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# WEAKLY CONTRACTIVE MAPS AND COMMON FIXED POINTS

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**Abstract.** We prove the existence of common coincidence point and common fixed point for the self mappings satisfying a generalized weak contractive condition. Consequently it is shown that sequence of Mann type iterations associated with two self-mappings on convex metric space converges to their common fixed point.

## 1. Introduction and preliminaries

When we study linearity, we deal with a vector space and particularly with a normed vector space. But many of linear concepts may be obtained by defining convexity instead. Since these concepts are based not on the linear structure, but on the convexity, it is natural to extend them to nonlinear spaces, which still carry some kind of convexity.

In 1970, Takahashi [23] succeeded to introduce a notion of convexity in a metric space and generalized some results regarding fixed points of nonlinear mappings in Banach spaces. This paper of Takahashi was followed by a spate of papers and a number of interesting results on fixed point theorems have been extended to metric spaces by various authors (e.g. see [3–5,7–13]). The aim of this paper is to prove the existence of coincidence points and common fixed points for a class of noncommuting mappings on convex metric spaces. Our results extend recent results of Beg and Abbas [6] and Rhoades [21].

Definition 1.1. [23] Let $X$ be a metric space and $I = [0, 1]$. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on $X$ if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda) d(u, y).$$

Metric space $X$ together with the convex structure $W$ is called a convex metric space.

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Let $X$ be a convex metric space. A nonempty subset $C \subset X$ is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$. Takahashi [23] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(x, y) \leq r\}$ are convex. Also if $\{C_\alpha : \alpha \in A\}$ is a family of convex subsets of $X$, then $\bigcap\{C_\alpha : \alpha \in A\}$ is convex.

All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see Takahashi [23]).

Jungck [14] introduced the concept of commuting mappings and improved the Banach contraction principle. Afterwards Sessa [22] generalized the concept of commuting mappings by calling self mappings $f, g$ of a metric space $X$ weakly commuting if and only if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$. However, since elementary function are not weakly commuting. Pant [18] introduced less restrictive concept of $R$-weakly commutativity of mappings (see Definition 1.2) and improved two important fixed points theorems for a pair of $R$-weakly commuting mappings. Recently Azam [2] used this concept of $R$-weakly commutativity of mappings and extended the results of Pant [18] to four mappings.

Definition 1.2. Let $(X, d)$ be a metric space. A pair of mappings $f, g \colon X \to X$ is called $R$-weakly commuting, provided there exists some positive real number $R$ such that

$$d(fgx, gfx) \leq Rd(fx, gx)$$

for each $x \in X$. For details see Pant [18].

We note that $R$-weakly commuting mappings commute at their coincidence points. Jungck and Rhoades [16] then defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points.

## 2. Main results

Alber and Guerre-Delabriere [1] obtained fixed point results in Hilbert spaces by introducing the concept of weakly contractive mappings (see Definition 2.1). Rhoads [21] extended their work in Banach spaces. Recently Beg and Abbas [6] proved a generalization of the corresponding theorems of Rhoads [21] for a pair of mapping in which one is weakly contractive with respect to the other.

Definition 2.1. Let $X$ be a metric space. A mapping $T \colon X \to X$ is called weakly contractive with respect to $f \colon X \to X$ if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy))$$

where $\varphi \colon [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and positive on $(0, \infty)$, $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$.

Definition 2.2. (modified Mann iterative scheme). Let $X$ be a convex complete metric space and let $T$ be a weakly contractive map with respect to $f$ on $X$.

Assume that $TX \subset fX$ and $fX$ is a convex subset of $X$. Define a sequence $\{y_n\}$ in $fX$ as

$$y_n = fx_{n+1} = W(Tx_n, fx_n, \alpha_n), \qquad x_0 \in X, \ \ n \geq 0,$$

where $\alpha_n \in I$ for each $n$. The sequence thus obtained is a modified Mann iterative scheme.

