continuous at $x = 0$. We introduce a new class of functions named as finitely split continuous functions through which we can include more functions in a

certain class, and split continuity become a special case of this new class of functions.

DEFINITION 2.1. Let X, Y be topological spaces. Then a function $f : X \rightarrow Y$ is said to be finitely split continuous at $x \in X$ if there exists a finite set $\{y_0, y_1, \dots, y_k\} \subset Y$ such that for all open sets $U_0, \dots, U_k$ containing $y_0, \dots, y_k$ respectively, the following properties hold:

(i) For every open set U containing x , $f(U) \cap U_i \neq \emptyset$ for every $i = \{0, \dots, k\}$;

(ii) There exists an open set W containing x such that $f(W) \subset \bigcup_{i=0}^k U_i$.

f is called finitely split continuous on X if it is finitely split continuous at each point of X .

REMARK 2.2. If f is finitely split continuous at x then there exists a finite set $I_x = \{y_0, \dots, y_k\}$ for which the conditions (i) and (ii) of Definition 2.1 hold. In this case we say that f is finitely split continuous at x with respect to I_x .

For a function $f : X \rightarrow Y$, we can define the corresponding multifunction $\bar{f} : X \rightrightarrows Y$ by $\bar{f}(x) = \{y \in Y : (x, y) \in \overline{G(f)}\}$, where the graph of f is given by $G(f) = \{(x, f(x)) : x \in X\}$. Our next result establishes the uniqueness of I_x .

PROPOSITION 2.3. Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function. If f is finitely split continuous at x with respect to I_x , then $I_x = \bar{f}(x)$ and thus I_x is unique and $f(x) \in I_x$.

Proof. Let $y \in I_x$. If $y = f(x)$, then clearly $y \in \bar{f}(x)$. Now, suppose $y \neq f(x)$, and let O be any open set containing (x, y) . Then, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y , respectively, such that $U \times V \subset O$ and $f(x) \notin V$. Since $y \in I_x$, it follows that $f(U) \cap V \neq \emptyset$. Thus, there exists a point $z (\neq x)$ in U such that $f(z) \in V$, implying that $(z, f(z)) \in (U \times V) \cap G(f) \subset O \cap G(f)$. This shows that $O \cap G(f) \neq \emptyset$, and therefore $y \in \bar{f}(x)$. Consequently, $I_x \subset \bar{f}(x)$.

Suppose that $y \in \bar{f}(x)$. Then for every pair of open sets $U \subset X$ and $V \subset Y$ containing x and y respectively, we have $(U \times V) \cap G(f) \neq \emptyset$, and consequently, $f(U) \cap V \neq \emptyset$.

Assume, to the contrary, that $y \notin I_x$. Since I_x is finite, take $I_x = \{y_0, y_1, \dots, y_k\}$. Then, we can always choose pairwise disjoint open sets V_i containing y_i for $i = 0, 1, \dots, k$, together with an open set V containing y such that $V \cap \bigcup_{i=0}^k V_i = \emptyset$. By property (ii) of Definition 2.1, there exists an open set W of x such that $f(W) \subset \bigcup_{i=0}^k V_i$. By taking $U = W$, we obtain $f(U) \cap V = \emptyset$, which is a contradiction. Hence $y \in I_x$, and therefore $\bar{f}(x) \subset I_x$. $\square$

If the set I_x is singleton for every $x \in X$, then the function is continuous. Furthermore, we can obtain the notion of split continuity by considering I_x as a two-point set for every $x \in X$. In particular, if f is split continuous at x , then $\bar{f}(x) = \{f(x), f^*(x)\}$.

A function $f : X \rightarrow Y$ is subcontinuous at $x \in X$, if for every net $(x_\lambda) \rightarrow x$, the net $(f(x_\lambda))$ has a cluster point (for more details, see, [8]). We establish that subcontinuity is, in essence, closely tied to finitely split continuity. We first recall the following result from [12]. A function $f : X \rightarrow Y$ is subcontinuous at x if and only if for every open cover $\mathcal{U}$ of Y there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ and a neighborhood V of x such that $f(V) \subset \bigcup \mathcal{F}$ (see [12, Theorem 2.1]). Therefore, the following proposition is a direct consequence of the property (ii) in the Definition 2.1.

PROPOSITION 2.4. *Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function. If f is finitely split continuous at x , then it is subcontinuous at x .*

An important fact about Definition 2.1 is that the condition (i) is not necessary, by omitting it from the Definition 2.1, the resulting property of “ f being split continuous at x ” is the same, just the set I_x is now non-unique. As a consequence, we conclude the following.

THEOREM 2.5 (Equivalent Definition of Finitely Split Continuity). *Let X, Y be two topological spaces and $x \in X$. Then a function $f : X \rightarrow Y$ is finitely split continuous at x , if and only if, there is a finite set $J_x \subset Y$ such that for every open set $U \subset Y$, with $J_x \subset U$, there is an open neighborhood W of x such that $f(W) \subset U$.*

Proof. We only need to prove the converse implication. Define

$$I_x = \{y \in J_x : f(U) \cap V \neq \emptyset \text{ for every pair of open sets } U \ni x \text{ and } V \ni y\}.$$

Then clearly, property (i) of Definition 2.1 is satisfied. If $I_x = J_x$, we are done. Hence, assume that $I_x \subsetneq J_x$. Then, for each $y \in J_x \setminus I_x$, there exist open sets U'_y containing x and V'_y containing y such that $f(U'_y) \cap V'_y = \emptyset$. Let $W_0 = \bigcap_{y \in J_x \setminus I_x} U'_y$. Then W_0 is an open neighbourhood of x , and $f(W_0) \cap V'_y = \emptyset$ for all $y \in J_x \setminus I_x$. Let V_y be an open subset of Y containing y for each $y \in I_x$, and take $V = \bigcup_{y \in I_x} V_y$. Then V is open and $I_x \subset V$. Define

$$\tilde{V} = V \cup \bigcup_{y \in J_x \setminus I_x} V'_y.$$

Then $\tilde{V}$ is open and $J_x \subset \tilde{V}$. There exists an open neighbourhood W of x such that $f(W) \subset \tilde{V}$. For the open neighbourhood $W \cap W_0$ of x , we obtain $f(W \cap W_0) \subset V$. Thus property (ii) of Definition 2.1 holds. It follows that f is finitely split continuous at x . $\square$

Recall that a multifunction $\Gamma : X \rightrightarrows Y$ is called upper semicontinuous at a point $p \in X$ if, for every open set U containing $\Gamma(p)$, there exists an open set W containing p such that, $x \in W$ implies $\Gamma(x) \subset U$. The multifunction Γ is called upper semicontinuous on X if it is upper semicontinuous at every point $p \in X$. For a function $f : X \rightarrow \mathbb{R}$, f is said to be upper semicontinuous (u.s.c) at $p \in X$ if the multifunction $x \mapsto (-\infty, f(p)]$ is upper semicontinuous at point p . On the other hand f is said to be lower semicontinuous (l.s.c) at $p \in X$ if the multifunction $x \mapsto [f(p), \infty)$ is upper semicontinuous at p . Then f is continuous at p if and only if it is both upper semicontinuous and lower semicontinuous at p . Let $\Gamma : X \rightrightarrows Y$ be a multifunction. A

function $f : X \rightarrow Y$ is called a selection function of Γ if, for every $x \in X$, we have $f(x) \in \Gamma(x)$. For more information one can see [3, 4].

In [5], Beer et al. established a relation between split continuity and upper semi-continuity in metric structure. They showed that a function $f : (X, d) \rightarrow (Y, \rho)$ is split continuous on X if and only if there exists another function $g : (X, d) \rightarrow (Y, \rho)$ such that the multifunction $x \mapsto \{f(x), g(x)\}$ is upper semicontinuous on X . Here, we move forward in that direction utilizing a more general approach that provides a characterization of arbitrary selection functions for upper semicontinuous multifunctions.

THEOREM 2.6. *Let X, Y be topological spaces, and let $\Gamma : X \rightrightarrows Y$ be a multifunction. If Γ is upper semicontinuous at $x \in X$ and $\Gamma(x)$ is finite, then every selection function of Γ is finitely split continuous at x .*

Proof. Let Γ be upper semicontinuous at $x \in X$ with $\Gamma(x)$ is a finite subset of Y and $f : X \rightarrow Y$ be a selection of Γ . Also, Let U be an open subset of Y containing $\Gamma(x)$. Since Γ is upper semicontinuous at x , there exists an open set W containing x such that $z \in W$ implies $\Gamma(z) \subset U$. Thus, $f(W) \subset U$. By Theorem 2.5, f is finitely split continuous at x . $\square$

The following is a corollary of [11, Corollary 3.12]. However, we include a proof for the reader's convenience.