THEOREM 2.3. *Let $X$ be a metric space and $S$, $T$, $f$ be self mappings of $X$ and for each $x, y$ in $X$*

$$d(Sx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)), \tag{2.1}$$

*where $\varphi$ is continuous, nondecreasing and positive on $(0, \infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. If $SX \cup TX \subset fX$ and $fX$ is a complete subspace of $X$, then there exists a point $p$ in $X$ such that $fp = Sp = Tp$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Choose a point $x_1$ in $X$ such that $fx_1 = Sx_0$. This can be done since $SX \subset fX$. Similarly choose $x_2 \in X$ such that $fx_2 = Tx_1$. In general, having chosen $x_n$ in $X$, we obtain $x_{n+1}$ in $X$ such that

$$fx_{2k+1} = Sx_{2k} \quad \text{and} \quad fx_{2k+2} = Tx_{2k+1},$$

where $k$ is any positive integer. Consider

$$\begin{aligned} d(fx_{2k-1}, fx_{2k}) = d(Sx_{2k-2}, Tx_{2k-1}) &\leq d(fx_{2k-2}, fx_{2k-1}) \\ - \varphi(d(fx_{2k-2}, fx_{2k-1})) &\leq d(f(x_{2k-2}, f(x_{2k-1})), \quad \text{and} \\ d(fx_{2k}, fx_{2k+1}) = d(Tx_{2k-1}, Sx_{2k}) &\leq d(fx_{2k-1}, fx_{2k}), \end{aligned}$$

which shows that $\{d(fx_n, fx_{n+1})\}$ is a nonincreasing sequence of positive real numbers. Using an argument similar to that used in the corresponding part of [6, Theorem 2.1], it follows that $d(fx_n, fx_m) \to 0$ as $m, n \to \infty$. As $fX$ is a complete subspace of $X$, therefore $\{fx_n\}$ has a limit $q$ in $fX$. Consequently, we obtain $p$ in $X$ such that $fp = q$. Thus

$$fx_{2k+1} = Sx_{2k} \to fp \quad \text{and} \quad fx_{2k+2} = Tx_{2k+1} \to fp.$$

Now using inequality (2.1), we obtain

$$d(fx_{2k+1}, Tp) = d(Sx_{2k}, Tp) \leq d(fx_{2k}, fp) - \varphi(d(fx_{2k}, fp)).$$

Taking limit $n \to \infty$, we obtain

$$d(q, Tp) \leq d(q, fp) - \varphi(d(q, fp)).$$

It follows that $fp = Tp$. By a similar argument we have $fp = Sp$. Hence $p$ is a solution of the functional equations $fx = Sx = Tx$. ∎

COROLLARY 2.4. [6] *Let $X$ be a metric space and let $T$ be a weakly contractive mapping with respect to $f$. If the range of $f$ contains the range of $T$ and $fX$ is a complete subspace of $X$, then $f$ and $T$ have a coincidence point in $X$.*

Theorem 2.5. *Let $X$ be a metric space and $S, T, f$ be self mappings of $X$ satisfying inequality (2.1). If the pairs $(f, S)$ and $(f, T)$ are weakly compatible (or R-weakly commuting) and $SX \cup TX \subset fX$ and $fX$ is a complete subspace of $X$, then $f$, $S$ and $T$ have a common fixed point.*

*Proof.* By Theorem 2.3, we obtain a point $p$ in $X$ such that $fp = Sp = Tp = q$ (say) which further implies that $fSp = Sfp$ and $fTp = Tfp$. Obviously, $Sq = fq = Tq$. We claim $q = fq$. For, if $q \neq fq$, consider

$$d(fq, q) = d(Sq, Tp) \leq d(fp, fq) - \varphi(d(fp, fq)),$$

$d(q, Tp) < d(q, fq)$. This contradiction leads to the result. ∎

Corollary 2.6. [6] *Let $X$ be a metric space and let $T$ be a weakly contractive mapping with respect to $f$. If $T$ and $f$ are weakly compatible and $TX \subset fX$ and $fX$ is a complete subspace of $X$, then $f$ and $T$ have a common fixed point in $X$.*

Theorem 2.7. *Let $X$ be a convex metric space and let $T\colon X \to X$ be a weakly contractive mapping with respect to $f\colon X \to X$. If the pair $(f, T)$ is weakly compatible (or R-weakly commuting) and $TX \subset fX$ and $fX$ is a convex and complete subspace of $X$, then the modified Mann iterative scheme with $\sum \alpha_n = \infty$ converges to a common fixed point of $f$ and $T$.*