THEOREM 2.7. *Let X, Y be topological spaces, with Y regular. If $f : X \rightarrow Y$ is finitely split continuous at $x \in X$, then the multifunction $\bar{f} : X \rightrightarrows Y$ is upper semicontinuous at $x \in X$.*

Proof. Suppose $f : X \rightarrow Y$ is a finitely split continuous function. Let $x \in X$ and $I_x = \{y_0 = f(x), y_1, \dots, y_k\}$. By Proposition 2.3, we have $I_x = \bar{f}(x)$. Let V be an open set containing $\bar{f}(x)$. Since Y is regular, there exist open sets V_i containing y_i for each $i = 0, 1, \dots, k$ such that $\bigcup_{i=0}^k V_i \subset \bigcup_{i=0}^k \bar{V}_i \subset V$. As f is finitely split continuous at x , there

exists an open set U containing x such that $f(U) \subset \bigcup_{i=0}^k V_i$. Let $p \in U$ and $z \in \bar{f}(p)$.

Then, there exists a net $(x_\lambda, f(x_\lambda))_{\lambda \in \Lambda}$ in $G(f)$ such that $(x_\lambda, f(x_\lambda))_{\lambda \in \Lambda} \rightarrow (p, z)$. This implies that $(x_\lambda)_{\lambda \in \Lambda} \rightarrow p$ and $(f(x_\lambda))_{\lambda \in \Lambda} \rightarrow z$. Since $p \in U$ and U is an open set, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. Consequently, $f(x_\lambda) \in \bigcup_{i=0}^k V_i$ for all $\lambda \geq \lambda_0$. Thus, $z \in \overline{\bigcup_{i=0}^k V_i} \subset V$, which implies that $\bar{f}(p) \subset V$. Therefore, $\bar{f}$ is upper semicontinuous at x . $\square$

By using the last two results, we can conclude the following significant characterization of finitely split continuous functions.

THEOREM 2.8. *Let X, Y be topological spaces, Y be regular and $f : X \rightarrow Y$ be a function. Then f is finitely split continuous at $x \in X$ if and only if, $\bar{f}$ is upper semicontinuous at x and $\bar{f}(x)$ is finite.*

COROLLARY 2.9. *Let X, Y be topological spaces, Y be regular. If $f : X \rightarrow Y$ is finitely split continuous at $x \in X$, then every selection function of the multifunction $\bar{f} : X \rightrightarrows Y$ is also finitely split continuous at $x \in X$.*

From these aforementioned results, one can easily observe some important properties of split continuous functions in the context of topological space.

COROLLARY 2.10. *Let X, Y be topological spaces, Y be regular. If $f : X \rightarrow Y$ is split continuous at $x \in X$ then f^* is also split continuous at $x \in X$.*

Proof. The proof follows from the fact that, in the case of split continuity, f^* is a selection function of $\bar{f}$. $\square$

COROLLARY 2.11. *Let X, Y be topological spaces, Y be regular and $f : X \rightarrow Y$ be a function. Then f is split continuous on X if and only if there exists another function $g : X \rightarrow Y$ such that the multifunction $\Gamma : X \rightrightarrows Y$ defined by $\Gamma(x) = \{f(x), g(x)\}$ is upper semicontinuous on X .*

We now present a characterization of finitely split continuity via nets, along with the corresponding statement for split continuity.

THEOREM 2.12. *Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is finitely split continuous at x , if and only if, for every net $(x_\lambda) \rightarrow x$, the net $(f(x_\lambda))$ has finitely many cluster points, but at least one.*

Proof. Let f be finitely split continuous at x with respect to the unique finite set I_x . Let $(x_\lambda) \rightarrow x$. Then from Proposition 2.4, f is subcontinuous and so the net $(f(x_\lambda))$ has at least one cluster point. Now since Y is Hausdorff, from property (ii) in Definition 2.1, we can conclude that the net $(f(x_\lambda))$ has no cluster point outside I_x . Therefore, the net $(f(x_\lambda))$ has finitely many cluster points, but at least one.

Conversely suppose that the condition holds. Let B_x be a fundamental system of neighborhoods of x . Let A be the set all cluster points of all such nets $(f(x_\alpha))_{\alpha \in B_x}$, where $(x_\alpha)_{\alpha \in B_x} \rightarrow x$. We claim that A is finite. If not, there exists an infinite set of distinct points $\{y_n : n \in \mathbb{N}\} \subset A$. Then without any loss of generality for every $n \in \mathbb{N}$, there exists a net $(x_\alpha^n)_{\alpha \in B_x} \rightarrow x$ such that $(f(x_\alpha^n))_{\alpha \in B_x}$ converges to y_n . Now, for each $n \in \mathbb{N}$, we can construct a set D_n such that there exists a bijection $g_n : D_n \rightarrow B_x$. Now for each $n \in \mathbb{N}$, define a relation ' $\geq_{D_n}$ ' on D_n by $d \geq_{D_n} d'$ if and only if $g_n(d) \subset g_n(d')$ for all $d, d' \in D_n$. Then $(D_n, \geq_{D_n})$ is a directed set for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, consider the net $(z_d^n)_{d \in D_n}$ defined by $z_d^n := x_{g_n(d)}^n$. Then $(z_d^n)_{d \in D_n} \rightarrow x$ and both $(f(z_d^n))_{d \in D_n}$ and $(f(x_\alpha^n))_{\alpha \in B_x}$ have the same cluster points.

Let $\Lambda = \bigsqcup_n D_n$ be the disjoint union of all D_n . Next, we define a relation ' $\geq$ ' on Λ such that $\lambda_1 \geq \lambda_2$ holds if one of the following conditions is satisfied:

(i) $\lambda_1, \lambda_2 \in D_n$ and $\lambda_1 \geq_{D_n} \lambda_2$ for some $n \in \mathbb{N}$;

(ii) $\lambda_1 \in D_{n_1}, \lambda_2 \in D_{n_2}$ and $g_{n_1}(\lambda_1) \subset g_{n_2}(\lambda_2)$ for some $n_1 \neq n_2$.

Clearly, $(\Lambda, \geq)$ is a directed set. Now we define a net $(s_\lambda)_{\lambda \in \Lambda}$ by $s_\lambda := z_\lambda^n$ if $\lambda \in D_n$. Then $(s_\lambda)_{\lambda \in \Lambda} \rightarrow x$ and $(f(s_\lambda))_{\lambda \in \Lambda}$ has infinitely many cluster points, which contradicts the given condition. Therefore, A is finite.

Let U be open in Y with $A \subset U$. Suppose that for every neighborhood W of x , we have $f(W) \not\subset U$. Consequently, we obtain a net $(x_\alpha)_{\alpha \in B_x} \rightarrow x$ such that $f(x_\alpha) \notin U$ for all $\alpha \in B_x$. Then from the construction of the set A , we can conclude that $(f(x_\alpha))$ has no cluster point. But this contradicts the given condition. So there exists an open neighborhood W of x , such that $f(W) \subset U$. Then from Theorem 2.5, f is finitely split continuous at x . $\square$

COROLLARY 2.13. *Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is split continuous at x , if and only if, for every net $(x_\lambda) \rightarrow x$, the net $(f(x_\lambda))$ has exactly one or two cluster points.*

A well-known characterization of continuity vis subcontinuity was established in [8]. For topological spaces X and Y , a function $f : X \rightarrow Y$ is continuous if and only if it is subcontinuous and has a closed graph. A localized version of this result also holds for each $x \in X$, where “having a closed graph at x ” can be formulated as $\bar{f}(x)$ is a singleton. Continuing this pattern, we provide a characterization of finitely split continuity via subcontinuity, and consequently, of split continuity.

THEOREM 2.14. *Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is finitely split continuous at $x \in X$, if and only if, f is subcontinuous at x and $\bar{f}(x)$ is finite.*

Proof. The proof follows from Proposition 2.3, Proposition 2.4 and Theorem 2.12. $\square$

THEOREM 2.15. *Let X, Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is split continuous at $x \in X$, if and only if, f is subcontinuous at x and $\bar{f}(x)$ has one or two elements.*

The proof of the following theorem is similar to the proof of [5, Proposition 2.5].