*Proof.* From Corollary 2.6, we obtain a common fixed point $q$ of $f$ and $T$. Now consider

$$\begin{aligned} d(y_n, q) &= d(fx_{n+1}, fp) = d(W(Tx_n, fx_n, \alpha_n), fp) \\ &\leq (1-\alpha_n)d(fx_n, fp) + \alpha_n d(Tx_n, fp) \leq (1-\alpha_n)d(fx_n, fp) + \alpha_n d(Tx_n, Tp) \\ &\leq (1-\alpha_n)d(fx_n, fp) + \alpha_n(d(fx_n, fp) - \varphi(d(fx_n, fp))) \\ &\leq d(fx_n, fp) - \alpha_n\varphi(d(fx_n, fp)) \leq d(y_{n-1}, q), \end{aligned}$$

which gives $\lim_{n\to\infty} d(y_n, q) = l \geq 0$. Now if $l > 0$, then for any positive integer $N$ we have

$$\sum_{n=N}^{\infty} \alpha_n\varphi(l) \leq \sum_{n=N}^{\infty} \alpha_n\varphi(d(y_n, q)) \leq \sum_{n=N}^{\infty} (d(y_{n-1}, q) - d(y_n, q)) < d(y_{N-1}, q),$$

which is a contradiction for the choice of $\alpha_n$. Hence, the modified Mann iterative scheme converges to a common fixed point of $f$ and $T$. ∎

Theorem 2.8. *Let $X$ be a convex metric space and let $T$ be a weakly contractive mapping with respect to $f$. If the pair $(f, T)$ is weakly compatible (or R-weakly commuting) and $TX \subset fX$ and $fX$ is a convex and complete subspace of $X$, suppose two sequences of mappings $\{y_n\}$ and $\{z_n\}$ are defined as*

$$z_n = fx_{n+1} = W(Tv_n, fx_n, \alpha_n), \quad y_n = fv_n = W(Tx_n, fx_n, \beta_n), \quad n = 0, 1, 2, \dots$$

*where $0 \leq \alpha_n, \beta_n \leq 1$, $\sum \alpha_n\beta_n = \infty$, and $x_0 \in X$. Then the iterative scheme $\{z_n\}$ converges to a common fixed point of $T$ and $f$.*

*Proof.* Let $q$ be a common fixed point of $T$ and $f$ (its existence follows from corollary 2.6). Now

$$\begin{aligned}
d(z_n,q) &= d(W(Tv_n,fx_n,\alpha_n),q) \le \alpha_n d(Tv_n,q) + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n d(Tv_n,Tp) + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n(d(fv_n,fp) - \varphi(d(fv_n,fp)) + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n d(fv_n,fp) - \alpha_n\varphi(d(fv_n,fp)) + (1-\alpha_n)d(f(x_n,q)\\
&\le \alpha_n d(fv_n,q) - \alpha_n\varphi(d(fv_n,q)) + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n d(W(Tx_n,fx_n,\beta_n,q) - \alpha_n\varphi(d(fv_n,q)) + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n(\beta_n d(Tx_n,q) + (1-\beta_n)d(fx_n,q)) - \alpha_n\varphi(d(fv_n,q))\\
&\qquad + (1-\alpha_n)d(f(x_n,q)\\
&\le \alpha_n\beta_n d(Tx_n,q) + \alpha_n(1-\beta_n)d(f(x_n,q) - \alpha_n\varphi(d(fv_n,q))\\
&\qquad + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n\beta_n d(Tx_n,Tp) + \alpha_n(1-\beta_n)d(fx_n,q) - \alpha_n\varphi(d(fv_n,q))\\
&\qquad + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n\beta_n(d(fx_n,q) - \varphi(d(fx_n,q)) + \alpha_n(1-\beta_n)d(fx_n,q) - \alpha_n\varphi(d(fv_n,q))\\
&\qquad + (1-\alpha_n)d(fx_n,q)\\
&\le \alpha_n\beta_n d(fx_n,q) - \alpha_n\beta_n\varphi(d(fx_n,q)) + \alpha_n d(fx_n,q) - \alpha_n\beta_n d(fx_n,q)\\
&\qquad - \alpha_n\varphi(d(fv_n,q)) + d(fx_n,q) - \alpha_n d(fx_n,q)\\
&\le d(fx_n,q) - \alpha_n\beta_n(d(fx_n,q)) - \alpha_n(d(fv_n,q)) \le d(fx_n,q).
\end{aligned}$$