THEOREM 2.16. *Let X be a topological space. If $f : X \rightarrow \mathbb{R}$ is split continuous at $x \in X$, then f is either upper semicontinuous or lower semicontinuous at x .*

It is natural to ask whether the class of finitely split continuous functions is closed under composition or not? In other words, if $f : X \rightarrow Y$ is finitely split continuous at $p \in X$ and $g : Y \rightarrow Z$ is finitely split continuous at $f(p)$, does it follow that $g \circ f$ is finitely split continuous at p ? The following example shows that this is not true in general. Consider the functions $f : [0, 1] \rightarrow [0, 1]$ and $g : [0, 1] \rightarrow \mathbb{R}$ defined as follows:

- $f(x) = x$ if $x \neq 0$, and $f(0) = 1$,
- $g(x) = n$ if $x = \frac{1}{n+1}$ for some integer $n \geq 0$, and $g(x) = 0$ otherwise.

In this case, f is finitely split continuous at 0, and g is finitely split continuous at $f(0) = 1$. However, the composition $g \circ f$ is not finitely split continuous at 0. Note that here g is not finitely split continuous at 0. Our next result asserts that if both functions are finitely split continuous on their respective domains, then their composition is also finitely split continuous. But in case of split continuity, composition of two split continuous functions may not be split continuous [5].

THEOREM 2.17. *Let $f : X \rightarrow Y$ be a function and $p \in X$. The following statements are equivalent:*

- (i) *f is finitely split continuous at p with respect to some $I_p = \{y_0, y_1, \dots, y_k\}$.*
- (ii) *Whenever Z is a topological space and $g : Y \rightarrow Z$ is finitely split continuous at each y_i , for $i = 0, 1, \dots, k$, the composition $g \circ f$ is finitely split continuous at p .*

Proof. (i) $\Rightarrow$ (ii) Suppose that f is finitely split continuous at p with respect to some $I_p = \{y_0, y_1, \dots, y_k\}$. Let Z be a topological space, and $g : Y \rightarrow Z$ be finitely split continuous at each y_i , with respect to $I_{y_i} = \{z_0^i, z_1^i, z_2^i, \dots, z_{i_k}^i\} \subset Z$ for $i = 0, 1, \dots, k$.

Define $I = \bigcup_{i=0}^k I_{y_i}$. Let V be an open set containing I . Since g is finitely split continuous at y_i , there exists an open neighborhood U_i of y_i such that $g(U_i) \subset V$, for each $i = 0, 1, \dots, k$.

On the other hand since f is finitely split continuous at p , there exists an open neighborhood W of p such that $f(W) \subset \bigcup_{i=0}^k U_i$. This implies that $g \circ f(W) \subset V$.

Hence, from Theorem 2.5, $g \circ f$ is finitely split continuous at p .

(ii) $\Rightarrow$ (i) holds trivially by choosing $g = id_Y$. □

If $f : X \rightarrow Y$ is finitely split continuous at $p \in X$ with respect to $I_p = \{y_0, y_1, \dots, y_k\}$, and $g : Y \rightarrow Z$ is finitely split continuous at each $y_i \in I_p$ with respect to I_{y_i} , then the finite set needed for the finitely split continuity of $g \circ f$ at p does not necessarily have to be $I = \bigcup_{i=0}^k I_{y_i}$. The following example illustrates this fact.

EXAMPLE 2.18. Note that the function $f : [0, 1] \rightarrow [0, 2]$, defined by

$$f(x) = \begin{cases} 1 & \text{if } x < 1, \\ 2 & \text{if } x = 1 \end{cases},$$

is finitely split continuous at 1 with $I_1^f = \{1, 2\}$, and the function $g : [0, 2] \rightarrow \mathbb{R}$, defined by

$$g(x) = \begin{cases} 1 & \text{if } x < 1, \\ 2 & \text{if } x = 1, \\ x + 2 & \text{if } x > 1 \end{cases},$$

is finitely split continuous at 1 and 2 with $I_1^g = \{1, 2, 3\}$ and $I_2^g = \{4\}$ respectively. However, the composition function $g \circ f$ is finitely split continuous at 1 with $I_1^{g \circ f} = \{2, 4\} \neq I = I_1^g \cup I_2^g = \{1, 2, 3, 4\}$.

COROLLARY 2.19. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are finitely split continuous on X and Y , respectively, then $g \circ f$ is finitely split continuous on X .*

In the following theorem we show that if a split continuous function between two topological spaces is followed by a continuous function into an arbitrary space, then the composition is also split continuous.

THEOREM 2.20. *Let $f : X \rightarrow Y$ be a function with $p \in X$. Then the following statements are equivalent.*

(i) *f is split continuous at p .*

(ii) *For any space W and for every $g \in C(Y, W)$, the composition $g \circ f$ is split continuous at p .*

Proof. (i) $\Rightarrow$ (ii) Suppose that f is split continuous at p with respect to the set $I_p = \{f(p), y_p\}$, and let B_p , B_{y_p} , and $B_{f(p)}$ be the fundamental systems of neighborhoods of p , y_p , and $f(p)$, respectively. Consider two fundamental systems of neighborhoods, $B_{g \circ f(p)}$ and $B_{g(y_p)}$, of $g \circ f(p)$ and $g(y_p)$, respectively. We aim to show that $g \circ f$ is split continuous at p with $I_p = \{g(f(p)), g(y_p)\}$. Let $U_1 \in B_{g \circ f(p)}$ and $U_2 \in B_{g(y_p)}$. Since g is continuous at both $f(p)$ and y_p , there exist $V_1 \in B_{f(p)}$ and $V_2 \in B_{y_p}$ such that $g(V_1) \subset U_1$ and $g(V_2) \subset U_2$. From split continuity of f at p , for any open set $U \in B_p$, $f(U) \cap V_1 \neq \emptyset$ and $f(U) \cap V_2 \neq \emptyset$. Consequently, $g \circ f(U) \cap U_1 \neq \emptyset$ and $g \circ f(U) \cap U_2 \neq \emptyset$. Also, from split continuity of f , there exists $U' \in B_p$ such that $f(U') \subset V_1 \cup V_2$. This implies $g \circ f(U') \subset U_1 \cup U_2$. Hence, $g \circ f$ is split continuous at p .

(ii) $\Rightarrow$ (i) holds trivially by choosing $g = id_Y$. □

3. Finitely split continuity on metric spaces

In this section, we analyze finitely split continuous functions in the context of metric spaces. For a metric space (X, d) , $B_d(x, \varepsilon)$ represents the open ball centered at x with radius ε . By Theorem 2.5, for a function $f : (X, d) \rightarrow (Y, \rho)$ and $x \in X$ if there exists a finite set $I \subset Y$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ satisfying $f(B_d(x, \delta)) \subset \bigcup_{y \in I} B_\rho(y, \varepsilon)$, then f is finitely split continuous at x , with respect to

some $I_x \subset I$. Also, from Theorem 2.12, $f : (X, d) \rightarrow (Y, \rho)$ is finitely split continuous at x if and only if for every sequence (x_n) in X with $x_n \rightarrow x$, the sequence $(f(x_n))$ has finitely many cluster points but at least one. In the following theorem we will present a stronger form of Theorem 2.17 in metric space structure.

THEOREM 3.1. *Let $f : (X, d) \rightarrow (Y, \rho)$ be a function and $p \in X$. The following statements are equivalent:*

- (i) f is finitely split continuous at p with respect to the set $I_p = \{y_0, y_1, \dots, y_k\}$.
- (ii) If $g : Y \rightarrow \mathbb{R}$ is finitely split continuous at each y_i , for $i = 0, 1, \dots, k$, the composition $g \circ f$ is finitely split continuous at p .
- (iii) Whenever $g \in C(Y, \mathbb{R})$, the composition $g \circ f$ is finitely split continuous at p .

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 2.17 and (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) On the contrary, suppose that f is not finitely split continuous at p . Then we have the following cases:

Case I: There exists a sequence (x_n) with $x_n \rightarrow p$ such that $(f(x_n))$ has infinitely many cluster points in Y .

Subcase I(a): There exists a sequence (y_n) consisting of cluster points of $(f(x_n))$ and $\varepsilon > 0$ such that $\rho(y_m, y_n) > \varepsilon$ for all $m \neq n \in \mathbb{N}$.

Then, for each $n \in \mathbb{N}$, there exists an increasing sequence (m_k^n) of natural numbers such that $f(x_{m_k^n}) \rightarrow y_n$. Consequently, for every n there exists some $K_n \in \mathbb{N}$ such that $f(x_{m_k^n}) \in B_\rho(y_n, \frac{\varepsilon}{4})$ for all $k \geq K_n$. Hence without any loss of generality for every n , we can choose the corresponding increasing sequence (m_k^n) in such a way that whenever $n \neq n'$ we have $f(x_{m_k^n}) \neq f(x_{m_{k'}^{n'}})$ for all $k, k' \in \mathbb{N}$. Thus, the set $A = \{y_n, f(x_{m_k^n}) : n, k \in \mathbb{N}\}$ is closed in Y . Consequently, the function $g' : A \rightarrow \mathbb{R}$, defined by $g'(y) = n$ if $y = y_n$ or $y = f(x_{m_k^n})$, is continuous on A .