Thus $\{d(z_n,q)\}$ is a nonincreasing nonnegative sequence which converges to the limit $l \ge 0$. Suppose $l > 0$, then for any fixed integer $N$ we have

$$\sum_{n=N}^{\infty}\alpha_n\beta_n\varphi(l) \le \sum_{n=N}^{\infty}\alpha_n\beta_n\varphi(d(z_n,q)) \le \sum_{n=N}^{\infty} d(z_n,q) - d(z_{n+1},q) \le d(z_N,q),$$

which contradicts $\sum \alpha_n\beta_n = \infty$. Hence the iterative scheme $\{z_n\}$ converges to a common fixed point of $T$ and $f$. ∎

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REFERENCES

[**1**] Ya. I. Alber and S. Guerr-Delabriere, *Principle of weakly contractive maps Hilbert spaces*, New Results in Operator Theory and its Applications (I.Gohberg and Yu. Lyubich, eds.), Oper. Theory Adv. Appl., vol. 98, Birkhäuser, Basel, 1997, pp. 7–22.

[**2**] A. Azam, *Noncommuting mappings and common fixed points*, J. Nat. Sci. Math. **45** (1& 2) (2005), 15–20.

[**3**] I. Beg, *Inequalities in metric spaces with applications*, Topological Methods in Nonlinear Analysis, **17** (1) (2001), 183–190.

[**4**] I. Beg, *An interation process for nonlinear mappings in uniformly convex linear metric spaces*, Czechoslovak Math. J., **53** (128) (2003), 405–412.

[**5**] I. Beg and M. Abbas, *Fixed points and best approximation in Menger convex metric spaces*, Arch. Math. **41** (2005), 389–397.

[**6**] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl. 2006 (2006), 1–7.

[**7**] I. Beg and A. Azam, *Construction of fixed points for generalized nonexpansive mappings*, Indian J. Math. **38**, 2 (1996), 161–170.

[**8**] I. Beg and A. Azam, *Fixed point theorems for Kannan mappings*, Indian J. Pure Appl. Math. **17**, 11 (1986), 1270–1275.

[**9**] I. Beg and A. Azam, *Fixed points on starshaped subset of convex metric spaces*, Indian J. Pure Appl. Math. **18** (1987), 594–596.

[**10**] I. Beg, A. Azam, F. Ali and T. Minhas, *Some fixed point theorems in convex metric spaces*, Rend. Circolo Mat. Palermo, **XL** (1991), 307–315.

[**11**] I. Beg, N. Shahzad and M. Iqbal, *Fixed point theorems and best approximation in convex metric spaces*, J. Approx. Theory ppl., **8** (4) (1992), 97–105.

[**12**] X. P. Ding, *Iteration process for nonlinear mappings in convex metric spaces*, J. Math. Anal. Appl. **132** (1988), 114–122.

[**13**] M. D. Guay, K. L. Singh and J. H. M. Whitfield, *Fixed point theorems for nonexpansive mappings in convex metric spaces*, Proceedings, Conference on Nonlinear Analysis (S. P. Singh and J. H. Burry, Eds), Marcel Dekker, Inc., New York. **80** (1982), 179–189.

[**14**] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.

[**15**] G. Jungck, *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc. **103** (1988), 977–983.

[**16**] G. Jungck and B.E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math. **29**, 3 (1998), 227–238.

[**17**] S. A. Naimpally, K. L. Singh and J. H. M. Whitfield, *Fixed points in convex metric spaces*, Math. Japonica, **29** (4) (1984), 585–597.

[**18**] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), 436–440.

[**19**] R. P. Pant, *Common fixed points for contractive maps*, J. Math. Anal. Appl. **226** (1998), 281–286.

[**20**] R. P. Pant, *R-weak commutativity and common fixed points*, Soochoo J. Math. **25** (1) (1999), 37–42.

[**21**] B. E.Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis **47**, 4 (2001), 2683–2693.

[**22**] S. Sessa, *On a weak commutativity condition of mappings in fixed considerations*, Publ. Inst. Math. **32** (1982), 149–153.

[**23**] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings I*, Kodai Math. Sem. Rep., **22** (1970), 142–149.

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