Subcase I(b): There exists a non-constant Cauchy sequence (y_n) consisting of cluster points of $(f(x_n))$.

Then, for each $n \in \mathbb{N}$, there exists an increasing sequence (m_k^n) of natural numbers such that $f(x_{m_k^n}) \rightarrow y_n$. Without any loss of generality, we can assume that no y_n is a cluster point of (y_n) . So for every $n \in \mathbb{N}$ there exists some $\varepsilon_n > 0$ such that $y_m \notin B_\rho(y_n, \varepsilon_n)$ for every $m \neq n$. On the other hand for every $n \in \mathbb{N}$, there exists some $K_n \in \mathbb{N}$ such that $f(x_{m_k^n}) \in B_\rho(y_n, \frac{\varepsilon_n}{4})$ for all $k \geq K_n$. Hence without any loss of generality for every n , we can choose the corresponding increasing sequence (m_k^n) in such a way that whenever $n \neq n'$ we have $f(x_{m_k^n}) \neq f(x_{m_{k'}^{n'}})$ for all $k, k' \in \mathbb{N}$. Now consider the closed set $\bar{A}$ in Y , where $A = \{y_n, f(x_{m_k^n}) : n, k \in \mathbb{N}\}$. Consequently, the function $g' : \bar{A} \rightarrow \mathbb{R}$, defined by

$$g'(y) = \begin{cases} \frac{1}{n} & \text{if } y = y_n \text{ or } y = f(x_{m_k^n}), \\ 0 & \text{otherwise} \end{cases},$$

is continuous on $\bar{A}$.

Case II: There exists a sequence (x_n) with $x_n \rightarrow p$ such that $(f(x_n))$ has no cluster points in Y .

Thus, the set $A = \{f(x_n) : n \in \mathbb{N}\} \cup \{f(p)\}$ is closed in Y . Consequently, the function $g' : A \rightarrow \mathbb{R}$, defined by

$$g'(y) = \begin{cases} n & \text{if } y = f(x_n), \\ 0 & \text{otherwise} \end{cases},$$

is continuous on A .

In all the cases and subcases, by Tietze's extension theorem, there exists $g \in C(Y, \mathbb{R})$ such that $g(y) = g'(y)$ on A . However, $g \circ f$ is not finitely split continuous at p , which contradicts our assumption. $\square$

We know that oscillation provides an insightful characterization of continuous functions. Recall that, for a function $f : (X, d) \rightarrow (Y, \rho)$ and $p \in X$, the oscillation of f at p , denoted by $\omega(f, p)$, is defined as the limit of the sequence $(\omega_n(f, p))$, where $\omega_n(f, p) := \text{diam}_\rho f(B_d(p, \frac{1}{n}))$. Thus, the oscillation is given by

$$\omega(f, p) = \lim_{n \rightarrow \infty} \omega_n(f, p).$$

It is well known that f is continuous at p if and only if $\omega(f, p) = 0$. In [5], the authors have proved that if $f : (X, d) \rightarrow (Y, \rho)$ is split continuous at $p \in X$, then the oscillation of f at p is given by $\rho(f(p), f^*(p))$, that is, $\omega(f, p) = \rho(f(p), f^*(p))$. Furthermore, it was shown that the oscillation function of a split continuous function is itself split continuous. We now turn our attention to studying the oscillation function of a finitely split continuous function.

THEOREM 3.2. *Let $f : (X, d) \rightarrow (Y, \rho)$ be finitely split continuous at $p \in X$, with $I_p = \{y_1, y_2, \dots, y_k\}$. Then, the oscillation of f at p , $\omega(f, p)$, is given by, $\omega(f, p) = \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\}$.*

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be given. For each $y_i \in I_p$, there exists $x_i \in B_d(p, \frac{1}{n})$ such that $\rho(f(x_i), y_i) < \frac{\varepsilon}{2}$. Therefore, for all $i, j = 1, 2, \dots, k$, we have $\rho(f(x_i), y_j) > \rho(y_i, y_j) - \frac{\varepsilon}{2}$. That is, for every $n \in \mathbb{N}$, there exist $x \in B_d(p, \frac{1}{n})$ and $y \in I_p$ such that

$$\rho(f(x), y) > \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\} - \frac{\varepsilon}{2}.$$

Since f is finitely split continuous at p , there exists $x' \in B_d(p, \frac{1}{n})$ such that $\rho(f(x'), y) < \frac{\varepsilon}{2}$. Consequently, for every $n \in \mathbb{N}$, there exist $x, x' \in B_d(p, \frac{1}{n})$ such that

$$\rho(f(x), f(x')) > \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\} - \varepsilon.$$

It follows that for every $n \in \mathbb{N}$,

$$\sup\{\rho(f(x_1), f(x_2)) : x_1, x_2 \in B_d(p, \frac{1}{n})\} \geq \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\}.$$

Thus, $\omega(f, p) \geq \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\}$. Now, since f is finitely split continuous at p with $I_p = \{y_1, y_2, \dots, y_k\}$, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$f(x) \in \bigcup_{i=1}^k B_\rho(y_i, \frac{\varepsilon}{2}) \quad \forall x \in B_d(p, \frac{1}{n}).$$

Thus, for all $x_1, x_2 \in B_d(p, \frac{1}{n})$, we have $\rho(f(x_1), f(x_2)) < \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\} + \varepsilon$. Consequently, $\omega(f, p) \leq \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\}$, and thus,

$$\omega(f, p) = \max\{\rho(y_i, y_j) : i, j = 1, 2, \dots, k\}.$$

□

THEOREM 3.3. *Let $f : (X, d) \rightarrow (Y, \rho)$ be finitely split continuous on X . Then, the oscillation function, $\omega(f, \cdot)$ is also finitely split continuous on X .*

Proof. Let $p \in X$ and I_p be determined by the finitely split continuity of f at p . Define $I'_p = \{\rho(y, y') : y, y' \in I_p\}$. Let $\varepsilon > 0$ be given. Since f is finitely split continuous at p , there exists $\delta > 0$ such that $f(B_d(p, \delta)) \subset \bigcup_{y \in I_p} B_\rho(y, \frac{\varepsilon}{2})$. Let $x \in B_d(p, \delta)$. Then, there exists $\delta_0 > 0$ such that $B_d(x, \delta_0) \subset B_d(p, \delta)$. It follows that

$$f(B_d(x, \delta_0)) \subset \bigcup_{y \in I_p} B_\rho(y, \frac{\varepsilon}{2}).$$

Thus, we must have, $r - \varepsilon \leq \omega(f, x) \leq r + \varepsilon$, for some $r \in I'_p$. That is, $\omega(f, x) \in \bigcup_{r \in I'_p} (r - \varepsilon, r + \varepsilon)$. Consequently, $\omega(f, B_d(p, \delta)) \subset \bigcup_{r \in I'_p} (r - \varepsilon, r + \varepsilon)$. Therefore, $\omega(f, \cdot)$ is finitely split continuous at p with a subset of I'_p . □

The oscillation function $\omega(f, \cdot)$ can be finitely split continuous even if the function f itself is not. For instance, the function f introduced in Example 3.4 is nowhere finitely split continuous, yet its oscillation function is continuous, since $\omega(f, \cdot) = 1$.

It is well known that the pointwise limit of a sequence of continuous functions need not be continuous. Moreover, as shown in [2], the limit may also fail to be split continuous. Also, in [2] it has been shown that uniform limit of a sequence of split continuous functions is also split continuous. But in case of finitely split continuity one can see a different picture. In the following example we will show that uniform limit of a sequence of finitely split continuous functions may not be finitely split continuous.

EXAMPLE 3.4. Let $\mathbb{R} = \bigcup_{n \in \mathbb{N}} X_n$, where X_n are pairwise disjoint and dense in $\mathbb{R}$. Let us define (f_n) :

$$f_n(x) = \begin{cases} \frac{1}{k}, & \text{if } x \in X_k \text{ and } k < n \\ \frac{1}{n}, & \text{otherwise} \end{cases}$$

Also, define f on $\mathbb{R}$ such that $f(x) = \frac{1}{k}$ if $x \in X_k$. Note that f_n are finitely split continuous and converge uniformly to f , which is nowhere finitely split continuous.

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